Questão B3) (Valor: 2.0)

a) Prove que: $\left|\sqrt{1+x} - \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\right)\right| \leq \frac{x^4}{16}$, $\forall x \geq 0$.

b) Prove que $\int_0^1 \sqrt{1+x^4}\, dx \approx 1 + \frac{1}{2.5} - \frac{1}{8.9} + \frac{1}{13.16}$, com erro, em módulo, inferior a $\frac{1}{200}$.

a) Sejam $f(x) = \sqrt{1+x}$ e $x_0 = 0$.

$f(x) = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2} \Rightarrow f'''(0) = \frac{3}{8}$

$f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2}$

$\Rightarrow P_3(x) = 1 + \frac{1}{2}x - \frac{1}{4}\cdot\frac{1}{2}x^2 + \frac{3}{8}\cdot\frac{1}{6}x^3 = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$

$\exists\ \bar{x}$ entre x e x_0 tal que $f(x) - P_3(x) = E(x)$, onde $E(x) = \frac{f^{(4)}(\bar{x})}{4!}x^4 = \frac{-15}{4!\,16}(1+\bar{x})^{-7/2}x^4$, $\forall x \geq 0$

$\Rightarrow |f(x) - P_3(x)| = |E(x)| = \left|\frac{-15}{24}\cdot\frac{1}{16}(1+\bar{x})^{-7/2}x^4\right| \leq \frac{x^4}{16}$, pois $\left|\frac{-15}{24}\right| < 1$ e $|1+\bar{x}|^{-7/2} < 1$, $\forall x > 0$.

b) $f(x^4) = \sqrt{1+x^4}$, $P_3(x^4) = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16}$, $|E(x^4)| \leq \frac{x^{16}}{16}$

$$\int_0^1 \sqrt{1+x^4}\,dx \simeq \int_0^1 P_3(x^4)\,dx = \left[x + \frac{x^5}{2.5} - \frac{x^9}{8.9} + \frac{x^{13}}{13.16}\right]_0^1 = 1 + \frac{1}{2.5} - \frac{1}{8.9} + \frac{1}{13.16}$$

$$\left|\int_0^1 \sqrt{1+x^4}\,dx - \int_0^1 P_3(x^4)\,dx\right| = \left|\int_0^1 E(x^4)\,dx\right| \leq \int_0^1 |E(x^4)|\,dx \leq \int_0^1 \frac{x^{16}}{16}\,dx =$$

$$= \frac{x^{17}}{16.17}\Bigg|_0^1 = \frac{1}{16.17} = \frac{1}{272} < \frac{1}{200}$$