Informally, imagine an object that falls apart randomly as time passes; the state of the system at some given time consists in the sequence of the sizes of the pieces, which are viewed as particles. Suppose that the evolution is Markovian and obeys the following rule. First, different particles evolve independently, in other words the so-called branching property is fulfilled. Second, there is a parameter $\alpha \in \mathbb{R}$, which will be referred to as the index of self-similarity in the sequel, such that each particle with size $s > 0$ has an exponential lifetime with parameter proportional to $s^\alpha$. At its death, this particle splits and there results a family of sub-fragments, say with sizes $(s_i, i \in \mathbb{N})$, where the sequence of the ratios $(s_i/s, i \in \mathbb{N})$ has the same distribution for all particles.

The purpose of this short course is to construct and present some basic properties of such self-similar fragmentation. In this direction, material on so-called branching Markov chains and randomly marked trees will provide very convenient frameworks for the study.

1 Preliminaries on (branching) Markov chains

In this section, we briefly present some basic elements on Markov chains and branching Markov chains in continuous time which will be useful for the construction and the study of certain more complex models in the sequel.

1.1 Markov chains

Let $(E, d)$ be a Polish space, that is a complete metric separable space, and $(q(x, \cdot), x \in E)$ a family of finite measures on $E$. We assume that the map $x \to q(x, \cdot)$ is weakly measurable, i.e. for every bounded continuous function $f : E \to \mathbb{R}$, the map $x \to \int_E f(y)q(x, dy)$ is measurable. For every $x \in E$, we write

$$\bar{q}(x, \cdot) = q(x, \cdot)/q(x, E)$$
with the convention that $\bar{q}(x, \cdot)$ is the Dirac point mass at $x$ in the case when $q(x, E) = 0$. So $(\bar{q}(x, \cdot), x \in E)$ a measurable family of probability measures on $E$, that is a Markov kernel. It is well-known that we can use $(\bar{q}(x, \cdot), x \in E)$ as the transition probabilities of a Markov sequence $Y = (Y(0), Y(1), \ldots)$, also called Markov chain in discrete times.

Next, let $\varepsilon_0, \varepsilon_1, \ldots$ a sequence of i.i.d. standard exponential variables, which is independent of $Y$. We associate to every sample path of $Y$ the additive functional

$$A(n) := \sum_{i=0}^{n} \varepsilon_i/q(Y(i), E), \quad n = 0, 1, \ldots$$

and its inverse

$$\alpha(t) = \min \{ n : A(n) > t \}, \quad t \geq 0,$$

which is well-defined for all $t \geq 0$ whenever

$$\sum_{i=0}^{\infty} \varepsilon_i/q(Y(i), E) = \infty \quad \text{a.s.}$$

In this direction, we recall that (1) holds if and only if

$$\sum_{i=0}^{\infty} 1/q(Y(i), E) = \infty \quad \text{a.s.}$$

Indeed, let $(y_i)$ be some sequence of points in $E$. On the one hand, the identity

$$\mathbb{E}\left( \sum_{i=0}^{\infty} \varepsilon_i/q(y_i, E) \right) = \sum_{i=0}^{\infty} 1/q(y_i, E)$$

shows that if the series in the right-hand side converges, then $\sum_{i=0}^{\infty} \varepsilon_i/q(y_i, E) < \infty$ a.s. Conversely, taking the Laplace transform, we get

$$\mathbb{E}\left( \exp\left( - \sum_{i=0}^{\infty} \varepsilon_i/q(y_i, E) \right) \right) = \prod_{i=0}^{\infty} \frac{q(y_i, E)}{1 + q(y_i, E)} = \exp\left( \sum_{i=0}^{\infty} \log \left( 1 - 1/q(y_i, E) \right) \right).$$

If the series $\sum_{i=0}^{\infty} \varepsilon_i/q(y_i, E)$ diverges a.s., then the right-hand side above is zero, and hence $\sum_{i=0}^{\infty} 1/q(y_i, E) = \infty$ a.s. We also observe that (2) is plainly fulfilled whenever

$$\sup_{x \in E} q(x, E) < \infty$$

otherwise checking whether (2) holds may be tedious.

Taking (2) for granted, we may construct by time-substitution a process in continuous times $X = (X(t), t \geq 0)$ by the identity

$$X(t) := Y(\alpha(t)), \quad t \geq 0.$$ 

It is easily seen that $X$ is a Markov process, known as a continuous time Markov chain. The time-substitution construction can be rephrased as a so-called hold-jump description: the states $x \in E$ with $q(x, E) = 0$ are absorbing, that is $\mathbb{P}(X(t) = x \mid X(0) =$
\( x = 1 \), and starting from some non-absorbing state \( x \in E \) with \( q(x, E) > 0 \), the process \( X \) stays at \( x \) up to the holding time \( \varepsilon_0/q(x, E) \) which has an exponential distribution with parameter \( q(x, E) \), and then jumps \(^1\) according to the probability distribution \( \tilde{q}(x, \cdot) \), independently of the holding time. However this induces no difficulty whatsoever, and it is convenient not to distinguish this exponential holding time is strictly positive, so strictly speaking there may be no jump after this first holding time. Hence it is easy to check from this description that for every continuous bounded function \( f : E \to \mathbb{R} \),
\[
\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}(f(X(t)) - f(X(0)) \mid X(0) = x) = \int_E (f(y) - f(x)) q(x, dy), \tag{4}
\]
which identifies the infinitesimal generator of \( X \). In particular, either the very construction of \( X \) or \( (4) \), shows that the family \( (q(x, \cdot), x \in E) \) can be thought of as the jump rates of \( X \), and entirely characterize the distribution of Markov chain \( X \).

**Example:** The so-called compound Poisson processes form one of the simplest and best known family of Markov chains in continuous time. Specifically, consider the special case when \( E \) is some Euclidean space and the jump rates \( (q(x, \cdot), x \in E) \) are translation invariant, i.e. for every \( x \in E \), \( q(x, \cdot) \) is the image of the some finite measure \( q(0, \cdot) \) by the translation \( y \to x + y \). The measure \( q(0, \cdot) \) is known as the \( \text{Lévy measure} \) of the compound Poisson process. Plainly, the transition probabilities \( (\tilde{q}(x, \cdot), x \in E) \) are also translation invariant, and hence the Markov sequence \( \{(Y(0), Y(1), \ldots) \} \) is a random walk with step distribution \( \tilde{q}(0, \cdot) \). Since \( c := q(x, E) \) does not depend on the state \( x \), the continuous time Markov chain with jump rates \( (q(x, \cdot), x \in E) \) can be obtained as the composition \( X = Y \circ N \) where \( N = (N_t, t \geq 0) \) is a Poisson process with intensity \( c \) which is independent of the random walk \( Y \). This construction triggers the denomination \( \text{compound Poisson process} \) for such Markov chains.

More generally, it is easy to see that a Markov chain in continuous times can be constructed as a Markov sequence time-changed by an independent Poisson process if and only if the jump rates are bounded, that is if and only if \( (3) \) holds.

### 1.2 Branching Markov chains and random walks

We call \( \text{finite point measure} \) on \( E \) any measure of the type \( m = \sum_{i=1}^n \delta_{x_i} \) where \( n = 0, 1, \ldots \) and \( x_i \in E \) (for \( n = 0 \), the measure \( m = 0 \) is trivial). We denote by \( \mathcal{M}_p(E) \) the space of finite point measures on \( E \), endowed with the distance
\[
\text{dist}(m, m') := \sup_{f \in \mathcal{L}} \left| \int_E f(x) m(dx) - \int_E f(x) m'(dx) \right|,
\]
where \( \mathcal{L} \) stands for the space of Lipschitz-continuous functions \( f : E \to \mathbb{R} \) with \( |f(x) - f(y)| \leq d(x, y) \) for every \( x, y \in E \). It is easy to check that this distance is equivalent to the Prohorov metric on the space of finite measures on \( E \), and that \( (\mathcal{M}_p(E), \text{dist}) \) is a Polish space.

\(^1\)In the case when \( q(x, \{x\}) > 0 \), the probability that process \( X \) stays at the state \( x \) after the exponential holding time is strictly positive, so strictly speaking there may be no jump after this first holding time. However this induces no difficulty whatsoever, and it is convenient not to distinguish this degenerate case.
Next, we introduce some notation related to finite point measures. Given \( m \in \mathcal{M}_p(E) \), we denote by \( \theta_m : \mathcal{M}_p(E) \to \mathcal{M}_p(E) \) the map defined by \( \theta_m(m') = m + m' \). By a slight abuse in notation, for every finite measure \( \nu \) on \( \mathcal{M}_p(E) \) and \( m \in \mathcal{M}_p(E) \), we denote by \( \theta_m(\nu) \) the image of \( \nu \) by the map \( \theta_m \).

We consider a family \((\nu_x, x \in \Xi)\) of finite measures on \( \mathcal{M}_p(E) \), which depends measurably on the variable \( x \). We may associate to this family, a measurable kernel \((q(m, \cdot), m \in \mathcal{M}_p(E))\) of finite measures on \( \mathcal{M}_p(E) \) defined as follows. First, \( q(0, \cdot) = 0 \), and if \( m = \sum_{i=1}^n \delta_{x_i} \) with \( n \geq 1 \), then

\[
q(m, \cdot) := \sum_{i=1}^n \theta_{m_i}(\nu_{x_i}),
\]

where \( m_i = \sum_{j \neq i} \delta_{x_j} \).

Let us first describe the evolution of the Markov chain \( Y \) in discrete times with transition probabilities \( q(m, \cdot) = q(m, \cdot)/q(m, \mathcal{M}_p(E)) \) for \( m \in \mathcal{M}_p(E) \). As \( q(0, E) = 0 \), 0 is an absorbing state for the chain. Then, let the chain start from some finite point measure \( m = \sum_{i=1}^n \delta_{x_i} \neq 0 \). The distribution of next state of the chain is obtained by picking an atom \( x \) at random among those of \( m \) with probability proportional to \( \nu_x(\mathcal{M}_p(E)) \), and replacing it by that of a random point measure with law \( q(x, \cdot) \).

Now, we should like to consider a continuous time Markov chain with jump rates \((q(m, \cdot), m \in \mathcal{M}_p(E))\), so that we need conditions ensuring (2) (observe that the stronger condition (3) cannot hold).

**Lemma 1** In the preceding notation, (2) is fulfilled whenever there is a finite constant \( c > 0 \) such that

\[
\int_{\mathcal{M}_p(E)} \nu_x(dm)q(m) \leq c \nu_x(\mathcal{M}_p(E)), \quad \forall x \in E,
\]

where \( q(m) \) is the total jump rate from state \( m \) to state \( m' \) in \( \mathcal{M}_p(E) \), i.e.

\[
q(m) := q(m, \mathcal{M}_p(E)) = \sum_{i=1}^n \nu_{x_i}(\mathcal{M}_p(E))
\]

if \( m = \sum_{i=1}^n \delta_{x_i} \).

**Proof:** Let the Markov sequence \( Y \) start from some finite point measure; we have to check that \( \sum_{i=0}^{\infty} 1/q(Y(i)) = \infty \) a.s. For every integer \( n = 1, \ldots, \), set

\[
\xi(n) := \sum_{i=1}^k \nu_{y_i}(\mathcal{M}_p(E)),
\]

where \( y_1, \ldots, y_k \) denote the particles which are created at the \( n \)-th step of \( Y \), and then \( \sigma(n) = \xi(1) + \cdots + \xi(n) \). Plainly, we have \( q(Y(n)) \leq q(Y(0)) + \sigma(n) \).

The condition (5) of the lemma ensures that \( \mathbb{E}(\xi(n)) \leq c \), so that \( \mathbb{E}(\sigma(n)) \leq cn \) for all \( n \). Thus, by Fatou’s lemma, we have

\[
\liminf_{n \to \infty} \frac{q(Y(0)) + \sigma(n)}{n} < \infty \quad \text{a.s.,}
\]
or equivalently
\[ \limsup_{n \to \infty} \frac{n}{q(Y(0)) + \sigma(n)} > 0 \quad \text{a.s.} \]

Since the random sequence \( \sigma(\cdot) \) is increasing, the series \( \sum_{n=1}^{\infty} 1/(q(Y(0)) + \sigma(n)) \) diverges a.s., and thus so does \( \sum_{n=1}^{\infty} 1/q(Y(n)) \).

Let us take the conditions of Lemma 1 for granted. The Markov chain (in continuous times) \( X = (X(t), t \geq 0) \) on \( \mathcal{M}_p(E) \) associated with the family \( (q(m, \cdot), m \in \mathcal{M}_p(E)) \) is called a branching Markov chain; let us describe informally its dynamics. The atoms \( x_1, \ldots, x_n \) of a finite point measure \( m = \sum_{i=1}^{n} \delta_{x_i} \) are thought of as particles on \( E \) which will have independent evolutions. A particle located at \( x \) lives for an exponential time with parameter \( \nu_x(M_p(E)) \) (so it is immortal when \( \nu_x(M_p(E)) = 0 \)). At its death, the particle is removed and replaced by a random cloud of particles, say \( y_1, \ldots, y_k \), where the finite point measure \( \sum_{i=1}^{k} \delta_{y_i} \) is distributed according to the probability measure \( \nu_x(\cdot)/\nu_x(M_p(E)) \) on \( \mathcal{M}_p(E) \).

**Proposition 1** (Branching property) Let \( X \) and \( X' \) two independent versions of a same branching Markov chain, started respectively from two point measures \( m \) and \( m' \). Then \( X + X' \) is a version of the branching Markov chain started from \( m + m' \).

**Proof:** It is easily checked that \( X + X' \) has the Markov property and that its transition rates coincide with those of the branching Markov chain.

**Branching random walks** form a special class of branching Markov chains which have been introduced by Uchiyama [27]. Specifically, we now assume that \( E \) is some Euclidean space (or, more generally, on some nice topological group), and we consider a finite measure \( \nu \) on the space \( \mathcal{M}_p(E) \) of finite point measure on \( E \). Next, to each site \( x \in E \), we associate to \( \nu \) its image by the translation of its atoms by \( x \). That is, for every finite point measure \( m = \sum_{i=1}^{n} \delta_{x_i} \), we denote the shifted measure by \( m_x := \sum_{i=1}^{n} \delta_{x+x_i} \), and \( \nu_x \) is the image of \( \nu \) by this shift. The branching Markov chain (in continuous times) with rates \( (\nu_x, x \in E) \), is called a branching random walk with branching measure \( \nu \).

In this direction, observe that the condition (5) of Lemma 1 reduces to
\[ \int_{\mathcal{M}_p(E)} \nu(dm)m(E) < \infty. \]

It is easy to see that the process of total mass \( (X(t)(E), t \geq 0) \) is a classical branching process. More precisely, each particle lives for an exponential lifetime with parameter \( \nu(M_p(E)) \), and at its death, it gives birth to a random number \( Z \) of children, where the offspring distribution is specified by
\[ \mathbb{P}(Z = n) = \frac{1}{\nu(M_p(E))} \int_{\mathcal{M}_p(E)} \nu(dm)1_{\{m(E) = n\}}. \]
2 Fragmentation processes

2.1 Definitions and first properties

Throughout the rest of this chapter, we shall work with the state space of decreasing numerical sequences bounded by 1

\[ S^↓ := \{ s = (s_1, s_2, \ldots) : 1 \geq s_1 \geq s_2 \geq \ldots \geq 0 \} , \]

deedowed with the distance

\[ \text{dist}(s, s') = \sum_{i=1}^{\infty} 2^{-n} |s_i - s'_i| \]

which makes \( S^↓ \) metrizable and locally compact. We shall think of a sequence \( s \in S^↓ \) as that of the sizes of the fragments resulting from the split of some object with unit size. The total sum \( \sum_{i=1}^{\infty} s_i \) of the series \( s \in S^↓ \) may equal 1 (which corresponds to the so-called conservative situation), may be less than 1 (dissipative case), or even greater than 1 (for instance when the size of an object is the measure of its diameter). In most cases of interest, \( \lim_{i \to \infty} s_i = 0 \) and then it is sometimes convenient to identify the sequence with a Radon point measure on \([0, \infty[\), \( \sum_{i: s_i > 0} \delta_{s_i} \).

In this chapter, we will be interested in a simple family of Markov process with càdlàg paths \( X = (X(t), t \geq 0) \) with values in the space \( S^↑ \), called fragmentation processes. Informally, the evolution of \( X = (X(t), t \geq 0) \) is given by a non-interacting particle system, in the sense that each particle in the system evolves independently of the others. The dynamics of each particle are the following. A particle lives for an exponential time with parameter depending only on its size, and at its lifetime, it is replaced by a cloud of smaller particles which is independent of the lifetime of the particle.

Example (Packing process): Let \( U_1, \ldots, U_n \) be a sequence of i.i.d. uniform variables, and \( \ell_1, \ell_2, \ldots \) a sequence of i.i.d. random variables, which is independent of the \( U_i \)'s. We throw successively the random intervals \([U_i, U_i + \ell_i]\) on \([0, 1[\) at the jump times \( T_1 < T_2 < \ldots \) of an independent Poisson process \( N = (N_t, t \geq 0) \). We construct a process of nested open subsets \( \theta := (\theta(t), t \geq 0) \) as follows. The process starts from \( \theta(0) := [0, 1[ \) and remains constant except at times \( T_1, T_2, \ldots \) where it may jump. Specifically, if \([U_i, U_i + \ell_i]\) is entirely contained into \( \theta(T_i-) \), then \( \theta(T_i) = \theta(T_i-)\setminus[U_i, U_i + \ell_i] \). Otherwise \( \theta(T_i) = \theta(T_i-) \). Elementary properties of Poisson point processes easily entail that the process of the ranked lengths of the interval components of \( \theta(\cdot) \) is a dissipative fragmentation process. See Baryshnikov and Gnedin [2] (in relation with a packing problem related to communication networks) for much more on this example.

Although the description of fragmentation processes above bears obvious similarities with that for branching Markov chains, it is not completely obvious that such a random evolution is well-defined, because the jump rates from a configuration \( S^↓ = (s_1, \ldots) \) with \( s_i > 0 \) for every \( i \in \mathbb{N} \) may be (and indeed often are) unbounded. We shall now explain precisely the construction of this model.

To start with, for every \( x \in [0, 1] \), let \( \nu_x \) be some finite measure on \( S^↓ \) such that \( \sum_{i=1}^{\infty} s_i \leq x \) for \( \nu_x(ds) \)-almost all \( s \in S^↓ \). We assume that this family depends in a
measurable way on the variable \( x \). As previously, the total mass \( \nu_x(S^1) \) is parameter of the exponential lifetime of the particle, and the probability law \( \nu_x(\cdot)/\nu_x(S^1) \) is the law of the cloud of particles resulting from the dislocation of \( x \). We shall now explain how, under the following mild condition on \( \nu_x \):

\[
\sup_{x \in S^1} \nu_x(S^1) < \infty \quad \text{for every } \varepsilon > 0,
\]

the evolution of the system is well-defined.

Call a sequence \( s = (s_1, \ldots) \in S^1 \) finite if \( s_j = 0 \) for all large enough index \( j \); clearly we may (and will) identify a finite sequence \( s \) as a finite point measure on \([0, 1]\), \( m = \sum \delta_{s_i} \), where the sum is taken over indices \( i \) such that \( s_i > 0 \). Next, we introduce operators of threshold which map \( S^1 \) on the space of finite sequences. Specifically, for every \( \varepsilon > 0 \), we write \( \varphi_\varepsilon : [0, 1] \to [0, 1] \) the function such that \( \varphi_\varepsilon(x) = x \) if \( x > \varepsilon \) and \( \varphi_\varepsilon(x) = 0 \) otherwise, and, by a slight abuse in notation, we still write \( \varphi_\varepsilon : S^1 \to S^1 \) for its obvious extension to \( S^1 \) (component by component).

Let \( \nu^\varepsilon_x \) the image of \( \nu_x \) by the threshold operator \( \varphi_\varepsilon \); so we can think of \( \nu^\varepsilon_x \) as a finite measure on the space of point measures on \([0, 1]\). In this framework, it is easy to see that whenever (6) holds, the conditions of Lemma 1 are fulfilled, and thus we can construct a branching Markov chain in continuous times \( X^\varepsilon = (X^\varepsilon(t), t \geq 0) \) with jump rates given by the family \((\nu^\varepsilon_x, x \in [0, 1])\). This branching Markov chain takes values in the space of finite point measures on \([0, 1]\), but we may also view it as a process with values in \( S^1 \) by the preceding identification.

**Lemma 2** In the notation above, for every \( 0 < \eta < \varepsilon \), the threshold operator \( \varphi_\varepsilon \) transforms the process \( X^\eta \) into a Markov process with the same distribution as \( X^\varepsilon \).

**Proof:** By construction, the size of a child particle is always smaller than the size of its father. As a consequence, for every \( s, t \geq 0 \), the conditional distribution of \( \varphi_\varepsilon(X^\eta(t + s)) \) given \( X^\eta(t) \) only depends on \( \varphi_\varepsilon(X^\eta(t)) \). It follows that \( \varphi_\varepsilon(X^\eta(\cdot)) \) is a Markov process, more precisely a Markov chain in continuous times, and since \( \nu^\varepsilon_x \) is the image of \( \nu^\eta_x \) by the threshold operator \( \varphi_\varepsilon \), the jump rates of \( \varphi_\varepsilon(X^\eta(\cdot)) \) are the same as those of \( X^\varepsilon(\cdot) \). Thus, the two processes have the same distribution. \( \square \)

Lemma 2 enables us to appeal to Kolmogorov’s projective theorem, and we obtain the existence of a family of process \((X^\varepsilon(\cdot), \varepsilon > 0)\) such that for every \( \varepsilon > \eta > 0 \), \( \varphi_\varepsilon(X^\eta(\cdot)) \) has the same law as \( X^\varepsilon(\cdot) \). Plainly, if we are given a family \((m^\varepsilon, \varepsilon > 0)\) of point measures on \([0, 1]\) such that for every \( \varepsilon > \eta > 0 \), \( m^\varepsilon \) is the image of \( m^\eta \) by the threshold operator \( \varphi_\varepsilon \), there exists a unique sigma-finite measure \( m \) on \([0, 1]\) such that \( m^\varepsilon \) is the image of \( m \) by \( \varphi_\varepsilon \). Thus the family of processes \((X^\varepsilon(\cdot), \varepsilon > 0)\) can be obtained as the images of a same process \( X(\cdot) \) by the threshold operators.

Let us now turn our attention to self-similarity. Suppose that \( \nu \) is some finite measure on \( S^1 \). For every \( x \in [0, 1] \), write \( \nu_x \) for the image of \( x^\alpha \nu \) by the shrink \( s \to xs \). Clearly, the condition (6) holds, and the fragmentation process \( X \) constructed as above from the dislocation rates \((\nu_x, x \in [0, 1])\), is self-similar with index \( \alpha \). Clearly, the law of such a
self-similar fragmentation process is entirely determined by the index of self-similarity \( \alpha \) and the finite measure \( \nu \) which will be called the dislocation measure. The evolution of the process can be described as follows: a fragment with size \( x \) lives for an exponential time with parameter \( x^{\alpha} \nu(S^1) \), and then splits and gives rise to a family of smaller fragments distributed as \( x \xi \), where \( \xi \) has the law \( \nu(\cdot)/\nu(S^1) \). We stress that for \( \alpha < 0 \), this description a priori makes sense only when the size \( x \) is non-zero; however by self-similarity, the children of a particle with size 0 all have size zero. Particles with size zero play no role, and the evolution is thus well-defined in all cases.

**Example** (Poissonian rain). Consider a Poisson point process with values in the unit interval and characteristic measure given by the Lebesgue measure on \([0,1]\). In other words, we have a sequence \( U_1, \ldots \) of i.i.d. uniform variables on \([0,1]\); the times when they appear are the jump times of some independent Poisson process \((N_t, t \geq 0)\). Now, think of these Poissonian points as drops of rains, and consider for every time \( t \geq 0 \) the subset of \([0,1]\) which is dry, viz. \( \theta(t) := [0,1] \setminus \{U_i : i \leq N_t\} \). So \( \theta(t) \) is a random open set which consists of \( N_t + 1 \) open intervals. If we write \( X_1(t) \geq \ldots \geq X_{N_t+1}(t) \) for the sequence of their lengths ranked in the decreasing order and \( X_i(t) = 0 \) for \( i > N_t + 1 \), then it is easy to check that the process \( X = (X(t), t \geq 0) \) is a (conservative) self-similar fragmentation with index \( \alpha = 1 \), with dislocation measure \( \nu \) given by the distribution of \((V,1_V,0,\ldots)\) with \( V \) uniformly distributed on \([1/2,1]\). This simple fragmentation process bears remarkable connections with the celebrated coalescent of Kingman [20], which is used in biology as a model for the genealogy of large populations. Recently, Bertoin and Goldschmidt [7] observed that this same process also arises when describing the genealogy of the Yule process.

Plainly, the behavior of a self-similar fragmentation depends crucially on its index of self-similarity. Informally, fragments get smaller as time passes; the rate of dislocations thus decreases when the index is positive, whereas it increases when \( \alpha < 0 \).

Throughout the rest of this chapter, we shall consider a self-similar fragmentation process \( X = (X(t), t \geq 0) \) with index \( \alpha \in \mathbb{R} \) and finite dislocation measure \( \nu \). In order to avoid uninteresting discussions of degenerated cases, we shall always implicitly assume that \( \nu \) is not a linear combination of the Dirac point masses at \( 0 := (0,\ldots) \) and \( 1 := (1,0,\ldots) \). It should be clear now that a self-similar fragmentation process fulfils the following basic properties.

**Proposition 2** For every \( x \in [0,1] \), we denote by \( \mathbb{P}_x \) the law of \( X \) started from the configuration \((x,0,\ldots)\), that is, at the initial time, there is just one particle with size \( x \).

(i) The fragmentation process \( X \) has the branching property, namely for every sequence \( s = (s_1,\ldots) \in S^1 \) and every \( t \geq 0 \), the distribution of \( X(t) \) given that \( X(0) = s \) is the same as that of the decreasing rearrangement of the terms of independent random sequences \( X^{(1)}(t), X^{(2)}(t), \ldots \), where for each \( i \in \mathbb{N} \), \( X^{(i)}(t) \) is distributed as \( X(t) \) under \( \mathbb{P}_{s_1} \).

(ii) The fragmentation process \( X \) has the scaling property, namely for every \( x \in [0,1] \), the distribution of the rescaled process \((xX(x^{\alpha}t), t \geq 0) \) under \( \mathbb{P}_1 \) is \( \mathbb{P}_x \).
2.2 The tree of generations and the intrinsic martingale

In this section, we point at a representation of fragmentation processes as random infinite marked trees. This representation can be viewed as a different parametrization of the process, in which the natural time is replaced by the generation of the different particles. Specifically, we consider the infinite tree

$$ \mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n, $$

with the convention $\mathbb{N}^0 = \{\emptyset\}$. In the sequel $\mathcal{U}$ will often be referred to as the genealogical tree; its elements are called nodes, they will be used to label the particles produced by a fragmentation process. For each $u = (u_1, \ldots, u_n) \in \mathcal{U}$, we call generation of $u$ and write $|u| = n$, with the obvious convention $|\emptyset| = 0$. When $n \geq 1$ and $u = (u_1, \ldots, u_n)$, we call $u^- := (u_1, \ldots, u_{n-1})$ the father of $u$. Similarly, we write $u, i = (u_1, \ldots, u_n, i) \in \mathbb{N}^{n+1}$ for the $i$-th child of $u$. Finally, we call mark any map from $\mathcal{U}$ to some (measurable) set.

Now, consider a self-similar fragmentation process $X = (X(t), t \geq 0)$ with index $\alpha \in \mathbb{R}$ and finite dislocation measure $\nu$. Suppose for simplicity that the process starts from a single fragment with size $x > 0$, that is $X(0) = (x, 0, \ldots)$. We associate to each path of the process a mark on the infinite tree $\mathcal{U}$; roughly the mark at a node $u$ is the triple $(\xi_u, a_u, \zeta_u)$ of the size, birth-time and lifetime of the particle with label $u$. More precisely, the initial particle $x$ corresponds to the ancestor $\emptyset$ of the tree $\mathcal{U}$, and the mark at $\emptyset$ is the triple $(x, 0, \zeta_\emptyset)$ where $\zeta_\emptyset$ is the lifetime of the initial particle (in particular, $\zeta_\emptyset$ has the exponential law with parameter $x^\alpha \nu(S^1)$). The nodes of the tree at the first generation are used as the labels of the particles arising at the first split. Again, the mark associated to each of the nodes $i \in \mathbb{N}_1$ at the first generation, is the triple $(\xi_i, a_i, \zeta_i)$, where $\xi_i$ is the size of the $i$-th child of the ancestor, $a_i = a_\emptyset + \zeta_\emptyset$ (the birth-time of a child particle coincides with the death-time of the father), and $\zeta_i$ stands for the lifetime of the $i$-th child. And we iterate the same construction with each particle at each generation.

Clearly, the description of the dynamics of fragmentation entails that its genealogical representation also enjoys the branching property. Specifically, the distribution of the random mark can be described recursively as follows.

**Proposition 3** Given the marks $((\xi_v, a_v, \zeta_v), |v| \leq n)$ of the first $n$ generations, the marks at generation $n + 1$ can be expressed in the form

$$ (\xi_u, a_u, \zeta_u) = (\tilde{\xi}_u \xi_{u-}, a_{u-} + \zeta_{u-}, \xi_u^0 e_u), \quad u = (u_1, \ldots, u_{n+1}), $$

where $u^- := (u_1, \ldots, u_n)$ is the father of $u$ and

- as $u-$ describes the nodes at the $n$-th generation, the sequences $\xi_{u-} := (\xi_{u_1}, \ldots, u_{n+1}), i \in \mathbb{N}$ are i.i.d. random variable in $S^1$ with the law $\nu(\cdot)/\nu(S^1)$,
- as $u$ describes the nodes at the $(n + 1)$-th generation, the $e_u$’s are i.i.d. exponential variables with parameter $\nu(S^1)$ which are independent of the sequences $\xi_{u-}$’s.

Now to every node $u$ of the genealogical tree, we can associate the interval $I_u := [a_u, a_u + \zeta_u]$ during which the particle labelled by $u$ is alive. Putting the pieces together,
we may express the fragmentation process at time $t$ as the ranked sequence of the particles which are alive at time $t$, that is in terms of random point measures

$$\sum_{i=1}^{\infty} \delta_{X_i(t)} = \sum_{u \in \ell_t} 1_{\{t \in \ell_u\}} \delta_{\xi_u}.$$ 

We now conclude this section by introducing the so-called intrinsic martingale which is naturally induced by the tree representation. This martingale will play a crucial role when we will investigate the asymptotic behavior of fragmentation processes. We start by introducing the notation

$$\bar{p} := \inf \left\{ p > 0 : \int_{S^1} \sum_{i=1}^{\infty} s_i^p \nu(ds) < \infty \right\}$$

and then for every $p \geq \bar{p}$

$$\kappa(p) := \int_{S^1} \left( 1 - \sum_{i=1}^{\infty} s_i^p \right) \nu(ds).$$

(7)

Note that $\kappa$ is always a strictly increasing function on $[\bar{p}, \infty[; \kappa(\bar{p})$ may be finite or equal to $-\infty$.

For instance, consider the so-called uniform stick-breaking scheme, that is, cut the unit interval at a uniformly distributed random variable, keep the left portion, cut the right one at an independent uniformly distributed random variable, keep the left portion, and so on. The sequence of the lengths of the resulting intervals (ordered from the left to the right) is thus $U_1, (1 - U_1)U_2, (1 - U_1)(1 - U_2)U_3, \ldots$, where $U_1, U_2, \ldots$ are i.i.d. uniform variables; in particular the $p$-th moment of the $k$-th length is thus $(1 + p)^k$. When the dislocation measure $\nu$ is the distribution of the sequence of the lengths ranked in the decreasing order, we get that $\bar{p} = 0$ and $\kappa(p) = 1 - 1/p$ for $p > 0$.

We now make the fundamental:

**Malthusian Hypotheses.** There exists a (unique) solution $p^* \geq \bar{p}$ to the equation $\kappa(p^*) = 0$, which is called the Malthusian exponent. Furthermore the integral

$$\int_{S^1} \left( \sum_{i=1}^{\infty} s_i^{p^*} \right)^p \nu(ds)$$

is finite for some $p > 1$.

Throughout the rest of this chapter, the Malthusian hypotheses will always be taken for granted. We may now state.

**Proposition 4** The process

$$\Sigma_n := \sum_{|a|=n} \xi_u^{p^*}, \quad n \in \mathbb{N}$$

is a martingale which is bounded in $L^p$ for some $p > 1$, and in particular is uniformly integrable.
In the sequel, $\left(\Sigma_n, n \in \mathbb{N}\right)$ will be referred to as the intrinsic martingale. Observe that in the important case when dislocations are conservative, in the sense that

$$
\nu \left( \left\{ x \in S^1 : \sum_{i=1}^{\infty} x_i \neq 1 \right\} \right) = 0,
$$

then $p^* = 1$ and $\Sigma_n = 1$ for all $n \in \mathbb{N}$, and the statement is trivial.

**Proof:** Denote $\mathcal{G}_n$ the sigma field generated by $(\xi_u, |u| \leq n)$, so $(\mathcal{G}_n)$ is a filtration. It should be plain from the description of the dynamics of the random marks that for every $p > p^*$,

$$
\mathbb{E} \left( \sum_{|u|=n+1} \xi_u^p \mid \mathcal{G}_n \right) = c(p) \sum_{|u|=n} \xi_u^p
$$

where

$$
c(p)\nu(S^1) = \int_{S^1} \sum_{i=1}^{\infty} s_i^p \nu(ds) = \nu(S^1)(1 - \kappa(p)).
$$

In particular for the Malthusian exponent $p = p^*$, one has $c(p^*) = 1$ and the martingale property is proven.

Recall now the second condition of the Malthusian hypotheses. In order to establish the boundedness of the martingale in $L^p(\mathbb{P})$, all that we need is to check that the sum of its jumps raised to the power $p$ has a finite mean $[23]$, i.e.

$$
\mathbb{E} \left( \sum_{n=1}^{\infty} |\Sigma_{n+1} - \Sigma_n|^p \right) < \infty.
$$

It this direction, we use Proposition 3 and express the $n$-th jump in the form

$$
\Sigma_{n+1} - \Sigma_n = \sum_{|u|=n} \xi_u^{p^*} \left( \left( \sum_{j=1}^{\infty} \tilde{\xi}^{p^*}_{u,j} \right) - 1 \right)
$$

where $(\tilde{\xi}_{u,\cdot}, |u| = n)$ is a family of i.i.d. variables with the law $\nu(\cdot)/\nu(S^1)$, which is independent of $\mathcal{G}_n$. The conditional expectation given $\mathcal{G}_n$ of this quantity raised to the power $q$ is thus

$$
\mathbb{E} \left( \sum_{|u|=n} \xi_u^{p^*} \right) = b \sum_{|u|=n} \xi_u^{p^*}
$$

where $b$ denotes the finite constant given by

$$
b\nu(S^1) := \int_{S^1} \left| 1 - \sum_{i=1}^{\infty} s_i^{p^*} \right|^p \nu(ds).
$$

Now we know from the first part of the proof that

$$
\mathbb{E} \left( \sum_{|u|=n} \xi_u^{p^*} \right) = c(p^*)^n,
$$

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and since $p^* p > p^*$ and $\kappa$ is strictly increasing, $\kappa(p^* p) > 0$ and thus $c(p^* p) < 1$. This entails (8) and completes the proof of the statement. \hfill \Box

The terminal value $\Sigma_\infty$ of the intrinsic martingale will appear in many limit theorems for the fragmentation. In general its distribution is not known explicitly. However, it is straightforward from the branching property that there is the identity in law

$$\Sigma_\infty \overset{(d)}{=} \sum \xi_j^j \Sigma_\infty^{(j)}$$

where $\xi = (\xi_j, j \in \mathbb{N})$ has the law $\nu(\cdot)/\nu(S^1)$, and $\Sigma_\infty^{(j)}$ are independent copies of $\Sigma_\infty$, also independent of $\xi$. It is known that under fairly general conditions, such identity characterizes the law $\Sigma_\infty$ uniquely, see e.g. [25].

The intrinsic martingale $\Sigma_n$ is indexed by the generations; it will be also convenient to consider its analog in continuous time, i.e.

$$\Sigma(t) := \sum_{i=1}^\infty X^p_i(t) = \sum_{u \in I_u} 1_{\{t \in I_u\}} \xi_u^p, \quad t \geq 0,$$

where in the right hand side, $I_u$ denotes the life-interval of the particle indexed by the node $u$. It is straightforward to check that $(\Sigma(t), t \geq 0)$ is again a martingale in the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of the fragmentation $(X(t), t \geq 0)$; and more precisely, we have the following.

**Proposition 5** Assume that the index of self-similarity $\alpha$ is nonnegative. Then

$$\Sigma(t) = \mathbb{E}(\Sigma_\infty \mid \mathcal{F}_t),$$

where $\Sigma_\infty$ is the terminal value of the intrinsic martingale $(\Sigma_n, n \in \mathbb{N})$, and $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the fragmentation. In particular $\Sigma(t)$ converges in $L^p(\mathbb{P})$ to $\Sigma_\infty$ for some $p > 1$.

**Proof:** We know that $\Sigma_n$ converges in $L^p(\mathbb{P})$ to $\Sigma_\infty$ as $n$ tends to $\infty$, so

$$\mathbb{E}(\Sigma_\infty \mid \mathcal{F}_t) = \lim_{n \to \infty} \mathbb{E}(\Sigma_n \mid \mathcal{F}_t).$$

On the other hand, it is easy to deduce from the Markov property applied at time $t$ that

$$\mathbb{E}(\Sigma_n \mid \mathcal{F}_t) = \sum_{i=1}^\infty X^p_i(t) 1_{\{G(X_i(t)) \leq n\}} + \sum_{|u|=n} \xi_u^p 1_{\{a_u + \zeta_u < t\}},$$

where $G(x)$ stands for the generation of the particle $x$ (i.e. $G(\xi_u) = |u|$), and $a_u + \zeta_u$ is the instant when the particle corresponding to the node $u$ splits. We can express the latter quantity in the form

$$a_u + \zeta_u = x_0 e_0 + x_1^{-\alpha} e_1 + \ldots + x_{|u|}^{-\alpha} e_{|u|}$$
where $e_0, \ldots$ is a sequence of independent exponential variables with parameter $\nu(S^1)$, which is also independent of $\xi_u$, and $x_i$ stands for the size of particle labelled by the ancestor of $u$ at the $i$-th generation. When the index of self-similarity is nonnegative, $x_i^{-\alpha} \geq 1$ and hence for each fixed node $u \in U$, $a_u + \zeta_u$ is bounded from below by the sum of $|u| + 1$ independent exponential variables with parameter $\nu(S^1)$ which are independent of $\xi_u$. It follows that

$$
\lim_{n \to \infty} E \left( \sum_{|u|=n} \xi_{p^u}^* 1_{(a_u + \zeta_u < t)} \right) = 0,
$$

and we conclude that $E(\Sigma | \mathcal{F}_t) = \Sigma(t)$. \hfill \Box

We stress that the statement fails when $\alpha < 0$; more precisely we shall see in Section 5 that then $\Sigma(t) = 0$ whenever $t$ is sufficiently large.

### 2.3 A randomly tagged branch

Throughout this section, we shall work with the representation of a self-similar fragmentation with finite dislocation measure $\nu$ as a marked infinite tree. Recall that we assumed that the Malthusian hypotheses hold, in particular there exists $p^* > 0$ such that $\kappa(p^*) = 0$. For the sake of simplicity, we shall also implicitly suppose that the fragmentation starts from a single fragment with unit size, i.e. we shall work under $\mathbb{P} = \mathbb{P}_1$.

One calls branch of the genealogical tree $U$ an infinite sequence of integers $b = (u_1, \ldots)$, which we can think of as the line of ancestors of some leave of the tree, in the sense that for each $n$, we associate to $b$ its ancestor $b_n := (u_1, \ldots, u_n)$ at the generation $n$. Following an original idea developed by Lyons, Pemantle and Peres [24], we shall enrich the probabilistic structure by distinguishing at random a branch, called the tagged branch.

Specifically, we consider a pair $(M, \beta)$ where $M: U \to [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+$ is a random mark on the genealogical tree and $\beta$ is a random branch of $U$, whose joint distribution denoted by $\mathbb{P}^*$ is specified as follows. Let $\mathcal{H}_n$ stand for the space of bounded functionals $\Phi$ which depend on the mark $M$ and the branch $\beta$ only up to the $n$-th generation, i.e. such that $\Phi(M, \beta) = \Phi(M', \beta')$ if $\beta_n = \beta'_n$ and $M(u) = M'(u)$ whenever $|u| \leq n$. For such functionals, it will be convenient to use the slightly abusing notation $\Phi(M, \beta) = \Phi(M, \beta_n)$. It is immediately seen from the dynamics of the random mark $M$ and the intrinsic martingale property that

$$
E \left( \sum_{|u|=n} \Phi(M, u) \xi_{p^u}^* \right) = E \left( \sum_{|v|=n+1} \Phi(M, v) \xi_{p^v}^* \right)
$$

for every $\Phi \in \mathcal{H}_n$. This allows us to define unambiguously a probability measure $\mathbb{P}^*$ viewed as the joint distribution of a random mark $M$ and a random branch $\beta$ by

$$
\mathbb{E}^* (\Phi(M, \beta)) = E \left( \sum_{|u|=n} \Phi(M, u) \xi_{p^u}^* \right), \quad \Phi \in \mathcal{H}_n.
$$
Note that when the intrinsic martingale \((\Sigma_n, n \in \mathbb{N})\) is uniformly integrable (cf. Proposition 4), the first marginal of \(\mathbb{P}^*\) is absolutely continuous with respect to the law of the random mark \(M\) under \(\mathbb{P}\), with density \(\Sigma_\infty\).

The sizes of particles on the tagged branch will play an important role in the analysis of fragmentation processes. More precisely, we write \(\chi_n = \xi_{\beta_n}\) for the size of the particle corresponding to the node \(\beta_n\) of the tagged branch at the \(n\)-th generation, and \(\chi(t)\) for the size of the tagged particle alive at time \(t\), viz.

\[
\chi(t) = \chi_n \quad \text{if} \quad a_{\beta_n} \leq t \leq a_{\beta_n} + \zeta_{\beta_n},
\]

where \(a_{\beta_n}\) and \(\zeta_{\beta_n}\) denote respectively the birth-time and lifetime of the particle labelled by tagged node \(\beta_n\).

The following lemma shows that the first moment of additive functionals of the fragmentation are easily expressed in terms of the tagged particle.

**Lemma 3** Let \(f : \mathbb{R}_+ \to \mathbb{R}_+\) be a measurable function. Then we have for every \(n \in \mathbb{N}\)

\[
\mathbb{E}^* (f(\chi_n)) = \mathbb{E} \left( \sum_{|u| = n} \xi^*_u f(\xi_u) \right),
\]

and for every \(t \geq 0\)

\[
\mathbb{E}^* (f(\chi(t))) = \mathbb{E} \left( \sum_{i=1}^{\infty} X^*_i(t) f(X_i(t)) \right).
\]

**Proof:** The identity (9) yields the first formula. The second derives readily from the first by conditioning on the generation of the tagged particle at time \(t\) and the intrinsic martingale property. \(\square\)

Lemma 3 will be useful to compute first moments for fragmentations in combination with the following proposition.

**Proposition 6** Under \(\mathbb{P}^*\),

\[
S_n := -\log \chi_n, \quad n \in \mathbb{N}
\]

is a random walk on \(\mathbb{R}_+\) with step distribution

\[
\mathbb{P} \left( \log \chi_n - \log \chi_{n+1} \in dy \right) = \hat{\nu}(dy)/\nu(S^1),
\]

where the finite measure \(\hat{\nu}\) is defined by

\[
\int_{\mathbb{R}_+} f(y)\hat{\nu}(dy) = \int_{S^1} \sum_{i=1}^{\infty} x_i^p f(-\log x_i) \nu(dx).
\]

Moreover, conditionally on \((\chi_n, n \in \mathbb{N})\) the sequence of the lifetimes \((\zeta_{\beta_0}, \zeta_{\beta_1}, \ldots)\) along the tagged branch is a sequence of independent exponential variables with respective parameters \(\chi_{\beta_0}^0 \nu(S^1), \chi_{\beta_1}^1 \nu(S^1), \ldots\).
For example, in the case of the Poissonian rain introduced on page 8, the dislocation measure $\nu$ is the law of $(V,1-V,0,\cdots)$ where $V$ is uniformly distributed on $[1/2,1]$. It is immediate to check that the Malthusian parameter is $p^* = 1$, and the step distribution of the random walk $S_n$ is the exponential law with parameter 2.

**Proof:** Consider a functional $\Phi \in \mathcal{H}_n$. We see from (9) that for every $q \geq 0$

$$
\mathbb{E}^* (\exp(-q(S_{n+1} - S_n)) \Phi(M,\beta)) = \mathbb{E}^* (\chi_{n+1}^q \chi_n^{-q} \Phi(M,\beta)) = \mathbb{E} \left( \sum_{|u| = n} \sum_{i=1}^{\infty} \tilde{\xi}_{u,i}^q (\xi_{u,i} \tilde{\xi}_{u,i})^{p^*} \Phi(M,u) \right),
$$

where $\tilde{\xi}_{u,i} = (\tilde{\xi}_{u,1},\ldots)$ are i.i.d random variable in $\mathcal{S}^\downarrow$ with law $\nu(\cdot)/\nu(\mathcal{S}^\downarrow)$, which are independent of $\Phi(M,u)$. This shows that under $\mathbb{P}^*$, $S_n$ is a random walk with step distribution given by

$$
\mathbb{E}^* (f(S_1)) = \int_{\mathcal{S}^\downarrow} \left( \sum_{i=1}^{\infty} f(-\log s_i) s_i^{p^*} \right) \frac{\nu(ds)}{\nu(\mathcal{S}^\downarrow)},
$$

and thus establishes the first claim. The second is obvious. \qed

In particular, in the so-called homogeneous case when the index of self-similarity is $\alpha = 0$, Proposition 6 shows that the tagged fragment $(\chi(t), t \geq 0)$ can be expressed in the form $\chi(t) = \exp(-\eta_t)$, where

$$
\eta_t = S \circ N_t, \quad t \geq 0,
$$

with $N$ a Poisson process with parameter $\nu(\mathcal{S}^\downarrow)$ independent of the random walk $S$. In other words, the tagged fragment is the exponential of the compound Poisson process $\eta = S \circ N$.

In the case $\alpha \neq 0$, the process $(\chi(t), t \geq 0)$ is Markovian and enjoys an obvious scaling property. More precisely, it can be expressed as

$$
\chi(t) = \exp(-\eta_{\tau(t)}), \quad t \geq 0,
$$

where $\eta$ is the compound Poisson defined above and $\tau$ the time-change given as the inverse of the functional

$$
t \to \int_0^t \exp(\alpha \eta_s) ds.
$$

The time change $\tau(t)$ is finite for all $t \geq 0$ when $\alpha \geq 0$, whereas when $\alpha < 0$,

$$
\tau(t) < \infty \iff t < I := \int_0^\infty \exp(\alpha \eta_s) ds,
$$

and then

$$
\chi(t) = 0 \quad \text{whenever } t \geq I.
$$

These observations enable us in particular to compute the moments of power sums of self-similar fragmentations.
Proposition 7 We have for every $p \geq p$ and $t \geq 0$:

(i) in the homogeneous case $\alpha = 0$,

$$
\mathbb{E} \left( \sum_{i=1}^{\infty} X^p_i(t) \right) = \mathbb{E}^*(\chi(t)^{p-p^*}) = \exp(-t\kappa(p)),
$$

(ii) in the case $\alpha > 0$ when the index of self-similarity is positive,

$$
\mathbb{E} \left( \sum_{i=1}^{\infty} X^p_i(t) \right) = \mathbb{E}^*(\chi(t)^{p-p^*}) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\gamma(n, p),
$$

where $\gamma(0, p) = 1$ and for $n \geq 1$

$$
\gamma(n, p) = \prod_{k=0}^{n-1} \kappa(p + \alpha k).
$$

Proof: (i) Thanks to Proposition 6, we have whenever $p + p^* > p$ that

$$
\mathbb{E}^*(\chi(t)^p) = \mathbb{E}^*(\exp(-pS_N))
= \exp(-t\nu(S^1)) \sum_{k=0}^{\infty} \frac{(t\nu(S^1))^k}{k!} \left( \int e^{-p\nu(dy)/\nu(S^1)} \right)^k
= \exp(-t\kappa(p + p^*)),
$$

In other words, the so-called Laplace exponent of the compound Poisson process $S \circ N = \eta$ is $\kappa(p^* + \cdot)$, and this characterizes its law. The stated formula now derives from Lemma 3.

(ii) We start by observing from Lemma 3 that for every sufficiently large $p$

$$
\int_0^{\infty} \mathbb{E} \left( \sum_{i=1}^{\infty} X_i^{p+p^*}(s) \right) ds = \mathbb{E}^* \left( \chi(t)^p \right) dt
= \mathbb{E}^* \left( \int_0^{\infty} \exp(-p\tau(t)) dt \right)
= \mathbb{E}^* \left( \int_0^{\infty} \exp(-(p - \alpha)\eta_h) dt \right)
= \int_0^{\infty} \exp(-t\kappa(p^* + p - \alpha) dt
= 1/\kappa(p^* + p - \alpha).
$$

Next, we apply the self-similarity and branching properties of the fragmentation to see that for every $t \geq 0$

$$
\mathbb{E} \left( \int_t^{\infty} ds \sum_{i=1}^{\infty} X_i^p(s) \right) = \mathbb{E} \left( \sum_{i=1}^{\infty} X_i^{p-\alpha}(t) \right) \mathbb{E} \left( \int_0^{\infty} ds \sum_{i=1}^{\infty} X_i^p(s) \right)
= \mathbb{E} \left( \sum_{i=1}^{\infty} X_i^{p-\alpha}(t) \right) /\kappa(p - \alpha),
$$

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where the second equality follows from the identity above. Taking the derivative in the variable $t$, we arrive at

$$\frac{d}{dt} \mathbb{E} \left( \sum_{i=1}^{\infty} X_i^{p-\alpha}(t) \right) = -\kappa(p - \alpha) \mathbb{E} \left( \sum_{i=1}^{\infty} X_i^p(t) \right).$$

This equation entails that whenever $p > p^*$, the function $t \rightarrow \mathbb{E} \left( \sum_{i=1}^{\infty} X_i^p(t) \right)$ is completely monotone, and thus, by Bernstein’s theorem, can be expressed as the Laplace transform of some measure $\mu$ on $\mathbb{R}_+$:

$$\mathbb{E} \left( \sum_{i=1}^{\infty} X_i^p(t) \right) = \int_{\mathbb{R}_+} e^{-tx} \mu(dx).$$

More precisely, as the moments of $\mu$ can be recovered from the derivatives of its Laplace transform, and we get

$$\int_{\mathbb{R}_+} x^k \mu(dx) = \kappa(p) \ldots \kappa(p + (k - 1)\alpha).$$

By the series expansion of $e^{-tx}$, we finally arrive at the expression

$$\mathbb{E} \left( \sum_{i=1}^{\infty} X_i^p(t) \right) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \int_{\mathbb{R}_+} x^k \mu(dx)$$

which yields the desired formula. 

\[ \square \]

### 3 Laws of large numbers for fragmentations

In this section, we shall provide sharp information of the large time asymptotic behavior of (a weighted version of) the empirical distribution of the fragments of self-similar fragmentations with nonnegative indices. The approach relies crucially on an extension of the classical weak law of large numbers that we now present.

#### 3.1 A variation of the law of large numbers

To start with, let us specify the setting. For each $t \geq 0$, let $\lambda(t) = (\lambda_i(t), i \in \mathbb{N})$ be a sequence of nonnegative random variables such that for some fixed $p > 1$

$$\sup_{t \geq 0} \mathbb{E} \left( \left( \sum_{i=1}^{\infty} \lambda_i(t) \right)^p \right) < \infty \quad \text{and} \quad \lim_{t \to \infty} \mathbb{E} \left( \sum_{i=1}^{\infty} \lambda_i^p(t) \right) = 0.$$

Let also $(Y_i(t), i \in \mathbb{N})$ be a sequence of random variables which are independent conditionally on $\lambda(t)$. Finally, assume there is a sequence $(\bar{Y}_i, i \in \mathbb{N})$ of independent and identically distributed variables in $L^p(\mathbb{P})$, which is independent of $\lambda(t)$ for each fixed $t$, and such that $|Y_i(t)| \leq \bar{Y}_i$ for all $i \in \mathbb{N}$ and $t \geq 0$. We can now state
Lemma 4 Under the preceding assumptions,
\[
\lim_{t \to \infty} \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - \mathbb{E}(Y_i(t) \mid \lambda(t))) = 0 \quad \text{in } L^p(\mathbb{P}).
\]

Proof: Let \(a > 0\) be an arbitrarily large real number. Introduce for every \(i \in \mathbb{N}\) and \(t \geq 0\) the truncated variables \(Y_i(t, s) = 1_{\{Y_i(t) < a\}} Y_i(t)\). To start with, there is the upperbound
\[
\left| \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - \mathbb{E}(Y_i(t) \mid \lambda(t))) \right| \leq \left| \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - Y_i(t, a)) \right| + \left| \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t, a) - \mathbb{E}(Y_i(t, a) \mid \lambda(t))) \right|.
\]

Consider the first series in the right-hand side. The bounds \(|Y_i(t) - Y_i(t, a)| \leq 1_{\{Y_i > a\}} Y_i\) and the independence of \((Y_i, i \in \mathbb{N})\) and \(\lambda(t)\) entail
\[
\mathbb{E} \left( \left| \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - Y_i(t, a)) \right|^p \right) \leq \mathbb{E}(1_{\{Y_i > a\}} Y_i^p) \mathbb{E} \left( \left( \sum_{i=1}^{\infty} \lambda_i(t) \right)^p \right),
\]
and the latter quantity converges to 0 as \(a \to \infty\), uniformly for \(t \geq 0\). The same argument also shows that
\[
\lim_{a \to \infty} \sup_{t \geq 0} \mathbb{E} \left( \left| \sum_{i=1}^{\infty} \lambda_i(t) \mathbb{E}(Y_i(t) - Y_i(t, a) \mid \lambda(t)) \right|^p \right) = 0.
\]

Finally, conditionally on \(\lambda(t)\), the variables \(Y_i(t, a) - \mathbb{E}(Y_i(t, a) \mid \lambda(t))\) are centered, independent, and bounded in absolute value by \(a\). Thus, conditionally on \(\lambda(t)\),
\[
\sum_{i=1}^{n} \lambda_i(t)(Y_i(t, a) - \mathbb{E}(Y_i(t, a)) \mid \lambda(t))
\]
is a martingale bounded in \(L^p\) and there exists a universal constant \(c_p\) such that
\[
\mathbb{E} \left( \left| \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t, a) - \mathbb{E}(Y_i(t, a) \mid \lambda(t))) \right|^p \mid \lambda(t) \right) \leq c_p a^p \sum_{i=1}^{\infty} \lambda_i^p(t).
\]

Our assumptions ensure that the latter quantity converges to 0 as \(t \to \infty\) in \(L^1(\mathbb{P})\), so putting the pieces together, this completes the proof of the statement.

Let us now explain how we shall apply Lemma 4 in the rest of this section. We shall be interested in limit theorems involving functionals of the fragmentation of the type
\[
A(t) := \sum_{i=1}^{\infty} X_i^p(t) g(X_i(t), t),
\]
where $p^*$ is the Malthusian exponent and $g$ a certain measurable function. The first step of the analysis consists in considering this functional at time $t + s$ and applying the branching property of the fragmentation at time $t$. This yields an expression of the form

$$A(t + s) = \sum_{k=1}^{\infty} \lambda_k(t) Y_k(t, s)$$

where $\lambda_k(t) = X_k^{p^*}(t)$ and $Y_k(t, s)$ is a quantity depending of the fragmentation started at time $t$ from a single particle with size $X_k(t)$. We then use Lemma 4 to reduce the study of the asymptotic behavior of the latter quantity as both $t, s \to \infty$ to that of

$$\sum_{k=1}^{\infty} \lambda_k(t) \mathbb{E}(Y_k(t, s) \mid X_k(t)).$$

Essentially, this reduction amount to getting estimates for the first moment of additive functionals of the fragmentation, which are then obtained by limit theorems for the tagged fragment (cf. Section 2.3).

### 3.2 The homogeneous case

We suppose throughout this section that $X = (X(t), t \geq 0)$ is a homogeneous fragmentation (i.e. the index of self-similarity is $\alpha = 0$) for which the Malthusian hypotheses of Proposition 4 hold. We are interested in the asymptotic behavior of the empirical distribution of the fragments as time tends to infinity. The following pathwise limit theorem extends a result due to Kolmogorov [21] in the conservative case, which seems to have appeared in the first probabilistic work on fragmentation ever. See also Asmussen and Kaplan [1] for a closely related result.

Introduce the first and second right-derivatives of $\kappa$ at the Malthusian exponent $p^*$,

$$\mu := \kappa'(p^*) \ , \ \sigma^2 := -\kappa''(p^*) ;$$

so that in terms of the splitting measure

$$\mu = -\int_{S^1} \left( \sum_{i=1}^{\infty} x_i^{p^*} \log x_i \right) \nu(dx) \ , \ \sigma^2 = \int_{S^1} \left( \sum_{i=1}^{\infty} x_i^{p^*} (\log x_i)^2 \right) \nu(dx).$$

Note that these quantities are finite since $p^* > p$. Recall that $\Sigma_\infty$ denotes the terminal value of the intrinsic martingale in Proposition 4.

**Theorem 1** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function.

(i) We have

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{-1} \log X_i(t)) = \Sigma_\infty f(-\mu) ,$$

in $L^1(\mathbb{P})$.  

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(ii) Denote by $\mathcal{N}(0, \sigma^2)$ a centered Gaussian distribution with variance $\sigma^2$. Then
\[
\lim_{t \to \infty} \sum_{i=1}^{\infty} X_i^p(t) f(t^{-1/2}(\log X_i(t) + \mu t)) = \Sigma_{\infty} \mathbb{E}(f(\mathcal{N}(0, \sigma^2)))
\]
in $L^1(\mathbb{P})$.

**Proof:** We shall apply Lemma 4 in the following situation. Let $f : \mathbb{R_+} \to [0, 1]$ be a continuous function; we are interested in
\[
\sum_{i=1}^{\infty} X_i^p(t) f \left( (t + t^2)^{-1} \log X_i(t) + \mu t \right) f(t - 12) \log X_i(t) + \mu t \right).
\]
By an application of the Markov property at time $t$ and self-similarity, we can re-express this variable in the form
\[
\sum_{i=1}^{\infty} \lambda_i(t) Y_i(t)
\]
where $\lambda_i(t) = X_i^p(t)$ and
\[
Y_i(t) = \sum_{j=1}^{\infty} X_{ij}^p(t) f \left( (t + t^2)^{-1} \log(X_i(t)X_{ij}(t^2)) \right),
\]
with $X_1, X_2, \ldots$ a sequence of i.i.d. copies of $X$ which is independent of $X(t) = (X_1(t), \ldots)$.

It follows from Proposition 5 that the sequence $(\lambda_i(t), i \in \mathbb{N})$ fulfils the requirement of Lemma 4. Let us now consider the sequence $(Y_i(t), i \in \mathbb{N})$ conditionally on $X(t)$. Plainly, it is given by independent variables, each of which is bounded from above by
\[
\bar{Y}_i := \sup_{s \geq 0} \sum_{j=1}^{\infty} X_{ij}^p(s).
\]
On the one hand, the $\bar{Y}_i$’s are clearly i.i.d. On the other hand, $\sum_{j=1}^{\infty} X_{ij}^p(s)$ is a martingale which is bounded in $L^p$ for some $p > 1$, it follows from Doob’s inequality that its overall supremum belongs to $L^p(\mathbb{P})$.

Thus we may apply Lemma 4, which reduces the study to that of the asymptotic behavior of
\[
\sum_{i=1}^{\infty} \lambda_i(t) \mathbb{E}(Y_i(t) \mid X(t))
\]
as $t$ tends to $\infty$. In this direction, we thus compute the conditional expectation of $Y_i(t)$ given $X(t)$; we easily get on the event $\{X_i(t) = x\}$ that
\[
\mathbb{E}(Y_i(t) \mid X(t)) = \mathbb{E} \left( \sum_{j=1}^{\infty} X_{ij}^p(t^2) f((t + t^2)^{-1} \log X_j(t^2) + \log x) \right).
\]

Now recall the notion of tagged fragment $(\chi(t), t \geq 0)$ in Section 2.3. In particular we know from Lemma 3 that there is the identity
\[
\mathbb{E} \left( \sum_{j=1}^{\infty} X_{ij}^p(t^2) f((t + t^2)^{-1} \log X_j(t^2) + \log x) \right) = \mathbb{E}^*(f((t + t^2)^{-1} \log \chi(t^2) + \log x)).
\]
Moreover, recall from Proposition 6 that the process of the logarithm of size of the tagged fragment is a compound Poisson process, $-\log \chi(t) = S(N_t)$, where $S$ is a random walk with step distribution $\tilde{\nu}(\cdot)/\nu(S^1)$ and $N$ an independent Poisson process with parameter $\nu(S^1)$. In particular, $S(1)$ has finite means $\mu/\nu(S^1)$, and it follows from the law of large numbers that

$$\lim_{t \to \infty} E^*(f((t + t^2)^{-1}(\log \chi(t^2) + \log x)) = f(-\mu),$$

where the limit is uniform in $x$ such that, say $-\log x \leq t^{3/2}$. On the other hand, using again Lemma 3, we have

$$E\left(\sum_{i=1}^{\infty} X_i^P(t)1_{\{-\log X_i(t) \leq t^{3/2}\}}\right) = \mathbb{P}^*(-\log \chi(t) \leq t^{3/2}),$$

and the latter quantity tends to 0 as $t \to \infty$.

Recall from Proposition 5 that $\sum_{i=1}^{\infty} \lambda_i(t)$ converges to $\Sigma_\infty$ in $L^p(\mathbb{P})$. Putting the pieces together, we get that as $t \to \infty$

$$\sum_{i=1}^{\infty} \lambda_i(t)E(Y_i(t) \mid X(t)) \sim f(-\mu) \sum_{i=1}^{\infty} \lambda_i(t) \sim \Sigma_\infty f(-\mu).$$

(ii) The proof is similar; the arguments above are easily adapted to reduce the proof to asymptotics for the first moment

$$E^*(f(t^{-1/2}(\log \chi(t) + t\mu)),$$

where $f$ is a continuous bounded function. We may then use the central limit theorem for the compound random walk $\chi(t)$ to show that the preceding quantity converges to $E(f(\mathcal{N}(0, \sigma^2))$ when $t \to \infty$. Details are left to the reader. 

\subsection{3.3 The case of a positive index of self-similarity}

We now suppose throughout this section that $X = (X(t), t \geq 0)$ is a self-similar fragmentation with index $\alpha > 0$, with a finite dislocation measure $\nu$. Again, we shall suppose that the Malthusian hypotheses of Proposition 4 hold and we will be interested in the asymptotic behavior of the process as time tends to infinity. As for positive indices of self-similarity, small fragments split more slowly than large fragments, we may expect some homogenization phenomenon.

We now state the main result of this section, which has been obtained first by Filippov [17] in the conservative case. Its extension to non-conservative self-similar fragmentation has been established recently by Bertoin and Gnedin [6].

\textbf{Theorem 2} Suppose that $\alpha > 0$ and that the step distribution of the random walk $S_n = -\log \chi_n$ is not arithmetic. Then for every bounded continuous function $f : \mathbb{R}_+ \to \mathbb{R}$

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} X_i^P(t)f(t^{1/\alpha}X_i(t)) = \Sigma_\infty \int_0^{\infty} f(y)\rho(dy), \quad \text{in } L^1(\mathbb{P}),$$

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where $\Sigma_\infty$ is the terminal value of the intrinsic martingale and $\rho$ is a deterministic probability measure. More precisely, $\rho$ is determined by the moments

$$
\int_{[0,\infty]} y^{\alpha k} \rho(dy) = \frac{(k-1)!}{\alpha' (p^*) \kappa(p^* + \alpha) \cdots \kappa(p^* + (k-1)\alpha)} \quad \text{for } k = 1, \ldots,
$$

(with the usual convention that the right-hand side above equals $1/(\alpha'(p^*))$ for $k = 1$).

For example, suppose that the dislocation measure $\nu$ is the distribution induced by the uniform stick-breaking scheme as described after equation (7). So $\kappa(p^*) = 1 - 1/p$, and the Malthusian exponent is $p^* = 1$. Moreover, the dislocation measure is conservative, and the intrinsic martingale is thus trivial, $\Sigma_n \equiv 1$. Suppose further that the index of self-similarity is $\alpha = 1$. One then gets

$$
\frac{(k-1)!}{\alpha' (p^*) \kappa(p^* + \alpha) \cdots \kappa(p^* + (k-1)\alpha)} = k!, \quad k \in \mathbb{N},
$$

so the probability measure $\rho$ appearing in Theorem 2 is simply the standard exponential distribution.

Just as in the homogeneous case, Lemma 4 reduces the proof to the analysis of the so-called tagged fragment; we shall only provide details on the latter aspect. Recall from Proposition 6 that the process $(\chi(t), t \geq 0)$ is a continuous time Markov chain, which enjoys an obvious scaling property. The classical renewal theory yields an important limit theorem for $\chi(t)$ as $t$ tends to infinity.

**Proposition 8** (Brennan and Durrett [14]) Suppose that the step distribution of the random walk $S_n = -\log \chi_n$ is not arithmetic. Then under $P^*$, $t^{1/\alpha} \chi(t)$ converges in distribution as $t \to \infty$ towards some variable $Y_\alpha$ which can be expressed in the form

$$
Y_\alpha = \left( \sum_{n=0}^{\infty} \exp(-\alpha R_n) e_n \right)^{1/\alpha},
$$

where $(R_0, R_1, \ldots)$ is a random walk with the same step distribution as $S$ and with initial distribution

$$
\frac{\nu(S_1)}{\kappa'(p^*)} y \nu(S_1 \in dy),
$$

and $e_0, e_1, \ldots$ a sequence of i.i.d. exponential variables with parameter $\nu(S_1)$, which is independent of the random walk $(R_0, R_1, \ldots)$.

This result can be proven by taking limits as $t \to \infty$ in the explicit moment calculations of Proposition 7(ii), using complex analysis and contour integral, see [6]. We develop below a more probabilistic approach.

**Proof:** There is no loss of generality in assuming that the fragmentation starts from a single fragment with unit size. Write $T(y) := \min \{ n \in \mathbb{N} : S_n > -\log y \}$ for every $y \in [0,1]$. We thus have

$$
P^*(\chi(t) < y) = P^* \left( \sum_{n=0}^{T(y)} \exp(\alpha S_n) e'_n \leq t \right),
$$

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where $e'_0, e'_1, \ldots$ is a sequence of i.i.d. exponential variables with parameter $\nu(S^1)$, which is independent of the random walk (so that $\zeta_n = \exp(\alpha S_n) e'_n$ is the lifetime of the tagged particle at the $n$-th generation). Reversing the indices, we may re-express this quantity in the form

$$\mathbb{P}^\ast(\chi(t) < y) = \mathbb{P}^\ast \left( \sum_{n=0}^{T(y)} \exp(-\alpha R_n(y)) e_n \leq y^\alpha \right),$$

where $R_n(y) := -\log y - S T(y) - n$ and $e_0, e_1, \ldots$ is a new sequence of i.i.d. exponential variables with parameter $\nu(S^1)$, which is again independent of $(R_n(y), n \in \mathbb{N})$. Rescaling thus gives

$$\mathbb{P}^\ast(\frac{R_0}{\alpha} \in dy) = \frac{y}{E(S_1)} \mathbb{P}^\ast(S_1 \in dy).$$

This easily entails the statement. \qed

We now complete the proof of Theorem 2 by pointing out that the distribution of the limiting variable $Y_\alpha$ can be characterized by its moments as follows. Recall that the function $\kappa$ is defined in (7).

**Proposition 9** For every integer $k \geq 1$, we have

$$\mathbb{E} \left( Y_\alpha^k \right) = \frac{(k-1)!}{\alpha \kappa'(p^\ast) \kappa(p^\ast + \alpha) \cdots \kappa(p^\ast + (k-1)\alpha)},$$

and this determines uniquely the law of $Y_\alpha$.

**Proof:** It is convenient combine Proposition 6 and 8 and re-express

$$Y_\alpha^\alpha = \exp(-\alpha R_0) \int_0^\infty e^{-\alpha \Upsilon_t} dt$$

(10)

where $\Upsilon = (\Upsilon_t, t \geq 0)$ is distributed as the increasing compound Poisson process $S \circ N$ where $N$ is a Poisson process with rate $\nu(S^1)$ which is independent of the random walk $S$ (recall the example at the end of Section 1.1 and the preceding lemma), which is independent of $R_0$. In particular, we have (from the Lévy-Khintchine formula) that the Laplace transform is given by

$$\mathbb{E} \left( e^{-q\Upsilon_t} \right) = \exp \left( -t \int_{[0, \infty[} \left( 1 - e^{-qy} \right) \sum_{i=1}^{\infty} y^{p^r} \nu(-\log x_i \in dy) \right)$$

$$= \exp \left( -t \int_{S^1} \nu(dx) \sum_{i=1}^{\infty} \left( x_i^{p^r} - x_i^{p^r+q} \right) \right),$$

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and finally
\[ \mathbb{E}\left(e^{-qY_t}\right) = \exp(-t\kappa(q + p^*)). \] (11)

On the one hand, it is immediate that
\[ \mathbb{E}(\exp(-qR_0)) = \frac{\kappa(q + p^*)}{q\kappa'(p^*)}, \quad q > 0. \] (12)

On the other hand we shall check that
\[ \mathbb{E}\left(\left(\int_0^\infty \exp(-\alpha Y_s) \, ds\right)^k\right) = \frac{k!}{\kappa(p^* + \alpha) \cdots \kappa(p^* + \alpha k)}, \quad k = 1, 2, \ldots \] (13)

Indeed, set
\[ I_t = \int_t^\infty \exp(-\alpha Y_s) \, ds \]
for every \( t \geq 0 \). On the one hand, for every positive real number \( r > 0 \), we have the identity
\[ I_0^r - I_t^r = r \int_0^t \exp(-\alpha Y_s)I_s^{r-1} \, ds. \] (14)

On the other hand, we may express \( I_s \) in the form \( I_s = \exp(-\alpha Y_s)I_0^r \), where
\[ I_0^r = \int_0^\infty \exp(-\alpha Y_u) \, du \quad \text{and} \quad Y_u = Y_{s+u} - Y_s. \] (15)

From the independence and stationarity of the increments of the Lévy process, we see that \( I_0^r \) has the same law as \( I_0 = I \) and is independent of \( Y_s \). Plugging this in (14) and taking expectations, we get using (11) that
\[ \mathbb{E}(I^r) (1 - \exp(-t\kappa(p^* + \alpha r))) = r \int_0^t \exp(-sk(p^* + \alpha r))\mathbb{E}(I^{r-1}) \, ds \]
\[ = \frac{r}{\kappa(1 + r)} \left(1 - e^{-t\kappa(p^* + \alpha r)}\right) \mathbb{E}(I^{r-1}). \]

Finally
\[ \mathbb{E}(I^r) = \frac{r}{\kappa(p^* + \alpha r)} \mathbb{E}(I^{r-1}), \]
and since \( \mathbb{E}(I^0) = 1 \), we get the formula (13) by iteration, taking \( r = k \in \mathbb{N} \). Combining (10), (12) and (13) completes the proof of the first statement.

Finally, as \( \lim_{q \to \infty} \kappa(q) = \nu(S^1) \), we see that \( Y_\alpha^\alpha \) possesses exponential moments of any order less than \( \nu(S^1) \), and therefore is determined by its entire moments. \( \Box \)

### 3.4 Another limit theorem via renewal theory

Finally, we turn our interest to question motivated the mining industry. More precisely, fragmentation is needed in the mining industry to reduce rocks into sufficiently small particles. For this purpose, rocks are broken in crushers and mills by a repetitive mechanism. Particles are screened so that when they become smaller than the diameter of the mesh of
a thin grid, they are removed from the process. Here, following a recent work of Bertoin and Martinez [8], we will be interested in the distribution of the small particles that can go across the grid. In other words, we would like to get information about the distribution of the state of a fragmentation process when we stop each particle at the instant when it becomes smaller than some small parameter \( \varepsilon > 0 \). Clearly, this does not depend on the index of self-similarity \( \alpha \), but only on the dislocation measure \( \nu \). More precisely, we use the genealogical tree representation and consider

\[
\varphi_\varepsilon(da) := \sum_{u \in U, u \neq \emptyset} 1_{\{\xi_u - \geq \varepsilon, \xi_u < \varepsilon\}} \xi_u^p \delta_{\xi_u/\varepsilon}(da),
\]

where \( u^- \) stands for the father of \( u \). So \( \varphi_\varepsilon \) is a random finite measure on \( ]0, 1[ \), which can be viewed as a weighted version of the empirical measure of the particles taken at the instant when then become smaller than \( \varepsilon \). In the sequel, we shall suppose that the fragmentation is not arithmetic as in Proposition 8. Recall also that \( \Sigma_\infty \) denotes the terminal value of the intrinsic martingale.

**Theorem 3** As \( \varepsilon \to 0 \), \( \varphi_\varepsilon \) converges in probability to \( \Sigma_\infty \varphi \), where \( \varphi \) is a deterministic probability measure on \( [0, 1] \) given by

\[
\varphi(da) = \left( \int S \sum_{i=1}^\infty 1_{\{s_i < a\}} s_i^p \nu(ds) \right) \frac{da}{ak^i(p^*)}.
\]

**Proof:** We start by considering the variables

\[
\sum_{u \in U, u \neq \emptyset} 1_{\{\xi_u - \geq \varepsilon, \xi_u < \varepsilon\}} \xi_u^p.
\]

Write \( G_\varepsilon \) for the sigma-field generated by the variables \( (1_{\{\xi_u - \geq \varepsilon\}} \xi_u, u^- \in U) \), so \( (G_\varepsilon)_{\varepsilon>0} \) is a reversed filtration. An easy application of the branching property of the marked tree, similar to that in Proposition 5, entails that

\[
\sum_{u \in U, u \neq \emptyset} 1_{\{\xi_u - \geq \varepsilon, \xi_u < \varepsilon\}} \xi_u^p = \mathbb{E}(\Sigma_\infty \mid G_\varepsilon),
\]

and it follows that

\[
\lim_{\varepsilon \to 0} \sum_{u \in U, u \neq \emptyset} 1_{\{\xi_u - \geq \varepsilon, \xi_u < \varepsilon\}} \xi_u^p = \Sigma_\infty \text{ in } L^p(\mathbb{P})
\]

for some \( p > 1 \).

Next, we turn our attention to first moment estimates. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function with support in \( ]0, 1[ \), and consider

\[
\langle \varphi_\varepsilon, f \rangle = \sum_{u \in U, u \neq \emptyset} 1_{\{\xi_u - \geq \varepsilon, \xi_u < \varepsilon\}} \xi_u^p f(\xi_u/\varepsilon).
\]
Fix $\eta > 0$ and work under $\mathbb{P}_\eta$, i.e. when at the initial time there is a unique fragment with size $\eta$. We compute the expectation of this variable by conditioning on the mark of the father $u^- = v$ of $u$ and applying the branching property. We get

$$
\mathbb{E}_\eta \left( \langle \varphi_\varepsilon, f \rangle \right) = \mathbb{E}_\eta \left( \sum_{v \in \mathcal{U}} 1\{\xi_v \geq \varepsilon\} \xi_v \sum_{i=1}^{\infty} 1\{\xi_i \leq \varepsilon\} \xi_i \sum_{i=1}^{\infty} 1\{\xi_i \leq \varepsilon/\xi_v\} \xi_i \sum_{i=1}^{\infty} 1\{\xi_i \leq \varepsilon/\xi_v\} \xi_i f(\xi_i/\varepsilon) \right),
$$

where $(\tilde{\xi}_i)_{i \in \mathbb{N}}$ has the law $\nu(\cdot)/\nu(S^1)$ and is independent of $\xi_v$. Integrating with respect to the latter gives

$$
\mathbb{E}_\eta \left( \langle \varphi_\varepsilon, f \rangle \right) = \mathbb{E}_\eta \left( \sum_{n=0}^{\infty} \sum_{|v|=n} 1\{\log \xi_v \geq \log \varepsilon\} \xi_v \sum_{i=1}^{\infty} 1\{\xi_i \leq \varepsilon/\xi_v\} \xi_i \sum_{i=1}^{\infty} 1\{\xi_i \leq \varepsilon/\xi_v\} \xi_i f(\xi_i/\varepsilon) \right),
$$

where for $a \geq 0$

$$
h(a) = \mathbb{E} \left( \sum_{i=1}^{\infty} 1\{\xi_i \leq e^{-a}\} \xi_i \sum_{i=1}^{\infty} 1\{\xi_i \leq e^{-a}\} \xi_i f(\xi_i e^{-a}) \right).
$$

Now we can evaluate this expression using the tagged branch and the random walk $S_n = -\log \chi_n$, thanks to Lemma 3:

$$
\mathbb{E}_\eta \left( \langle \varphi_\varepsilon, f \rangle \right) = \sum_{n=0}^{\infty} \mathbb{E}^* \left( 1\{S_n \leq -\log \varepsilon\} h(\log \varepsilon - S_n - \log \eta) \right).
$$

Our assumptions enable us to apply the renewal theorem to the renewal process $S_n$, and we get the estimate

$$
\lim_{\varepsilon \to 0} \mathbb{E}_\eta \left( \langle \varphi_\varepsilon, f \rangle \right) = \frac{1}{\mathbb{E}^*(S_1)} \int_0^\infty h(a) da,
$$

where the convergence is uniform in $\eta$ as long as $\varepsilon/\eta \to 0$.

Now using the extension of the law of large numbers stated in Lemma 4 in a similar way as in the proof of Theorem 1 (again technical details are left to the reader), we can check that

$$
\lim_{\varepsilon \to 0} \langle \varphi_\varepsilon, f \rangle = \frac{\sum_{n=0}^{\infty} \mathbb{E}^*(S_n)}{\mathbb{E}^*(S_1)} \int_0^\infty h(a) da \quad \text{in } L^1(\mathbb{P}).
$$

Finally, we already know that $\mathbb{E}^*(S_1) = \kappa'(p^*)/\nu(S^1)$, and on the other hand, an easy computation gives

$$
\int_0^\infty h(a) da = \int_0^1 f(b)b^{-1} \mathbb{E} \left( \sum_{i=1}^{\infty} 1\{\xi_i < b\} \hat{\xi}_i \right) db,
$$

which completes the proof. \qed
4 Additive martingales (homogeneous case)

In this section, we consider a homogeneous fragmentation process (i.e. self-similar with index \( \alpha = 0 \)), say \( X = (X(t), t \geq 0) \) with a finite dislocation measure \( \nu \). Before resuming the study of such processes, we point at an important connection with branching random walks in continuous time: In the case when \( \nu \) only charges finite sequences, the process

\[
Z(t)(dx) := \sum \delta_{-\log X_i(t)}(dx),
\]

where the sum is taken over the fragments with strictly positive size, is a branching random walk with branching measure the image of \( \nu \) by the map \( x \rightarrow -\log x \). The proof of this result is elementary and left to the reader; the result itself has a number of interesting consequences as it essentially reduces the study of the class of homogeneous fragmentations associated to a finite dislocation measure, to that of branching random walks in continuous times, for which a great deal of results are known. Indeed, all the properties that we shall establish in this section can be interpreted as parts of the folklore of the theory of branching random walks. Recall that the Malthusian hypotheses are enforced.

4.1 Convergence of additive martingales

A crucial fact for the study of homogeneous fragmentation is that there is a simple formula for the moments of the process. Indeed, we know from Proposition 7(i) and self-similarity that for every \( p > p^\ast \) and \( t \geq 0 \),

\[
E_x \left( \sum_{i=1}^{\infty} X_i^p(t) \right) = x^p \exp(-t\kappa(p)).
\]

It follows immediately from the branching property and scaling that:

**Corollary 1** For every \( p > p^\ast \), the process

\[
M(p, t) := \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t)
\]

is a nonnegative martingale which converges a.s.

In order to investigate the asymptotic behavior of homogeneous fragmentations, it is crucial to know if the limit of the martingale above is strictly positive or zero. A first step in the analysis is the following elementary lemma.

**Lemma 5** The function \( p \rightarrow \kappa(p)/p \) reaches its maximum at a unique location \( \bar{p} > p^\ast \), which is the unique solution to the equation

\[
p\kappa'(p) = \kappa(p).
\]

More precisely, the function \( p \rightarrow \kappa(p)/p \) increases on \( [p, \bar{p}] \) and decreases on \( [\bar{p}, \infty[ \), and the value of its maximum is \( \kappa'(
\bar{p}) = \kappa(\bar{p})/\bar{p} \).
Proof: We first point out that the function \( \kappa \) is concave and increasing. It follows that the function \( p \to p \kappa'(p) - \kappa(p) \) decreases on \([1, \infty[\). (16)

Indeed, this function has derivative \( p \kappa''(p) \), which is negative since \( \kappa \) is concave. Recall that \( \kappa(p^*) = 0 \) by the definition of the Malthusian exponent \( p^* \). On the other hand, it is obvious that \( \lim_{q \to \infty} \kappa(q)/q = 0 \), hence the function \( p \to \kappa(p)/p \) has the same limit at \( p^* \) and at \( \infty \). Finally, the derivative must be zero at \( \bar{p} \), which entails that the overall maximum is given by \( \kappa'(\bar{p}) = \kappa(\bar{p})/\bar{p} \).

We may now state the main result of this section.

**Theorem 4** Assume that there exists some constants \( a, b > 0 \) such that

\[
\nu \left( \sum_{i=1}^{\infty} s_i^q > b \right) = 0. \quad (17)
\]

Then for every \( p \in ]p, \bar{p}[ \), the martingale \( M(p, \cdot) \) is bounded in \( L^1(\mathbb{P}) \). Moreover its terminal value is strictly positive a.s. whenever

\[ \nu(s_1 = 0) = 0 \]

(i.e. no particle disappears completely at its lifetime; it always gives rise to at least one new particle).

Proof: The proof uses the same route as that of Proposition 4. Recall that all that we need is to check that the sum of the jumps of the martingale raised to the power \( q \) has a finite mean, i.e.

\[
\mathbb{E} \left( \sum_{t>0} |M(p, t) - M(p, t^-)|^q \right) < \infty. \quad (18)
\]

It is convenient to re-express the left-hand side in terms of the generation of the fragments. Specifically, denote by \( \xi^{(k)}_1, \ldots \) the fragments of the \( k \)-th generation, and by \( T_i^{(k)} \) the instant when \( \xi^{(k)}_i \) splits. The jump of \( M(p, t) \) at time \( T_i^{(k)} \) is

\[
\exp(\kappa(p)T_i^{(k)})|\xi^{(k)}_i|^p \left( \sum_{j=1}^{\infty} \tilde{\xi}^p_j - 1 \right)
\]

where \( \tilde{\xi} \) is independent of \( T_i^{(k)} \) and \( \xi^{(k)}_i \) and has the law \( \nu(\cdot)/\nu(S^1) \). The conditional expectation of this quantity raised to the power \( q \), given the splitting time \( T_i^{(k)} \) and \( \xi^{(k)}_i \) is

\[
c \exp(q\kappa(p)T_i^{(k)})|\xi^{(k)}_i|^p q^q, \text{,}
\]

where

\[
c := \int_{S^1} \left| 1 - \sum_{i=1}^{\infty} s_i^q \right|^q \nu(ds)/\nu(S^1) \text{.}
\]
We point that \( c < \infty \); indeed (17) and Jensen’s inequality yield
\[
\left| \sum_{i=1}^{\infty} s_i^p \right|^q \leq b^q \left| \sum_{i=1}^{\infty} \frac{s_i^a}{s_i^p-a} \right|^q \leq b^q-a \sum_{i=1}^{\infty} s_i^{q(p-a)+a},
\]
which entails the claim since \( q(p-a)+a > p \) provided that \( q \) is chosen sufficiently close to 1.

On the one hand, we know that \( T_i^{(k)} \) is the sum of \( k+1 \) independent exponential variables with parameter \( \nu(S^1) \); in other words it has the gamma distribution with parameters \( k+1 \) and \( \nu(S^1) \). In particular,
\[
E \left( \exp(q\kappa(p)T_i^{(k)}) \right) = \left( \frac{\nu(S^1)}{\nu(S^1) - q\kappa(p)} \right)^{k+1}.
\]
On the other hand, we have already seen in (21) that
\[
E \left( \sum_{i=1}^{\infty} \left| \xi_i^{(k)} \right| \nu \right) = \left( \frac{\nu(S^1) - \kappa(pq)}{\nu(S^1)} \right)^k.
\]

Because \( p < \bar{p} \), thanks to Lemma 5 we may chose \( q > 1 \) small enough so that \( q\kappa(p) < \kappa(pq) \), and then the series
\[
\sum_{k=0}^{\infty} E \left( \sum_{i=1}^{\infty} \exp(q\kappa(p)T_i^{(k)}) \left| \xi_i^{(k)} \right| \nu \right)
\]
converges, which completes the proof of (18).

Now checking that the terminal value \( M(p, \infty) \) is strictly positive is easy. First, we know that \( P(M(p, \infty) > 0) > 0 \) since \( M(p, \infty) \) is nonnegative and has expectation 1. The assumption \( \nu(s_1 = 0) = 0 \) ensures that each particle has at least one child. By applying the branching property at the first branching time, we see that
\[
P(M(p, \infty) = 0) \leq P(M(p, \infty) = 0)^2,
\]
and hence \( P(M(p, \infty) = 0) = 0. \)

\[\square\]

### 4.2 Some applications

In this section, we develop some applications of the preceding theorem to the asymptotic behavior of homogeneous fragmentation. First, we specify the rate of decay of the largest fragment.

**Corollary 2** The assumptions are the same as in Theorem 4. Then it holds with probability one that
\[
\lim_{t \to \infty} \frac{1}{t} \log X_1(t) = -\kappa'(\bar{p}) = -\frac{\kappa(\bar{p})}{\bar{p}}.
\]
\textbf{Proof:} For every $p > 1$, we have
\[
\exp(t \kappa(p)) X^p(t) \leq \exp(t \kappa(p)) \sum_{i=1}^{\infty} X^p_i(t)
\]
and the right-hand side remains bounded as $t$ tends to infinity. Hence
\[
\limsup_{t \to \infty} \frac{1}{t} \log X_1(t) \leq - \frac{\kappa(p)}{p},
\]
and optimizing over $p$ yields
\[
\limsup_{t \to \infty} \frac{1}{t} \log X_1(t) \leq - \frac{\kappa(p)}{p}.
\]

On the other hand, for every $p \in [1, \bar{p}]$ and $\varepsilon > 0$ sufficiently small, we have the lower bound
\[
\exp(t \kappa(p)) \sum_{i=1}^{\infty} X^p_i(t) \leq X^\varepsilon(t) \exp(t \kappa(p)) \sum_{i=1}^{\infty} X^{p-\varepsilon}_i(t).
\]
We know that both limits
\[
\lim_{t \to \infty} \exp(t \kappa(p)) \sum_{i=1}^{\infty} X^p_i(t) \quad \text{and} \quad \lim_{t \to \infty} \exp(t \kappa(p - \varepsilon)) \sum_{i=1}^{\infty} X^{p-\varepsilon}_i(t)
\]
are finite and strictly positive a.s., and we deduce that
\[
\liminf_{t \to \infty} \frac{1}{t} \log X_1(t) \geq - \frac{\kappa(p) - \kappa(p - \varepsilon)}{\varepsilon}.
\]
We take the limit of the right-hand side as $\varepsilon \to 0+$ and then as $p$ tends to $\bar{p}$ to conclude that
\[
\liminf_{t \to \infty} \frac{1}{t} \log X_1(t) \geq -\kappa'(\bar{p}).
\]
Now, this quantity coincides with $-\kappa(\bar{p})/\bar{p}$, as we know from Lemma 5.

We point out that the argument of the proof above also shows that the martingale $M(p, t)$ converges to 0 a.s. (and a fortiori is not uniformly integrable) for $p > \bar{p}$. In fact, it can even be shown that the same remains true for $p = \bar{p}$.

Finally we conclude this section by an application to the asymptotic behavior of homogeneous fragmentations which is easily deduced from the martingales that we considered and classical large deviations techniques. Again, we shall focus for the sake of simplicity on the case when the dislocation measure $\nu$ is conservative. Further, it will be convenient here to represent the random sequence $X(t) = (X_1(t), \ldots)$ by the empirical distribution,
\[
\rho_t(dy) := \sum_{i=1}^{\infty} \delta_{\frac{1}{t} \log X_i(t)}(dy).
\]
(19)
Define the convex decreasing function $\Lambda$ on $]p, \infty[ \, ]$ by
\[
\Lambda(p) = \begin{cases} 
-\kappa(p) & \text{if } p < p < \bar{p}, \\
-p\kappa'(\bar{p}) & \text{if } p \geq \bar{p}.
\end{cases}
\]
Corollary 3 The assumptions are the same as in Theorem 4. It holds a.s. that

$$\lim_{t \to \infty} \frac{1}{t} \log \int_{\mathbb{R}} e^{tp} \rho_t(dy) = \Lambda(p)$$

for every $p > \bar{p}$.

Proof: Let us first prove the statement for a fixed $p > \bar{p}$. Observe that

$$\int_{\mathbb{R}} e^{tp} \rho_t(dy) = \sum_{i=1}^{\infty} X_{i}^{p}(t).$$

The case $\bar{p} < p < \bar{p}$ follows from Theorem 4, so suppose that $\bar{p} \leq p$. We use the bounds

$$X_{i}^{p}(t) \leq \sum_{i=1}^{\infty} X_{i}^{p}(t) \leq X_{i}^{p-\bar{p}}(t) \sum_{i=1}^{\infty} X_{i}^{\bar{p}}(t).$$

Recall first from Corollary 2 that $\log X_{1}(t) \sim -t\kappa'(\bar{p})$ as $t \to \infty$, and then from Lemma 5 and Corollary 1 that $e^{t\bar{p}\kappa'(\bar{p})} \sum_{i=1}^{\infty} X_{i}^{\bar{p}}(t)$ is a martingale which converges a.s. It follows immediately that

$$\lim_{t \to \infty} \frac{1}{t} \log \sum_{i=1}^{\infty} X_{i}^{p}(t) = \lim_{t \to \infty} \frac{1}{t} \log \int_{\mathbb{R}} e^{tp} \rho_t(dy) = -p\kappa'(\bar{p}) = \Lambda(p) \quad \text{a.s.}$$

The limit above holds a.s. simultaneously for every rational number $p > \bar{p}$, and by an immediate monotonicity argument, the proof is complete. $\square$

Pathwise large deviation estimates for the random measures $\rho_t$ follow from Corollary 3. Introduce the Fenchel-Legendre transform of $\Lambda$,

$$\Lambda^*(a) = \sup_{p > \bar{p}} (ap - \Lambda(p))$$

Note that $\Lambda^*(a) = \infty$ for every $a > -\kappa'(\bar{p})$ and that $\Lambda^*$ is left-continuous at $-\kappa'(\bar{p})$.

Corollary 4 The assumptions are the same as in Theorem 4. The following holds a.s.

(i) For any closed set $F \subseteq [\bar{p}, \infty[$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \rho_t(F) \leq -\inf \{\Lambda^*(p), p \in F\}.$$

(ii) For any open set $G \subseteq \mathbb{R}$,

$$\liminf_{t \to \infty} \frac{1}{t} \log \rho_t(G) \geq -\inf \{\Lambda^*(p), p > -\kappa'(\bar{p}+), p \in G\}.$$

(iii) If moreover $\kappa'(\bar{p}+) = \infty$, then $(\rho_t)$ satisfy the LDP with the good convex rate function $\Lambda^*$.  

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Proof: We aim at applying the Gärtner-Ellis theorem (see Section 2.3 in Dembo and Zeitouni [16]). The fundamental condition on the behavior of the Laplace transform of $\rho_t$ (see Assumption 2.3.2 in [16]), is the conclusion of Corollary 3. Note that the assumption $p < 1$ ensures that 0 belongs to the interior of the domain of $\Lambda$. According to Lemma 2.3.9 in [16], every $x \in ]-\kappa'(\rho^+), -\kappa'(\rho)[)$ is a so-called exposed point of the Fenchel-Legendre transform $\Lambda^*$, and since $\Lambda^*(p) = \infty$ for every $p > -\kappa'(\rho)$, the statements (i) and (ii) merely rephrase Theorem 2.3.6 in [16]. The last statement follows from the first two and Lemma 2.3.9 in [16].

Finally, we mention that in the setting of branching random walks, Biggins [13] also used the remarkable martingales analogous to those in Theorem 4 to establish precise large deviation estimates for the empirical measure of a branching random walk (see also [11]).

5 Conservation or dissipation of mass

In this section, we shall again implicitly assume that at the initial time, the fragmentation process starts from a single particle with unit size. We are interested in the situation where the dislocation measure is conservative, i.e.

$$\nu \left( \sum_{i=1}^{\infty} s_i \neq 1 \right) = 0.$$  \hfill (20)

In other words, each time a particle splits, the sum of the sizes of the children particles is the same as the size of their father. In this setting, it is convenient to think of the size of a particle as a mass.

It is easy to deduce by iteration that for every $n \in \mathbb{N}$, the total mass of particles at the $n$-th generation is conserved, i.e.

$$\sum_{|u|=n} \xi_u = 1, \quad \text{a.s.}$$

Turning our interest to the total mass of particles at time $t$, one readily sees that there is always the inequality

$$\sum_{i=1}^{\infty} X_i(t) \leq 1, \quad \text{a.s.}$$

Indeed, if $G(x)$ denotes the generation of a particle $x$, then for every integer $n$, one easily gets that

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{ G(X_i(t)) \leq n \}} X_i(t) \leq 1, \quad \text{a.s.,}$$

and thus taking the limit as $n$ increases to infinity yields our claim. More generally, a similar argument shows that the total mass is a non-increasing process as time increases.

One could be tempted to believe that the inequality above is in fact an equality; in this direction, the argument in the proof of Proposition 5 shows that this indeed the case when
the index of self-similarity of the fragmentation is nonnegative. Nonetheless, conservation of mass fails when the index is negative; the intuitive explanation is as follows: fragments with small sizes are subject to intense fragmentation, and this makes them vanish entirely quickly. Here is a formal statement (which does not require the assumption (20)).

**Proposition 10**

(i) For $\alpha < 0$, it holds with probability one that

$$\inf \{ t \geq 0 : X(t) = (0, \ldots) \} < \infty.$$ 

(ii) When $\kappa(-\alpha) > 0$, it holds with probability one that for almost every $t > 0$

$$\text{Card} \{ j \in \mathbb{N} : X_j(t) > 0 \} < \infty.$$ 

We stress that in general, no matter what the value of $\alpha$ is, there may exist random instants $t$ at which

$$\text{Card} \{ j \in \mathbb{N} : X_j(t) > 0 \} = \infty.$$ 

For instance in the case when the dislocation measure fulfills

$$\nu(x_j > 0 \text{ for all } j \in \mathbb{N}) > 0,$$

then with probability one, there occur infinitely many sudden dislocations in the fragmentation process $X^{(\alpha)}$, each of which produces infinitely many terms. This does not induce any contradiction with Proposition 10 (ii) when $\alpha < -1$, because informally, as the index of self-similarity is negative, we know that fragments with small size vanish quickly.

**Proof:** Recall $\{\xi_{u}, |u| = k\}$ denote the set of particles at the $k$-th generation. We get from Lemma 3 and Proposition 6 that

$$\mathbb{E} \left( \sum_{|u|=k}^{\infty} |\xi_{u}|^p \right) = \gamma(p)^k,$$  

where

$$\gamma(p) = \int_{S} \sum_{i=1}^{\infty} s_i^p \nu(ds) / \nu(S) = \frac{\nu(S^1) - \kappa(p)}{\nu(S)}.$$ 

In the sequel, we will choose $p$ sufficiently large such that $\kappa(p) > 0$, i.e. $\gamma(p) < 1$. In particular, the series

$$\sum_{k=1}^{\infty} \sum_{|u|=k}^{\infty} |\xi_{u}|^p$$

has a finite mean and thus converges a.s., and a fortiori

$$\lim_{k \to \infty} \max_{|u|=k} |\xi_{u}| = 0.$$  

Now suppose for simplicity that $\nu(S^1) = 1$, pick $a > 0$ arbitrary and consider the event that for some generation $k$, there exists at least one particle $\xi_{u}, |u| = k$, with lifetime $\zeta_{u} > a/k^2$. Because each particle with size $x$ has a lifetime which is exponentially
distributed with parameter $x^\alpha$, the probability of this event can be bounded from above by

$$
\sum_{k=1}^{\infty} \mathbb{E}\left( \sum_{|u|=k} \exp \left( -ak^{-2} |\xi_u|^\alpha \right) \right) \leq c_p a^{-p} \sum_{k=1}^{\infty} k^{2p} \mathbb{E}\left( \sum_{|u|=k} |\xi_u|^{-\alpha p} \right),
$$

where $c_p$ is some constant which depends only on $p$. Now use (21) and take $p$ sufficiently large, so that the right hand side can be bounded from above by $c'_p a^{-p}$, where $c'_p$ is some constant which depends only on $p$ and $\nu$.

We see that provided that $a$ is chosen large enough, the probability that for all $k$ there are no particle of the $k$-th generation alive at time

$$
t := \zeta_\emptyset + a \sum_{k=1}^{\infty} k^{-2},
$$

can be made as close to 1 as we wish. Recalling (22), this completes the proof of (i).

Next, suppose $\kappa(-\alpha) > 0$. In this situation, we have by (21) that

$$
\mathbb{E}\left( \sum_{u \in U} \zeta_u \right) = \mathbb{E}\left( \sum_{k=0}^{\infty} \sum_{|u|=k} \zeta_u^{-\alpha} \right) < \infty.
$$

So, if $I_u$ denotes the time-interval during which the particle with label $u$ is alive (so the length of $I_u$ is the lifetime $\zeta_u$ of this particle), we have

$$
\mathbb{E}\left( \int_0^{\infty} dt \sum_{u \in U} 1_{\{t \in I_u\}} \right) = \mathbb{E}\left( \sum_{u \in U} \zeta_u \right) < \infty,
$$

which entails that for almost every $t \geq 0$, there are only finitely many particles alive at time $t$. \hfill \Box

Proposition 10 shows in particular that for negative indices of self-similarity, the total mass in a self-similar fragmentation vanishes at a finite time a.s., even when the dislocation measure is conservative. This phenomena of dissipation of mass can be viewed as the formation of dust (infinitesimal particles), the mass of dust at time $t$ being $1 - \sum_{i=1}^{\infty} X_i(t)$. We refer to the recent works of Haas [18, 19] for many deep results about the formation of dust in self-similar fragmentations with negative indices.
References


