Erdős-Renyi Random Graphs

The first random graph model was introduced by Erdős and Rényi in the late 1950’s. To define the model, we begin with the set of vertices \( V = \{1, 2, \ldots, n\} \). For \( 1 \leq x < y \leq n \) let \( \eta_{x,y} \) be independent = 1 with probability \( p = \lambda / n \) and 0 otherwise. Let \( \eta_{y,x} = \eta_{x,y} \). If \( \eta_{x,y} = 1 \) there is an edge from \( x \) to \( y \).

A large Erdős-Renyi random graph has a degree distribution that is Poisson with mean \( \lambda \). However in many technological and social networks, the degree distribution \( p_k \) follows a power law: \( p_k \sim C k^{-\alpha} \).

Fixed Degree Distributions

Molloy and Reed (1995) were the first to construct graphs with specified degree distributions. We will use the approach of Newman, Strogatz, and Watts (2001, 2002) to define the model. Let \( d_1, \ldots, d_n \) be independent and have \( P(d_i = k) = p_k \). Since we want \( d_i \) to be the degree of vertex \( i \), we condition on \( E_n = \{d_1 + \cdots + d_n \text{ is even}\} \).

If the probability \( P(E_1) \in (0,1) \) then \( P(E_n) \to 1/2 \) as \( n \to \infty \) so the conditioning will have little effect on the finite dimensional distributions.

Barabási and Albert (1999)

Barabási and Albert (1999) introduced a simple model that a power law graph with \( \alpha = 3 \).

At every time step, we add a new vertex with \( m \) edges that link the new vertex to \( m \) vertices already present in the system. The connections are chosen according to the \textit{preferential attachment rule}: we assume that the probability \( \pi_i \) that a new vertex will be connected to a vertex \( i \) depends on the degree of that vertex, so that \( \pi_i = d_i / \sum_j d_j \).

The limiting degree distribution is

\[
p_k = \frac{2m(m+1)}{k(k+1)(k+2)} \quad \text{for } k = m, m+1, \ldots
\]
Contact Process

Consider the contact process on a power-law random graph. In this model
- infected individuals become healthy at rate 1 (and are again
susceptible to the disease)
- susceptible individuals become infected at a rate $\lambda$ times the number
of infected neighbors.

Pastor-Satorras and Vespigniani (2001a, 2001b, 2002) have made an
extensive study of this model using mean-field methods (See Section 4.8.).

Contact Process Conjectures

Let $\lambda_c$ be the critical value for prolonged persistence. If $\lambda > \lambda_c$ there will
be a quasi-stationary distribution with density $\rho(\lambda) \sim C(\lambda - \lambda_c)^\beta$.
- If $\alpha \leq 3$ then $\lambda_c = 0$.
- If $3 < \alpha < 4$, $\lambda_c > 0$ but the critical exponent $\beta > 1$.
- If $\alpha > 4$ then $\lambda_c > 0$ and $\beta = 1$.

Problem. Berger, Borgs, Chayes, Saberi (2005) prove persistence for time $\exp(c n^{1/2})$ for any $\lambda > 0$ when $\alpha = 3$. Does it last for $\exp(cn)$?

Voter models

Vertex voter model. Each vertex $x$ changes at rate 1. Pick a neighbor at
random and set $\xi(x) = \xi(y)$. Genealogical process jumps at rate 1.
Stationary distribution $\pi(x) = cd(x)$.

Edge voter model. Each edge becomes active at rate 1. Flip a coin to
give it an orientation $(x, y)$ then set $\xi_t(x) = \xi_t(y)$. Genealogical process
of a site is a random walk that jumps to a randomly chosen neighbor at
rate $d(x)$. Stationary distribution is uniform.

Small worlds

In interacting particle systems it is traditional to model space using a
regular lattice such as $\mathbb{Z}^d$ and have interactions between sites $x$ and $y$
when $y - x \in \mathcal{N}$, the neighborhood set. However in reality, people interact
not only with those who live near them but also with those they see at
work or school, creating long range connections.

BC small world

The first to create a small world were Bollobás and Chung (1988), who
added a random matching to a ring of $n$ vertices with nearest neighbor
connctions and showed that the resulting graph had diameter $\sim \log_2 n$.
This graph (the BC small world) is not a good model of a social network
but it is nice to study because it looks locally like the tree in which each
vertex has degree 3.
Watts and Strogatz

Watts and Strogatz (1998) are famous for popularizing this concept. They started with a ring and rewired a fraction of existing connections. It is more convenient to use Newman and Watts (1999) approach. The NW small world is a ring with nearest neighbor connections plus an Erdos-Renyi(\(\rho/n\)), where \(\rho\) is small. Looks locally like a random tree.

Contact Process

The contact process on the BC small world has two phase transitions, \(\lambda_1 < \lambda_2\), which correspond to the critical values for local and global survival on the tree.

\[
\begin{align*}
\lambda_1 &= \inf \{ \lambda : P(|\xi_0^\ell| = 0 \text{ eventually}) < 1 \} \\
\lambda_2 &= \inf \{ \lambda : \liminf_{t \to \infty} P(0 \in \xi_{|t\ell|}) > 0 \}
\end{align*}
\]

Almost Theorem. Prolonged persistence for time \(\exp(cn)\) occurs for \(\lambda > \lambda_1\). Almost refers to the fact that in addition to the quenched long range connections, we use some annealed ones.

Intermediate phase on small world

\(\lambda_1 < \lambda < \lambda_2\) the contact process survives but dies out locally.

Conjecture. When \(\lambda_1 < \lambda < \lambda_2\) there is a \(c_0\) so that if we start with a single infected at 0 then

\[
\begin{align*}
P(0 \in \xi^0(c \log n)) &\to 0 \quad \text{for } 0 < c < c_0 \\
P(0 \in \xi^0(c \log n)) &\to \rho^2 \quad \text{for } c > c_0
\end{align*}
\]

where \(\rho\) is the survival probability for the contact process on the 3-tree.

Voter Model

Theorem. Put two particles at random locations on the BC small world with \(n\) points and let them perform independent random walks until they hit at time \(T_n\). Then \(T_n/2n\) converges in distribution to an exponential with mean 1.

Conjecture. The voter model on the BC small world has a one parameter family of quasi-stationary distributions \(\xi^\infty_{|p\ell|}, 0 \leq p \leq 1\). If \(0 < p < 1\) these stationary distributions last for time \(O(n)\) before entering an absorbing state. [Lower bound is known.]

Bigger Small Worlds

BC(\(d\)). Consider the torus \((\mathbb{Z} \mod L)^d\) where \(L\) is even and pair the vertices at random to produce long range connections. Locally the graph looks like the free product \(\mathbb{Z}^d \ast \{0, 1\}\) which has elements \(z_01z_11\cdots 1z_m\).

Erdos Renyi Random Graphs

Vertices \(V = \{1, 2, \ldots, n\}\). For \(1 \leq x < y \leq n\) let \(\eta_{x,y}\) be independent \(= 1\) with probability \(p = \lambda/n\) and 0 otherwise. Let \(\eta_{y,x} = \eta_{x,y}\). If \(\eta_{x,y} = 1\) there is an edge from \(x\) to \(y\).

To begin the construction of the cluster containing 1, let

\[
S_0 = \{2, 3, \ldots, n\} \quad \text{susceptibles} \\
l_0 = \{1\} \quad \text{infected} \\
R_0 = \emptyset \quad \text{removed}
\]

In graph terms, we have already examined the connections of all sites in \(R_t\), \(l_t\) are the sites to be investigated on this turn, and \(S_t\) are unexplored.
We have already examined the connections of all sites in $R_t$, $I_t$ are the sites to be investigated on this turn, and $S_t$ are unexplored. These sets evolve as follows:

$$R_{t+1} = R_t \cup I_t$$
$$I_{t+1} = \{ y \in S_t : \eta_{x,y} = 1 \text{ for some } x \in I_t \}$$
$$S_{t+1} = S_t - I_{t+1}$$

The cluster containing 1, $C_1 = \bigcup_{i=0}^{\infty} I_t$.

**Heuristic.** When $t$ is not too large, $|I_t|$ is a branching process in which each individual has a Poisson($\lambda$) number of offspring.

### Erdős Renyi Results 2: Diameter

The average distance between two points on the giant component is

$$\sim \log n / (\log \lambda)$$

[Physics proof: $|I_t| \sim W^t \lambda^t = n$ when $t \sim \log \lambda$.]

The diameter of the giant component (i.e., $\max d(x,y)$) is $\geq c \log n$ where $c > 1 / (\log \lambda)$ due to dangling ends of length $O(\log n)$: a vertex of degree 1 connected to a path of other vertices with degree 2 (see Section 2.4 for more details.)

### Random Walk Viewpoint

Let $R_0 = \emptyset$, $U_0 = \{2, 3, \ldots, n\}$, and $A_0 = \{1\}$. $R_t$ is the set of removed sites, $U_t$ are the unexplored sites and $A_t$ is the set of active sites. At time $\tau = \inf \{ t : A_t = \emptyset \}$ the process stops. If $A_t \neq \emptyset$, pick $i_t$ from $A_t$ according to some rule that is measurable with respect to $A_t = \sigma(A_0, \ldots, A_t)$ and let

$$R_{t+1} = R_t \cup \{ i_t \}$$
$$A_{t+1} = A_t - \{ i_t \} \cup \{ y \in U_t : \eta_{i_t,y} = 1 \}$$
$$U_{t+1} = U_t - \{ y \in U_t : \eta_{i_t,y} = 1 \}$$

This time $|R_t| = t$ for $t \leq \tau$, so the cluster size is $\tau$. $S_t = |R_t|$ is almost a random walk with jump distribution $-1 + \text{Poisson}(\lambda)$. (In proof we work with a lower bound $W_t$ on this random walk.)

### Threshold for connectivity

**Theorem 2.8.1.** Consider $G = ER(n, \lambda/n)$ with $\lambda = a \log n$. The probability $G$ is connected tends to 0 if $a < 1$ and to 1 if $a > 1$.

Probability $d(x) = 0$ is $\approx \exp(-a \log n) = n^{-a}$. If $a < 1$, a second moment computation shows that there are about $n^{1-a}$ isolated vertices.

$a > 1$. With probability $1 - o(n^{-1})$, $x$ has at least 14 neighbors, 7 log $n$ at distance 2, and connects to giant component.

**2.8.3.** $\lambda = \log n + b + o(1)$. $P(G$ is connected $) \to \exp(-e^{-b})$. 
Diameter of connected Erdős-Renyi graphs

**Theorem 2.8.6.** If \( \lim \inf np/(\log n) = c > 1 \) and \( (\log p)/(\log n) \to 0 \) then the diameter, \( D(n,p) \sim (\log n)/(\log np) \).

In words, the physicist’s computation gives the right answer. Proof uses large deviations estimates to control the growth of the branching process.

If \( p = n^{(1/d)-1}(\log(n^2/c))^{1/d} \) then in the limit the diameter is \( d \) with probability \( e^{-c/2} \) and \( d + 1 \) with probability \( 1 - e^{-c/2} \).

Fixed Degree Distributions

**Newman, Strogatz, Watts model.** Let \( d_1, \ldots, d_n \) be independent and have \( P(d_i = k) = p_k \). Since we want \( d_i \) to be the degree of vertex \( i \), we condition on \( E_n = \{d_1 + \cdots + d_n \text{ is even}\} \). Attach \( d_i \) half-edges to vertex \( i \) and then pair the half-edges at random. For the moment we will suppose \( E(d_i^2) < \infty \).

Let \( G_0(z) = \sum_k p_k z^k \) and \( G_1(z) = \sum_k q_k z^k \) be the generating functions for the first, and second and subsequent generations.

**Theorem 3.1.3.** The condition for the existence of a giant component is \( \nu > 1 \). In this case the fraction of vertices in the giant component is asymptotically \( 1 - G_0(\rho_1) \) where \( \rho_1 \) is the smallest fixed point of \( G_1 \) in \([0,1]\).

**Proof.** Look at cluster growth as a random walk. See Section 3.2 for details.

Subcritical Phase

**Conjecture 3.3.1.** If \( p_k \sim ck^{-\gamma} \) with \( \gamma > 3 \) and \( \nu < 1 \) then the largest component is \( O(n^{(1/\gamma - 1)}) \).

Why? The tail of the distribution \( \sum_{k=0}^\infty p_k \sim cK^{1-\gamma} \) so the largest degree present in a graph with \( n \) vertices is \( O(n^{1/(\gamma - 1)}) \).

Chung and Lu model

Assign weights \( w_i \) to the vertices. The probability of an edge from \( i \) to \( j \) is \( w_i w_j / \sum_k w_k \). Loops from \( i \) to \( i \) are allowed so the expected degree at \( i \) is

\[
\sum_j \frac{w_i w_j}{\sum_k w_k} = w_i
\]

Of course for this to make sense we need \( (\max_i w_i)^2 < \sum_k w_k \).
Let \( d = (1/n) \sum_k w_k \) be the average degree. As in the fixed degree model, if we follow an edge from \( i \), vertices are chosen proportional to their weights, i.e., \( j \) is chosen with probability \( w_j / \sum_k w_k \). Thus the relevant quantity for connectedness of the graph is the second order average degree

\[
\bar{d} = \sum_j w_j \frac{w_j}{\sum_k w_k}
\]

Theorem 3.3.2 Let \( \text{vol}(S) = \sum_{i \in S} w_i \). If \( \bar{d} < 1 \) then all components have volume at most \( C \sqrt{n} \) with probability at least

\[
1 - \frac{d \bar{d}^2}{C^2(1 - d)}
\]

Proof. Let \( x \) be the probability that there is a component with volume \( x > C \sqrt{n} \). Let \( \gamma = 1/\sum_k w_k \). Pick two vertices at random with probabilities proportional to their weights. The probability \( \pi \) a randomly chosen pair of vertices is in the same component has

\[
x(C \sqrt{n})^2 \leq \pi
\]

van der Hofstad, Hooghiemstra, and Van Mieghem (2004)

Consider a Newman-Strogatz-Watts random graph with \( P(D \geq x) \leq x^{-\beta + 1} \) with \( \beta > 3 \). Let \( \mu = ED \) and let \( \nu = E(D(D - 1))/ED \) be the mean of the size biased distribution. Let \( Z_t \) be the two-phase branching process and let

\[
W = \lim_{t \to \infty} Z_t / \mu \nu^{t-1}.
\]

Suppose \( \nu > 1 \). Let \( H_n \) be the distance between 1 and 2 in the random graph on \( n \) vertices. By physics heuristic

\[
H_n \sim \log_\nu n
\]

Theorem 3.4.1. For \( k \geq 1 \), let \( a(k) = \lfloor \log_\nu k \rfloor - \log_\nu \in (-1, 0] \). As \( n \to \infty \)

\[
P(H_n = |H_n| < \infty) = P(R_{a(n)} = k) + a(1)
\]

If \( k = \mu(\nu - 1) \) then for \( a \in (-1, 0] \)

\[
P(R_a > k) = E(\exp(-\nu k^{\nu+1} W_1 W_2) | W_1 W_2 > 0)
\]

where \( W_1 \) and \( W_2 \) are independent copies of \( W \).

Sketch of proof. Taking turns grows each cluster by one branching step,

\[
P(H_n > k) \approx E \exp \left( - \sum_{i=2}^{k+1} \frac{Z_i^2 \bar{Z}_i^2}{\mu^2 n} \right)
\]

The sum of the degrees is \( \sim \mu n \). Now use the branching process limit theorem.
Start with $G_1 = 0$ and 1 connected by an edge. At time $(t + 1) \geq 2$ we add vertex $t + 1$ and connect it one other vertex $i \leq t$ with probability proportional to $d_i + a$ where $a > -1$.

Sum of the weights is $M_t(t) = S_t = 2t + (t + 1)a$. Let $N_k(t)$ be the expected number of vertices of degree $k$ at time $t$.

$$N_k(t + 1) - N_k(t) = \frac{1}{S_t} [(k - 1 + a)N_{k-1}(t) - (k + a)N_k(t)] + \delta_{k,1}$$

where $\delta_{k,1} = 1$ if $k = 1$ and 0 otherwise, and we set $N_0(t) \equiv 0$. This the master equation of Dorogovstev, Mendes, and Samukhin (2000).

We apply the result to $X_s = E(Z(k,t)|F_s)$. We claim that $|X_s - X_{s-1}| \leq 2$. To see this, note that whether we attach the vertex $s$ to either $v$ or $v'$ does not affect the degrees of $w \neq v,v'$, or the probabilities they will be chosen later, so it follows that $|X_s - X_{s-1}| \leq 2$. Since $Z(k,0) = E(Z(k,t))$ taking $x = \sqrt{\log t}$ we have

$$P(|Z(k,t) - E(Z(k,t))| > \sqrt{\log t}) \leq t^{-1/8} \rightarrow 0$$

Theorem 4.1.4. As $t \rightarrow \infty$, $Z(k,t)/t \rightarrow p_k$ in probability.

We denote by $M^k_n$ the sum of the weights. From this, we get $\mu_1^k = 2k + 1$.

$S_t/t \rightarrow \mu = 2 + a$. When $k = 1$

$$N_k(t + 1) = 1 + \frac{1 - \frac{1 + a}{S_t}}{\mu} N_k(t)$$

Iterating one can show (Lemma 4.1.2)

$$N_k(t) \rightarrow \frac{\mu}{a + k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{a + j}\right)^{-1}$$

and then using induction

$$N_k(t) \rightarrow n_k = \frac{\mu}{a + k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{a + j}\right)^{-1}$$

With the asymptotics for the mean in hand, the rest is easy thanks to the inequality of Azuma (1967) and Hoeffding (1963).

Lemma 4.1.3. Let $X_t$ be a martingale with $|X_{s} - X_{s-1}| \leq c$ for $1 \leq s \leq t$. Then

$$P(|X_t - X_0| > x) \leq \exp(-x^2/2c^2t)$$

Let $Z(k,t)$ be the number of vertices of degree $k$ at time $t$, and let $F_s$ denote the $\sigma$-field generated by the choices up to time $s$. We apply the result to $X_s = E(Z(k,t)|F_s)$.

Mori’s (2004) martingales. Let $X[n,j]$ be the weight of vertex $j (= d_j + a)$ after the $n$-th step, let $\Delta[n+1,j] = X[n+1,j] - X[n,j]$, and let $F_n$ denote the $\sigma$-field generated by the first $n$ steps. If $j \leq n$ then

$$P(\Delta[n+1,j] = 1|F_n) = X[n,j]/S_n$$

Recall $S_n = 2n + (n + 1)a$ is the sum of the weights. From this, we get

$$E(X[n+1,j]|F_n) = X[n,j] \left(1 + \frac{1}{S_n}\right)$$

so $c_nX[n,j]$ will be a martingale if $c_n/c_{n+1} = S_n/(1 + S_n)$. 

Being a nonnegative martingale \( c_n X[n,j] \) converges to a limit. Using some other martingales involving products of moments of the \( X[n,j] \) one can show that the maximal degree in the random tree after \( n \) steps, \( M_n \) has \((\beta = a)\)

**Theorem 4.3.2.** With probability one, \( n^{-1/(2+\alpha)} M_n \rightarrow \mu \).

When \( a = 0 \), Barabási-Albert, \( M_n = O(n^{1/2}) \).

The power \( 0 < \beta - 2 < 1 \), and \( q_k \) is concentrated on the nonnegative integers so \( q_k \) is in the domain of attraction of a one-sided stable law with index \( \alpha = \beta - 2 \).

**Theorem 4.5.1.** Davies (1978). Consider a branching process with offspring distribution \( \xi \) with \( P(\xi > k) \sim B_n k^{-\alpha} \) where \( \alpha = \beta - 2 \in (0,1) \). As \( n \rightarrow \infty \),

\[
\alpha^{-1} \log(Z_n + 1) \rightarrow W
\]

with \( P(W = 0) = \rho \) the extinction probability for the branching process.

We will prove a result about distance in the Chung-Lu model. The probability of having degree \( \geq K \) is \( \sim BK^{-\beta+1} \) where \( 1/B = (\beta - 1)\zeta(\beta) \). Assuming the weights are decreasing we have

\[
w_i = K \text{ when } i/n = BK^{-\beta+1}
\]

Solving gives \( w_i = (i/nB)^{-1/(\beta-1)} \)

**Theorem 4.5.2.** Consider Chung and Lu’s power law graphs with \( 2 < \beta < 3 \). Then the distance between two randomly chosen vertices in the giant component, \( H_n \) is asymptotically at most

\[
(2 + o(1)) \log \log n / (\log(\beta - 2))
\]

Distances for power laws graphs

Consider a Newman-Strogatz-Watts random graph with \( p_k = k^{-\beta}/\zeta(\beta) \) where \( 2 < \beta < 3 \) and \( \zeta(\beta) = \sum_{k=1}^{\infty} k^{-\beta} \) is the constant need to make the sum 1. In this case, \( q_{k-1} = kp_k/\mu = k^{1-\beta}/\zeta(\beta - 1) \), so the mean is infinite and the tail of the distribution

\[
\sum_{k=K}^{\infty} q_k \sim \frac{1}{\zeta(\beta - 1)(\beta - 2)} k^{2-\beta}
\]

To study the average distance between two randomly chosen points, we will first investigate the behavior of the branching process in order to figure out what to guess.

To determine the distances using physicists reasoning, we note that our limit theorem says

\[
\log(Z_t + 1) \approx \alpha^{-1} W
\]

so \( Z_t + 1 \approx \exp(\alpha^{-1} W) \). Replacing \( Z_t + 1 \) by \( n \) and solving gives

\[
\log n = \alpha^{-1} W
\]

Discarding the \( W \) and writing \( \alpha^{-1} = \exp(\alpha(t) \log \alpha) \) we get

\[
t \sim \frac{\log \log n}{\log(1/\alpha)}
\]

\( H_n \) is asymptotically at most

\[
(2 + o(1)) \log \log n / (\log(\beta - 2))
\]

van der Hofstad, Hooghiemstra, and Znamenski (2005a) have shown that this gives the correct asymptotics for the Newman-Strogatz-Watts model and have shown that the fluctuations are \( O(1) \). Note that the correct asymptotics are twice the heuristic that comes from growing one cluster to size \( n \), but matches the guess that comes from growing two clusters to size \( \sqrt{n} \). The world gets very small when \( \alpha \leq 2 \), see van der Hofstad, Hooghiemstra, and Znamenski (2005b).
**Sketch of proof.** Let \( t = n^{1/\sqrt{\log n}} \) and consider the vertices
\( H_0 = \{ i : w_i \geq t \} \). By comparing with an Erdős-Rényi random graph in which two vertices in \( H \) are connected with probability \( p \), where \( np \to \infty \), faster than \( \log n \) so the probability \( H_0 \) is connected tends to 1 and the diameter of \( H_0 \) is of order \((\log \log n)^{3/2}\). (See Section 2.8 for results we mentioned earlier.)

Since \( H_0 \) is connected it follows that each \( H_k \) with \( k \leq \ell \equiv \inf \{ i : t^{i+1} < (\log n)^{1/\epsilon} \} \) is connected. Let \( m = (\log \log n)/(\log \alpha) \). This is chosen so that
\[
\alpha^{-m} \log n < 1
\]
and \( \alpha^{-m} < e \), so \( \ell \leq m \).

\( H_0 \) has diameter \( O(\sqrt{\log \log n}) \), so at this point we have shown that \( H_k \) is connected and has diameter smaller than \( 2m + O(\sqrt{\log \log n}) \). To connect the remaining points we use the lemma again with \( S = \{ j \} \) and \( t = H_k \).

**Problem.** Can you derive Theorem 4.6.1 by considering the branching process with offspring distribution \( q_{k-1} = k^{-2}/c(2) \)? There is a large literature on branching processes with infinite mean, but it does not seem to be useful for concrete examples like this one. It is my guess that
\[
\log(1 + Z_1)/(\log t) \to 1.
\]

**Lemma 4.5.3.** Let \( \text{Vol}(S) = \sum_{i \in S} w_i \). If \( S \cap T = \emptyset \) and \( \text{Vol}(S) \text{Vol}(T) \geq c \text{Vol}(G) \) then the distance from \( S \) to \( T \) satisfies
\[
P(d(S, T) > 1) \leq e^{-c}.
\]

Let \( H_k = \{ i : w_i > t^{\alpha} \} \) and suppose \( t^{\alpha} \geq (\log n)^{1/\epsilon} \) where \( 0 < \epsilon < \alpha < (\beta - 2) \) and now \( \alpha > \beta - 2 \). Using the previous lemma with \( S = \{ j \} \) and \( T = H_j \), we see that if \( j \in H_{k+1} \) then with probability \( \geq 1 - n^{-2} \), \( j \) is connected to a point in \( H_k \).

Let \( G'_m \) be the version of the Barabási-Albert model in which the new vertex added has \( m \) connections to existing vertices. The main result to Bollobás and Riordan (2004b) is:

**Theorem 4.6.1.** Let \( m \geq 2 \) and \( \epsilon > 0 \). Then with probability tending to 1, \( G'_m \) is connected and
\[
(1 - \epsilon)(\log n)/(\log \log n) \leq \text{diameter}(G'_m) \leq (1 + \epsilon)(\log n)/(\log \log n)
\]
The case \( m = 1 \) is excluded because the upper bound is false in this case. Due to dangling ends the average pairwise distance is \( O(\log n) \) in this case.

**Lower bound in Theorem 4.6.1.**

To do this, we will consider \( G'_N \) with \( N = nm \). The idea behind the proof is to compare \( G'_N \) with a random graph in which an edge from \( i \) to \( j \) is present with probability \( c/\sqrt{\beta} \). Let \( g_j \) be the vertex to which \( j \) sends an edge when it is added to the graph.

**Lemma 4.6.2**
(a) If \( 1 \leq i < j \) then \( P(g_j = i) \leq C_1(ij)^{-1/2} \).
(b) If \( 1 \leq i < j < k \) then \( P(g_j = i, g_k = i) \leq C_2kj^{-1}(jk)^{-1/2} \).
Lemma 4.6.3. Let \( E \) and \( E' \) be events of the form
\[
E = \cap_{j=1}^{n} \{ e_{ij} \} \quad E' = \cap_{j'=1}^{n} \{ e'_{ij'} \}
\]
where \( i < j \) and \( i' < j' \) for all \( s \). If the sets \( \{i_1, \ldots, i_r\} \) and \( \{i'_1, \ldots, i'_{r'}\} \) are disjoint then \( P(E \cap E') \leq P(E)P(E') \).

Lemma 4.6.4. Let \( S \) be a graph on \( \{1, 2, \ldots, N\} \) in which each vertex is joined to at most two later vertices. Let \( B = \{G_1, \sqrt{C_2}\} \). If \( E(S) \) denotes the edges in \( S \) then
\[
P(S \subset G^m_n) \leq B^{E(S)} \prod_{j \in E(S)} \frac{1}{\sqrt{V_j}}
\]

Percolation. Calculations with generating functions for the branching process suggest that for site percolation

Conjecture 4.7.2. Given a degree distribution with \( p_k \sim c k^{-\gamma} \) with \( 2 < \gamma \leq 3 \), the critical percolation probability is 0. If \( 2 < \gamma < 3 \) the size of the giant component is \( \exp(\frac{c p}{(1 + o(1))}) \) as \( p \to 0 \). If \( \gamma = 3 \) the size is \( \exp(-1 + o(1)) \).

The key to proving that \( \lambda_c = 0 \) for the contact process on power law graphs is:

Lemma 4.8.2. Let \( G \) be a star graph with center 0 and leaves \( 1, 2, \ldots, k \). Let \( A_t \) be the set of vertices infected at time \( t \) when \( A_0 = \emptyset \). If \( k \lambda^2 \to \infty \) then \( P(\lambda^2(A_t) \neq 0) \to 1 \).

Write the state of the system as \( (m, n) \) where \( m \) is the number of infected leaves and \( n = 1 \) if the center is infected and 0 otherwise. It is straightforward to prove this by considering what happens to \( m \) in the alternating periods when \( n = 0 \) and \( n = 1 \).

Consider a self-avoiding path \( V = v_0 = n, v_1, \ldots, v_t = n-1 \). Recalling that \( G^m_n \) is obtained by identifying the vertices of \( G^m_n \) in groups of \( m \), this corresponds to a graph \( S \) that consists of edges \( x_t y_{t+1}, t = 0, 1, \ldots, \ell - 1 \) with \( |x_t/m| = |y_t/m| = v_t \), where \( \lfloor x \rfloor \) rounds up to the next integer.

Lemma 4.6.4 implies \( S \) is present in \( G^m_n \) with probability
\[
\leq B^\ell \prod_{t=0}^{\ell-1} \frac{1}{\sqrt{v_t y_{t+1}}} \leq B^\ell \prod_{t=0}^{\ell-1} \frac{1}{\sqrt{v_t y_{t+1}}} = B^\ell \prod_{t=1}^{\ell-1} \frac{1}{v_t}
\]

There are at most \( m^\ell \) graphs \( S \) that correspond to our path \( V \) so
\[
P(V \subset G^m_n) \leq \left(\frac{B m^\ell}{\sqrt{v_0}}\right) \prod_{t=1}^{\ell-1} \frac{1}{v_t}
\]

Now sum over \( v \) and \( \ell \leq L = \log n/(\log \log n) \).

Bollobás and Riordan (2004) and Riordan (2004) have done a rigorous analysis of percolation on the Barabási-Albert preferential attachment graph, which has \( \beta = 3 \).

Theorem 4.7.3. Suppose that newly added vertices connect to \( m \geq 2 \) existing vertices. For \( 0 < p \leq 1 \) there is a constant \( c_m \) and a function
\[
\lambda_m(p) = \exp \left( -c_m(1 + o(1)) \right)
\]
so that with probability \( 1 - o(1) \) the size of largest component is \( (\lambda(p) + o(1))n \) and the second largest is \( o(n) \).

Contact Process

The key to proving that \( \lambda_c = 0 \) for the contact process on power law graphs is:

If we have a Newman-Watts-Strogatz graph with \( p_k \sim C k^{-\alpha} \) then \( \sum_{k \leq K} p_k \sim C' k^{1-\alpha} \) so \( \max_{i \leq n} d_i \approx n^{\gamma/(\alpha - 1)} \). When \( \lambda \) is held fixed the survival time is at least \( \exp(cn^{\gamma/(\alpha - 1)}) \) with high probability.

Problems.
1. When is the survival time at least of order \( \exp(cn) \)?
2. When do we have \( \lambda_c = 0 \) in the sense of existence of a quasi-stationary distribution with asymptotic positive density.
Small worlds

In the BC small world we add a random matching to a ring of \( n \) vertices with nearest neighbor connections. Bollobás and Chung (1988) showed that the resulting graph had diameter \( \sim \log_2 n \).

Recall that the NW small world is a nearest neighbor ring + ER(\( \frac{\rho}{n} \)). Barbour and Reinert (2001) have done a rigorous analysis of the average distance between points in a continuum model in which there is a circle of circumference \( L \) and a Poisson mean \( \frac{L}{\rho} \) number of random chords. The chords are the short cuts and have length 0.

The first step in their analysis is to consider an upper bound model that ignores intersections of growing arcs and that assumes each arc sees independent Poisson processes of shortcut endpoints. Let \( S(t) \) be size, i.e., the Lebesgue measure, of the set of points within distance \( t \) of a chosen point and let \( M(t) \) be the number of intervals. Under our assumptions

\[
S'(t) = 2M(t)
\]

while \( M(t) \) is a branching process in which there are no deaths and births occur at rate \( 2\rho \).

At time \( t = (2\rho)^{-1}(1/2) \log(L\rho) \), \( ES(t) = (L/\rho)^{1/2} - 1 \). Ignoring the \(-1\) we see that if we have two independent clusters run for this time then the expected number of connections between them is

\[
\sqrt{L\rho} \cdot \rho \cdot \sqrt{L\rho} = 1
\]

since the middle factor gives the expected number of shortcuts per unit distance and the last one is the probability a short cut will hit the second cluster.

Epidemics (Percolation)

Our next topic, following Moore and Newman (2000) are epidemic models on the small world, which are essentially percolation processes. We will consider bond percolation in which all individuals are susceptible and there is a probability \( p \) that an infected individual will transmit the infection to a neighbor. On the BC small world \( p_c = 1/2 \).

Physics proof #1. Introduce an infinite graph associated with the small world, that we call the “Big World.” We begin with a copy of the integers, \( \mathbb{Z} \). To each integer we attach a Poisson mean \( \rho \) long range bonds that lead to a new copy of \( \mathbb{Z} \) on which we repeat the previous construction. The first copy of \( \mathbb{Z} \) we call level zero. The levels of other copies are equal to the number of long range bonds we need to traverse to get to them.
they have degree number of edges. Collapsing the components of the ring to single vertices,

It follows that the conditions for a giant component is

\[ 1 + 2p/(1 - p) = (1 + p)/(1 - p) \]

Edges have to be open in order to reach the next level so \( E(M|N) = p\rho N \) and \( EM = p\rho \nu \). The critical value for percolation occurs when \( p_c\rho\nu = 1 \) or

\[
p\rho\nu = 1 + p_c = 1 
\]

Solving we have \( p_c^2 + (\rho + 1)p_c - 1 = 0 \) or

\[
p_c = \frac{-(\rho + 1) + \sqrt{(\rho + 1)^2 + 4\rho}}{2p} 
\]

Let \( p_0(n) \) be the probability 0 is connected to \( n \) sites on Level 0.

\[
p_0(n) = \sum_{j=0}^{n-1} p^j (1 - p) \cdot p^{n-1-j}(1 - p) = np^{n-1}(1 - p)^2 
\]

and the mean number of sites reached on level 0 is

\[
\nu = 1 + 2p/(1 - p) = (1 + p)/(1 - p) 
\]

Now each site in the cluster is connected to a Poisson mean \( \lambda = \rho p \) number of edges. Collapsing the components of the ring to single vertices, they have degree \( S_N = X_1 + \cdots + X_N \) where the \( X_i \) are i.i.d. Poisson(\( \lambda \)) and \( N \) is geometric with success probability \( r \).

\[
ES_N = \text{EXEN} = \lambda/r \\
\text{var}(S_N) = E\text{N} \cdot \text{var}(X) + (EX)^2 \cdot \text{var}(N) \\
= \frac{1}{r} \cdot \lambda + \lambda^2 \cdot \frac{1 - r}{r^2} 
\]

so the mean of the size biased distribution

\[
\frac{E(S_N(S_N - 1))}{ES_N} = \frac{\lambda^2 - r}{r} 
\]

It follows that the conditions for a giant component is

\[ \rho p(1 + p)/(1 - p) > 1 \] for bond percolation

\[ \text{Physics proof } #2. \] We will now give another derivation of the bond percolation critical value based on the fact that, seen from a fixed vertex, the NW small world is locally tree like. Color vertices blue if they are reached by a long range edge and red if they are reached by a short range edge. Ignoring collisions the growth of the cluster is a two-type branching process with mean matrix

\[
\begin{pmatrix}
B & R \\
B & \rho \\
R & \rho \\
\end{pmatrix} 
\]

The growth rate of this system is dictated by the largest eigenvalue of this matrix, which solves

\[
(\rho - \lambda)(1 - \lambda) - 2\rho = \lambda^2 - (\rho + 1)\lambda - \rho 
\]

Comparing with the previous quadratic equation we see that \( p_c = 1/\lambda \).

\[ \text{Rigorous proof of critical values.} \] Rather than take our usual approach of showing that the branching process accurately models the growth of the cluster, we will prove the result by reducing to a model with a fixed degree distribution. The reduction is based on the following picture

If we only use the connections around the ring then we get connected components that have a geometric distribution with success probability \( r \) where \( r = (1 - p) \) for bond percolation.

\[ \text{Ising model} \]

Using the FK representation (a.k.a., random cluster model) one can show

\[ \text{Theorem 5.4.6.} \] For the BC small world or the nearest neighbor NW small world, the critical value for the Ising model has \( \tanh(\beta_I) = p_c \).

Does the spin-glass transition for the Ising model on the tree have any implications for the behavior of the Ising model on the small world?
Contact process

Our version of the BC small world, which we will call BC_m, will be as follows. We start with a ring \( \mathbb{Z} \mod L \) where \( L \) is even and connect each vertex to all other vertices within distance \( m \), the npa relement so t h e npa relement so ft h e ring at random to make long range connections. We define the “big world” graph \( \mathcal{B}_m \) to consists of all vectors \( (z_1, \ldots, z_n) \) with \( n \geq 1 \) components with \( z_j \in \mathbb{Z} \) and \( z_j \neq 0 \) for \( j < n \). Neighbors in the positive half-space are defined as follows: a point \( + (z_1, \ldots, z_n) \) is adjacent to \( + (z_1, \ldots, z_n + y) \) for all \( y \) with \( 0 < |y| \leq m \) (these are the short-range neighbors of \( + (z_1, \ldots, z_n) \)). The long-range neighbor is \( + (z_1, \ldots, z_n, 0) \) if \( z_n \neq 0 \) \( + (z_1, \ldots, z_n-1) \) if \( z_n = 0, n > 1 \) \( -(0) \) if \( z_n = 0, n = 1 \).

We define two critical values:

\[
\lambda_1 = \inf \{ \lambda : \mathbb{P}(|A^0_\lambda| = 0 \text{ eventually}) < 1 \} \tag{1}
\]

\[
\lambda_2 = \inf \{ \lambda : \liminf_{t \to \infty} \mathbb{P}(0 \in A^0_\lambda) > 0 \}.
\]

\( \lambda_1 \) is the weak survival critical value

\( \lambda_2 \) is the strong survival critical value.

To obtain a lower bound \( \lambda_2 \), we use the fact that strong survival of the contact process on \( \mathcal{B}_m \) implies strong survival of the branching random walk on \( \mathcal{B}_m \). To bound \( \lambda^{brw}_{brw}(m) \), the strong survival critical value of the branching random walk, we define the “comb” of degree \( m \):

\[
\begin{array}{cccccc}
+(-3) & +(-2) & +(-1) & +(0) & +(1) & +(2) & +(3) \\
+(-3,0) & +(-2,0) & +(-1,0) & -(0) & +(1,0) & +(2,0) & +(3,0)
\end{array}
\]

Particles on the top row = type 1, bottom row = type 2, gives a two type branching process with mean matrix:

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & 0
\end{pmatrix}
\]

We will consider the discrete-time contact process. On either the small world or the big world,

- An infected individual lives for one unit of time.
- A site infects itself or its short-range neighbors with probability \( \alpha/(2m+1) \).
- It infects its long-range neighbor with probability \( \beta \).
- All infection events are independent, and each site that receives at least one infection is occupied with an infected individual at the next time.

To have a one parameter family of models we think of fixing \( r = \alpha/\beta \) and varying \( \lambda = \alpha + \beta \).

Results for multitype branching processes imply that the branching random walk on the comb survives if the largest eigenvalue of the matrix is larger than 1. Solving the quadratic equation \( (\alpha - \lambda)(-\lambda) - \beta^2 = 0 \) the largest root is

\[
\alpha + \sqrt{\alpha^2 - 4\beta^2} \over 2
\]

A little algebra shows that this is larger than 1 exactly when \( \alpha^2 - 4\beta^2 > (2 - \alpha)^2 \) or \( \alpha + \beta^2 > 1 \). This is an upper bound on the strong survival critical value of the branching process when what we need is a lower bound but it motivates the following:

**Theorem 5.5.2.** If \( \alpha + \beta^2 < 1 \) then there is no strong survival in the contact process on the big world for large \( m \).

**Conjecture.** For any range \( m \) and ratio \( r = \alpha/\beta \) we have \( \lambda_1 < \lambda_2 \).
Since the small world is a finite graph, the infection will eventually die out. However, by analogy with results for the $d$-dimensional contact process on a finite set, we expect that if the process does not become extinct quickly, it will survive for a long time.

Durrett and Liu (1988) showed that the supercritical contact process on $[0, L]$ survives for an amount of time of order $\exp(cL)$ starting from all ones.

Mountford (1999) showed that the supercritical contact process on $[0, L]^d$ survives for an amount of time of order $\exp(cL^d)$.

Consider the following modification of the small world contact process: each infected site infects its short-range neighbors with probability $\alpha/(2m+1)$ and its long-range neighbor with probability $\beta$, but now in addition, it infects a random neighbor (chosen uniformly from the grid) with probability $\gamma > 0$.

**Theorem 5.5.3.** Consider the modified small world model on $\mathbb{Z}^{d}$. If $\lambda > \lambda_1$ and we start with all infected individuals then there is a constant $c > 0$ so that the probability the infection persists to time $\exp(cL)$ tends to 1 as $L \to \infty$.

### Random Walks

Consider a Markov chain transition kernel $K(i,j)$ on $\{1, 2, \ldots, n\}$ with reversible stationary distribution $\pi$, i.e., $\pi_i K(i,j) = \pi_j K(j,i)$. To measure convergence to equilibrium we will use the relative pointwise distance

$$\Delta(t) = \max_{i,j} \left| \frac{K^t(i,j)}{\pi_j} - 1 \right|$$

which is larger than the total variation distance.

**Cheeger’s inequality.** Suppose for the moment that we have a general reversible transition probability, write $Q(x,y) = \pi(x) K(x,y)$, and define

$$h = \min_{\pi(S) \leq 1/2} \frac{\text{vol}(S \cup \partial S)}{\pi(S)}$$

where $Q(S, S^c) = \sum_{x \in S, y \in S^c} Q(x,y)$. Since this is the size of the boundary of $S$ when edge $(x,y)$ is assigned weight $Q(x,y)$, we will sometimes write this as $|\partial S|$. Let $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\Delta(t) \leq \frac{\lambda_{\text{max}}}{\pi_{\text{min}}} t$$

Let $G$ be a finite connected graph, $d(x)$ be the degree of $x$, and write $x \sim y$ if $x$ and $y$ are neighbors. We can define a transition kernel by $K(x,y) = 1/2$, $K(x,y) = 1/2d(x)$ if $x \sim y$ and $K(x,y) = 0$ otherwise. The 1/2 probability of staying put means that we don’t have to worry about periodicity or negative eigenvalues. Our $K$ can be written $(I + p)/2$ where $p$ is another transition probability, so all of the eigenvalues of $K$ are in $[0,1]$, and $\lambda_{\text{max}} = \lambda_1$.

$$\pi(x) = \frac{d(x)}{D} \text{ where } D = \sum_{y \in G} d(y),$$

defines a reversible stationary distribution since $\pi(x) K(x,y) = 1/2D = \pi(y) K(y,x)$. Letting $e(S, S^c)$ is the number of edges between $S$ and $S^c$, and vol($S$) be the sum of the degrees in $S$, we have

$$h = \frac{1}{2} \min_{\pi(S) \leq 1/2} \frac{e(S, S^c)}{\text{vol}(S)}$$
Fast mixing on random graphs

**Theorem 6.3.2.** Gkantis, Mihail, and Saberi (2003). Consider a random graph with a fixed degree distribution in which the minimum degree is \( r \geq 3 \). There is a constant \( \alpha_0 > 0 \) so that \( h \geq \alpha_0 \).

Bollobás (1988) proved this for random regular graphs. By comparing with the random 3-regular case one can show that that \( h \geq \alpha_1 \) in this case as well, hence:

**Theorem 6.3.4.** The random walk on the BC small world mixes in time \( O(\log n) \).

Also true for the Barabási-Albert model \( G_{n,d} \) in which each added vertex has \( d \geq 2 \) connections. False for \( d = 1 \) due to dangling ends of length \( O(\log n) \) in the tree. (Section 6.4.)

---

NW small world

**Theorem 6.6.1.** The random walk on the NW small world mixes in time at least \( O(\log^2 n) \) and at most \( O(\log^3 n) \).

Conjecture. Lower bound is correct answer.

**Proof.** The conductance \( \geq C/\log n \), a bound which cannot be improved since we have paths of length \( O(\log n) \) in which every vertex has degree 2. The conductance bound implies a spectral gap \( \geq C/\log^2 n \) and gives convergence time \( \log^3 n \).

---

Only degrees 2 and 3.

We consider a model \( G_{23} \) that is closely related to a NW small world in which a fraction \( p \) of the sites in the ring have a long range neighbor, and to the fixed degree distribution graphs with \( p_3 = p \) and \( p_2 = 1 - p \) but is easier to study. We start with a random 3-regular graph \( H \) with \( pn \) vertices, and produce a new graph \( G \) by replacing each edge by a path with a geometric number of edges with success probability \( r \), i.e., with probability \( (1 - r)^{-1}r \) we have \( j \) edges. The number of vertices of degree 2 in one of these paths has mean \((1/r) - 1\) so if we pick \( r \) so that \( 3p((1/r) - 1) = 1 - p \), we asymptotically have the desired degree distribution.

**Theorem 6.7.1.** The mixing time of the lazy random walk on \( G_{23} \) is \( O(\log^2 n) \).

---

Hitting times for random walks.

Consider the random walk on the BC small world. Pick two starting points \( x_1 \) and \( x_2 \) at random according to the stationary distribution \( \pi \), which in this case is uniform, and define independent continuous time random walks \( X_1^t \) and \( X_2^t \) that jump at rate one and have \( X_1^0 = x_1 \) and \( X_2^0 = x_2 \). Let \( A = \{ (x, x) \} \) and \( T_A = \inf \{ t \geq 0 : X_1^t = X_2^t \} \) be the first hitting time of \( A \) by \( (X_1^t, X_2^t) \).

**Theorem 6.8.1.** If the mixing time \( t_n = o(n), n\pi(A) \to b \), and \( E_\pi(T_A) \sim cn \), then under \( P_\pi \), \( T_A/n \) converges weakly to an exponential with mean \( c \).

**Proof.** The positions randomize in a time \( o(n) \), so the limit of \( T_A/n \) has the lack of memory property.

---

We now need to show \( E_\pi(T_A) \sim cn \). We use a version of Aldous’ Poisson clumping heuristic. Consider the discrete time version \( X_n \) of the two particle chain in which at each step we pick a particle at random and let it jump. Writing \( P_A \) for \( P_\pi(|X_0 \in A) \), a theorem of Kac implies that

\[ E_A(T_A) = 1/\pi(A) \]

Starting from the diagonal, the two particles may hit in a time that is \( O(t_n) \). The expected value on this event makes a contribution that is \( o(n) \) to the expected value. When the two particles don’t hit in \( O(t_n) \) the chain is close to equilibrium, so

\[ 1/\pi(A) \approx P_A(T_A \gg t_n)E_\pi(T_A) \]

and we have

\[ E_\pi(T_A) = \frac{1}{\pi(A)} \cdot \frac{1}{P_A(T_A \gg t_n)} \]
Consensus time. On a finite set the voter model will eventually reach an absorbing state in which all voters have the same opinion. Cox (1989) studied the voter model on a finite torus \((Z \mod N)^d\) and showed:

**Theorem 6.9.2.** Let \(\xi^p\) denote the voter model starting from product measure with density \(p \in (0, 1)\). The time to reach consensus \(\tau_N\) satisfies

\[
\tau_N = O(s_N) \quad \text{where} \quad s_N = \begin{cases} 
N^2 & d = 1 \\
N^3 \log N & d = 2 \\
N^d & d \geq 3
\end{cases}
\]

and \(E\tau_N \sim c_d[-p \log p - (1 - p) \log(1 - p)]s_N\), where \(c_d\) is a constant that depends on the dimension. In \(d \geq 3\) the finite system looks like the stationary distribution for the infinite system at times that are large but \(o(s_N)\).

It is natural to conjecture that the consensus time will be asymptotically \(c_G n\) where \(c_G\) is a constant that depends on the random graph. However, in order to prove this we would have to understand the behavior of the coalescing random walk starting from all sites occupied. The next result is a first step in that direction. For simplicity, we consider only the easiest model.

**Theorem 6.9.4.** Consider the BC small world and sample \(m\) points at random. The number of particles in the coalescing random walk at time \(n\) converges to Kingman’s coalescent in which transitions from \(k\) to \(k - 1\) occur at rate \((\binom{k}{2})\).

Let \(N_k(t)\) be the expected number of components of size \(k\) at time \(t\).

\[
N_k(t+1) = N_k(t) + 1 - 2\delta \frac{N_k(t)}{t}
\]

**Theorem.** As \(t \to \infty\), \(N_k(t)/t \to a_k\) where \(a_1 = 1/(1 + 2\delta)\) and

\[
a_k = \frac{\delta}{1 + 2\delta k} \sum_{j=1}^{k-1} ja_j \cdot (k - j)a_{k-j} \quad \text{for} \ k \geq 2
\]

Let \(h(x) = \sum_{k=1}^{\infty} x^k a_k\) and \(g(x) = \sum_{k=1}^{\infty} x^k k a_k\).

\[
h(x) + 2\delta g(x) = x + \delta g^2(x)
\]

Since \(h' = g(x)/x\)

\[
g'(x) = \frac{1}{2\delta x} \cdot \frac{x - g(x)}{1 - g(x)}
\]

(i) If \(g(1) < 1\) then \(\sum_{k=1}^{\infty} kb_k = g'(1) = 1/2\delta\).

(ii) If \(g(1) = 1\) then \(g'(1) = (1 - \sqrt{1 - 8\delta})/4\delta\).

Corollary. \(\delta_c = 1/8\).
A Simpler Model

Suppose a Poisson mean $\delta$ number of vertices are added at each step. If we let $A_{i,j,k}$ be the event no $(i,j)$ edge is added at time $k$ then

$$P(\cap_{k=j}^{n}A_{i,j,k}) = \prod_{k=j}^{n} \exp\left(-\frac{2\delta}{k(k-1)}\right)$$

$$= \exp\left(-2\delta \left(\frac{1}{j} - \frac{1}{n}\right)\right) \quad \#1$$

The last formula is not simple, so we will also consider two approximations

$$\approx 1 - 2\delta \left(\frac{1}{j} - \frac{1}{n}\right) \quad \#2$$

$$\approx 1 - \frac{2\delta}{j} \quad \#3$$

Proof of $\delta_c \geq 1/8$

Model #3 is biggest. Compare with multitype branching process.

$$m_{i,j,k} = 2\delta / (j \lor k)$$

Following Shepp (1989) we note that

$$\sum_{1 \leq j \neq i}^{4} \frac{1}{j^{1/2}} \leq \frac{8}{3^{1/2}}$$

This implies $\sum_{1}^{j} m_{i,j} j^{-1/2} \leq 8\delta$, and

$$\frac{1}{n} \sum_{i=1}^{n} E|C_i| \leq \frac{1}{n}$$

Subcritical estimates

Consider model #3 in which an edge from $x$ to $y$ is open with probability

$$h(x,y) = c / (x \lor y)$$

Let $V_{i,j}$ be the expected number of self-avoiding paths from $i$ to $j$ in the random graph.

Lemma 7.3.1. Suppose $c < 1/4$ and let $r = \sqrt{1 - 4\delta}/2$. If $1 \leq i < j$

$$P(i \to j) \leq EV_{i,j} \leq \frac{c}{2r^{1/2}(j^{1/2} + r)}$$

Proof of Lemma 7.3.1.

$$h(x,y) = c / (x \lor y)$$

$$EV_{i,j} \leq \sum_{m=0}^{\infty} \int_{0}^{1} \cdots \int_{0}^{1} dx_{1} \cdots \int_{0}^{1} dx_{m} h(i,x_{1}) h(x_{1},x_{2}) \cdots h(x_{m-1},x_{m}) h(x_{m},j)$$

Changing variables $x_{i} = e^{y_{i}}$, $dx_{i} = e^{y_{i}} dy_{i}$ and introducing

$$p(x,y) = \frac{1}{2} e^{-|x-y|/2}$$

we have

$$EV_{i,j} \leq \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} (4c)^{n} p^{n}(\log i, \log j)$$

The bilateral exponential random walk and the Cauchy are a dual pair of ch.f.

$$\sum_{n=1}^{\infty} (4c)^{n} p^{n}(0, x) = \frac{c}{\sqrt{1 - 4c}} e^{-c|x|/\sqrt{1 - 4c}/2}$$

$$r = \sqrt{1 - 4\delta}/2, \ x = \log j - \log i$$

Proof. see Theorem 7.3.2.
Kosterlitz-Thouless transition

To investigate the size of the giant component, CHKNS integrated the differential equation for the generating function $g$ near $\delta = 1/8$. Letting $S(\delta) = 1 - g(1)$ the fraction of vertices in the infinite component they plotted $\log(-\log S)$ vs $\log(\delta - 1/8)$ and concluded that

$$S(\delta) \sim \exp(-\alpha(\delta - 1/8)^{-\beta})$$

where $\alpha = 1.132 \pm 0.008$ and $\beta = 0.499 \pm 0.001$. Based on this they conjectured that $\beta = 1/2$.

Inspired by CHKNS’ conjecture Dorogovstev, Mendes, and Samukhin (2001) computed that as $\delta \downarrow 1/8$,

$$S \equiv 1 - g(1) \approx c \exp(-\pi/\sqrt{8\delta - 1})$$

To derive their formula DMS change variables $u(\xi) = 1 - g(1 - \xi)$ in the differential equation for $g$ to get

$$u'(\xi) = -\frac{1}{2\delta(1 - \xi)} \cdot \frac{u(\xi) - \xi}{u(\xi)}$$

Dropping the $1 - \xi$ they solve the equation

$$-\frac{1}{\sqrt{8\delta - 1}} \arctan \left( \frac{4\delta u(\xi)/\xi - 1}{\sqrt{8\delta - 1}} \right) - \ln \sqrt{\xi^2 - u(\xi)\xi + 2\delta u^2(\xi)} = C$$

and analyze the solution to get the result.

Proof of the upper bound

Let $G_n(c)$ be the random graph with $p_{ij} = c/(i \vee j)$.

**Theorem 7.4.1.** If $\eta > 0$ then for small $\epsilon$ the expected size of the largest component for $c = 1/4 + \epsilon$ is $\leq \exp(-(1-\eta)/2\sqrt{\epsilon})$.

The first step is to write the random graph $G_n(1/4 + \epsilon)$ as an edge disjoint sum of $G_1 = G_n(1/4 - \epsilon)$ and $G_2 = G_n(2\epsilon)$. To do this, we flip a coin for each edge with probability $(1/4 - \epsilon)/(1/4 + \epsilon)$ of heads and $2\epsilon/(1/4 + \epsilon)$ of tails.

Our task now is to estimate the probability a late vertex is connected to an early vertex. The constant from Lemma 7.3.1 is

$$r = \sqrt{1 - 4\epsilon}/2 = \sqrt{\epsilon} = \delta$$

and

$$r = \sqrt{1/4 - 2\epsilon} = \gamma$$

if $c = 1/4 - \epsilon$

Letting $N_1(i,j)$ be the expected number of self-avoiding paths from $i$ to $j$ in graph $G_i$ $i < j$

$$N_1(i,j) \leq (1/8 + o(1))^{i-1/2}j^{-1/2-\delta}$$

$$N_2(i,j) \leq (2 + o(1))^{i-1/2}j^{-1/2-\delta}$$

$N_1(i,j) + N_2(i,j) \leq \frac{1}{4\sqrt{\epsilon}}$
If \( s \) is even then the expected number of paths from \( a \) that end with the early vertex \( b \) is

\[
\leq 2 \sum_{\rho n < y_2, y_4, \ldots, y_{2s-2} \leq n} \prod_{i=0}^{s/2-1} \frac{(1 + o(1))\delta}{\sqrt{y_{2i+2}}^2} \\
= \frac{2}{\sqrt{ab}} \{(1 + o(1))\delta\}^{s/2} \sum_{\rho n < y_2, y_4, \ldots, y_{2s-2} \leq n} \prod_{i=1}^{s/2-1} y_{2i}^{-1} \\
= \frac{2}{\sqrt{ab}} \{(1 + o(1))\delta\}^{s/2} \left( \sum_{\rho n \leq n} z^{-1} \right)^{s/2-1} \\
\leq \frac{(2 + o(1))\delta}{\sqrt{ab}} \{(1 + o(1))\delta \log(1/\rho)\}^{s/2-1} \leq \frac{1}{\delta \eta \sqrt{ab}}
\]

Now sum over \( b \leq \rho n \) and \( a > \rho n \).