INTRODUCTION TO SYMPLECTIC MECHANICS: LECTURES I-II-III

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Contents

Introduction

These are the Lecture Notes for a short course on Hamiltonian mechanics from the symplectic point of view given by the author at the University of São Paulo during May-June 2006. It is my great pleasure to thank Professor P. Piccione for his kind invitation.

Chapter 1

Symplectic Vector Spaces

We will exclusively deal with finite-dimensional real symplectic spaces. We begin by discussing the notion of symplectic form on a vector space. Symplectic forms allow the definition of symplectic bases, which are the analogues of orthonormal bases in Euclidean geometry.

1.1 Generalities

Let E be a real vector space; its generic vector will be denoted by z. A symplectic form (or: *skew-product*) on E is a mapping $\omega : E \times E \longrightarrow \mathbb{R}$ which is

- linear in each of its components:

$$\omega(\alpha_1 z_1 + \alpha_2 z_2, z') = \alpha_1 \omega(z_1, z') + \alpha_2 \omega(z_2, z')$$
$$\omega(z, \alpha_1 z_1' + \alpha_2 z_2', z') = \alpha_1 \omega(z, z_1') + \alpha_2 \omega(z, z_2')$$

  for all $z, z', z_1, z_1', z_2, z_2'$ in E and $\alpha_1, \alpha_1', \alpha_2, \alpha_2'$ in $\mathbb{R}$;

- antisymmetric (one also says *skew-symmetric*):

$$\omega(z, z') = -\omega(z', z) \text{ for all } z, z' \in E$$

  (equivalently, in view of the bilinearity of ω: $\omega(z, z) = 0$ for all $z \in E$):

- non-degenerate:

$$\omega(z, z') = 0 \text{ for all } z \in E \text{ if and only if } z' = 0.$$

Definition 1 *A real symplectic space is a pair (E, ω) where E is a real vector space on $\mathbb{R}$ and ω a symplectic form. The dimension of (E, ω) is, by definition, the dimension of E.*

The most basic – and important – example of a finite-dimensional symplectic space is the *standard symplectic space* $(\mathbb{R}^{2n}_z, \sigma)$ where σ (the *standard symplectic form*) is defined by

$$\sigma(z, z') = \sum_{j=1}^{n} p_j x'_j - p'_j x_j \tag{1.1}$$

when $z = (x_1, ..., x_n; p_1, ..., p_n)$ and $z' = (x'_1, ..., x'_n; p'_1, ..., p'_n)$. In particular, when $n = 1$,

$$\sigma(z, z') = -\det(z, z').$$

In the general case $\sigma(z, z')$ is (up to the sign) the sum of the areas of the parallelograms spanned by the projections of z and z' on the coordinate planes x_j, p_j.

Here is a coordinate-free variant of the standard symplectic space: set $X = \mathbb{R}^n$ and define a mapping $\xi : X \oplus X^* \longrightarrow \mathbb{R}$ by

$$\xi(z, z') = \langle p, x' \rangle - \langle p', x \rangle \tag{1.2}$$

if $z = (x, p), z' = (x', p')$. That mapping is then a symplectic form on $X \oplus X^*$. Expressing z and z' in the canonical bases of X and X^* then identifies $(\mathbb{R}^{2n}_z, \sigma)$ with $(X \oplus X^*, \xi)$. While we will only deal with finite-dimensional symplectic spaces, it is easy to check that formula (1.2) easily generalizes to the infinite-dimensional case. Let in fact X be a real Hilbert space and X^* its dual. Define an antisymmetric bilinear form ξ on $X \oplus X^*$ by the formula (1.2) where $\langle \cdot, \cdot \rangle$ is again the duality bracket. Then ξ is a symplectic form on $X \oplus X^*$.

Remark 2 *Let Φ be the mapping $E \longrightarrow E^*$ which to every $z \in E$ associates the linear form Φ_z defined by*

$$\Phi_z(z') = \omega(z, z'). \tag{1.3}$$

The non-degeneracy of the symplectic form can be restated as follows:

$$\omega \text{ is non-degenerate} \Longleftrightarrow \Phi \text{ is a monomorphism } E \longrightarrow E^*.$$

We will say that two symplectic spaces (E, ω) and (E', ω') are isomorphic if there exists a vector space isomorphism $s : E \longrightarrow E'$ such that

$$\omega'(s(z), s(z')) = \omega'(z, z')$$

for all z, z' in E; two isomorphic symplectic spaces thus have same dimension. We will see below that, conversely, two finite-dimensional symplectic spaces are always isomorphic in the sense above if they have same dimension; the proof of this property requires the notion of symplectic basis, studied in next subsection

Let (E_1, ω_1) and (E_2, ω_2) be two arbitrary symplectic spaces. The mapping

$$\omega = \omega_1 \oplus \omega_2 : E_1 \oplus E_2 \longrightarrow \mathbb{R}$$

defined by

$$\omega(z_1 \oplus z_2; z_1' \oplus z_2') = \omega_1(z_1, z_1') + \omega_2(z_2, z_2') \tag{1.4}$$

for $z_1 \oplus z_2, z_1' \oplus z_2' \in E_1 \oplus E_2$ is obviously antisymmetric and bilinear. It is also non-degenerate: assume that

$$\omega(z_1 \oplus z_2; z_1' \oplus z_2') = 0 \text{ for all } z_1' \oplus z_2' \in E_1 \oplus E_2;$$

then, in particular, $\omega_1(z_1, z_1') = \omega_2(z_2, z_2') = 0$ for all (z_1', z_2') and hence $z_1 = z_2 = 0$. The pair

$$(E, \omega) = (E_1 \oplus E_2, \omega_1 \oplus \omega_2)$$

is thus a symplectic space; it is called the *direct sum* of (E_1, ω_1) and (E_2, ω_2).

Example 3 *Let $(\mathbb{R}_z^{2n}, \sigma)$ be the standard symplectic space. Then we can define on $\mathbb{R}_z^{2n} \oplus \mathbb{R}_z^{2n}$ two symplectic forms $\sigma^{\oplus}$ and $\sigma^{\ominus}$ by*

$$\sigma^{\oplus}(z_1, z_2; z_1', z_2') = \sigma(z_1, z_1') + \sigma(z_2, z_2')$$
$$\sigma^{\ominus}(z_1, z_2; z_1', z_2') = \sigma(z_1, z_1') - \sigma(z_2, z_2').$$

The corresponding symplectic spaces are denoted $(\mathbb{R}_z^{2n} \oplus \mathbb{R}_z^{2n}, \sigma^{\oplus})$ and $(\mathbb{R}_z^{2n} \oplus \mathbb{R}_z^{2n}, \sigma^{\ominus})$.

Here is an example of a nonstandard symplectic structure. Let B be an antisymmetric (real) $n \times n$ matrix: $B^T = -B$ and set

$$J_B = \begin{bmatrix} -B & I \\ -I & 0 \end{bmatrix}.$$

We have

$$J_B^2 = \begin{bmatrix} B^2 - I & -B \\ B & -I \end{bmatrix}$$

hence $J_B^2 \neq -I$ if $B \neq 0$. We can however associate to J_B the symplectic form σ_B defined by

$$\sigma_B(z, z') = \sigma(z, z') - \langle Bx, x' \rangle ; \tag{1.5}$$

this symplectic form intervenes in the study of electromagnetism (more generally in the study of any Galilean invariant Hamiltonian system). The scalar product $-\langle Bx, x' \rangle$ is therefore sometimes called the "magnetic term".

1.2 Symplectic Bases

We begin by observing that the dimension of a finite-dimensional symplectic vector is always even: choosing a scalar product $\langle \cdot, \cdot \rangle_E$ on E, there exists an endomorphism j of E such that $\omega(z, z') = \langle j(z), z' \rangle_E$ and the antisymmetry of ω is then equivalent to $j^T = -j$ where ^{T} denotes here transposition with respect to $\langle \cdot, \cdot \rangle_E$; hence

$$\det j = (-1)^{\dim E} \det j^T = (-1)^{\dim E} \det j.$$

The non-degeneracy of ω implies that $\det j \neq 0$ so that $(-1)^{\dim E} = 1$, hence $\dim E = 2n$ for some integer n, as claimed.

Definition 4 *A set* $\mathcal{B}$ *of vectors*

$$\mathcal{B} = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$$

of E *is called a "symplectic basis" of* (E, ω) *if the conditions*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad , \quad \omega(f_i, e_j) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n \tag{1.6}$$

hold (δ_{ij} *is the Kronecker index:* $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *if* $i \neq j$*).*

We leave it to the reader to check that the conditions (1.6) automatically ensure the linear independence of the vectors e_i, f_j for $1 \leq i, j \leq n$ (hence a symplectic basis is a basis in the usual sense).

Here is a basic (and obvious) example of a symplectic basis: define vectors $e_1, ..., e_n$ and $f_1, ..., f_n$ in $\mathbb{R}^{2n}_z$ by

$$e_i = (c_i, 0) \quad , \quad e_i = (0, c_i)$$

where (c_i) is the canonical basis of $\mathbb{R}^n$. (For instance, if $n = 1$, $e_1 = (1, 0)$ and $f_1 = (0, 1)$). These vectors form the canonical basis

$$\mathcal{B} = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$$

of the standard symplectic space $(\mathbb{R}^{2n}_z, \sigma)$; one immediately checks that they satisfy the conditions $\sigma(e_i, e_j) = 0$, $\sigma(f_i, f_j) = 0$, and $\sigma(f_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. This basis is called the canonical symplectic basis.

Taking for granted the existence of symplectic bases (it will be established in a moment) we can prove that all symplectic vector spaces of same finite dimension $2n$ are isomorphic: let (E, ω) and (E', ω') have symplectic bases $\{e_i, f_j; 1 \leq i, j \leq n\}$ and $\{e'_i, f'_j; 1 \leq i, j \leq n\}$ and consider the linear isomorphism $s : E \longrightarrow E'$ defined by the conditions $s(e_i) = e'_i$ and $s(f_i) = f'_i$ for $1 \leq i \leq n$. That s is symplectic is clear since we have

$$\begin{aligned}
\omega'(s(e_i), s(e_j)) &= \omega'(e'_i, e'_j) = 0 \\
\omega'(s(f_i), s(f_j)) &= \omega'(f'_i, f'_j) = 0 \\
\omega'(s(f_j), s(e_i)) &= \omega'(f'_j, e'_i) = \delta_{ij}
\end{aligned}$$

for $1 \leq i, j \leq n$.

The set of all symplectic automorphisms $(E, \omega) \longrightarrow (E, \omega)$ form a group $\mathrm{Sp}(E, \omega)$ – the symplectic group of (E, ω) – for the composition law. Indeed, the identity is obviously symplectic, and so is the compose of two symplectic transformations. If $\omega(s(z), s(z')) = \omega(z, z')$ then, replacing z and z' by $s^{-1}(z)$ and $s^{-1}(z')$, we have $\omega(z, z') = \omega(s^{-1}(z), s^{-1}(z'))$ so that s^{-1} is symplectic as well.

It turns out that all symplectic groups corresponding to symplectic spaces of same dimension are isomorphic:

Proposition 5 *Let* (E, ω) *and* (E', ω') *be two symplectic spaces of same dimension* $2n$*. The symplectic groups* $\mathrm{Sp}(E, \omega)$ *and* $\mathrm{Sp}(E', \omega')$ *are isomorphic.*

Proof. Let Φ be a symplectic isomorphism $(E,\omega) \longrightarrow (E',\omega')$ and define a mapping $f_\Phi : \mathrm{Sp}(E,\omega) \longrightarrow \mathrm{Sp}(E',\omega')$ by $f_\Phi(s) = f \circ s \circ f^{-1}$. Clearly $f_\Phi(ss') = f_\Phi(s)\Phi(s')$ hence f_Φ is a group monomorphism. The condition $f_\Phi(S) = I$ (the identity in $\mathrm{Sp}(E',\omega')$) is equivalent to $f \circ s = f$ and hence to $s = I$ (the identity in $\mathrm{Sp}(E,\omega)$); f_Φ is thus injective. It is also surjective because $s = f^{-1} \circ s' \circ f$ is a solution of the equation $f \circ s \circ f^{-1} = s'$. ∎

These results show that it is no restriction to study finite-dimensional symplectic geometry by singling out one particular symplectic space, for instance the standard symplectic space, or its variants. This will be done in next section.

Note that if $\mathcal{B}_1 = \{e_{1i}, f_{1j}; 1 \leq i,j \leq n_1\}$ and $\mathcal{B}_2 = \{e_{2k}, f_{2\ell}; 1 \leq k,\ell \leq n_2\}$ are symplectic bases of (E_1,ω_1) and (E_2,ω_2) then

$$\mathcal{B} = \{e_{1i} \oplus e_{2k}, f_{1j} \oplus f_{2\ell} : 1 \leq i,j \leq n_1 + n_2\}$$

is a symplectic basis of $(E_1 \oplus E_2, \omega_1 \oplus \omega_2)$.

1.3 Differential Interpretation of σ

A differential two-form on a vector space $\mathbb{R}^m$ is the assignment to every $x \in \mathbb{R}^m$ of a linear combination

$$\beta_x = \sum_{i<j\leq m} b_{ij}(x) dx_i \wedge dx_j$$

where the b_{ij} are (usually) chosen to be C^∞ functions, and the wedge product $dx_i \wedge dx_j$ is defined by

$$dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i$$

where $dx_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ is the projection on the i-th coordinate. Returning to $\mathbb{R}^{2n}_z$, we have

$$dp_j \wedge dx_j(z,z') = p_j x'_j - p'_j x_j$$

hence we can identify the standard symplectic form σ with the differential 2-form

$$dp \wedge dx = \sum_{j=1}^n dp_j \wedge dx_j = d(\sum_{j=1}^n p_j dx_j);$$

the differential one-form

$$pdx = \sum_{j=1}^n p_j dx_j$$

plays a fundamental role in both classical and quantum mechanics; it is sometimes called the (reduced) *action form* in physics and the *Liouville form* in mathematics[1].

[1] Some authors call it the *tautological one-form*.

Since we are in the business of differential form, let us make the following remark: the exterior derivative of $dp_j \wedge dx_j$ is

$$d(dp_j \wedge dx_j) = d(dp_j) \wedge dx_j + dp_j \wedge d(dx_j) = 0$$

so that we have

$$d\sigma = d(dp \wedge dx) = 0.$$

The standard symplectic form is thus a closed non-degenerate 2-form on $\mathbb{R}^{2n}_z$. This remark is the starting point of the generalization of the notion of symplectic form to a class of manifolds: a symplectic manifold is a pair (M, ω) where M is a differential manifold M and ω a non-degenerate closed 2-form on M. This means that every tangent plane $T_z M$ carries a symplectic form ω_z varying smoothly with $z \in M$. As a consequence, a symplectic manifold always has even dimension (we will not discuss the infinite-dimensional case).

One basic example of a symplectic manifold is the cotangent bundle $T^*\mathbb{V}^n$ of a manifold $\mathbb{V}^n$; the symplectic form is here the "canonical 2-form" on $T^*\mathbb{V}^n$, defined as follows: let $\pi : T^*\mathbb{V}^n \longrightarrow \mathbb{V}^n$ be the projection to the base and define a 1-form λ on $T^*\mathbb{V}^n$ by $\lambda_z(X) = p(\pi_*(X))$ for a tangent vector $\mathbb{V}^n$ to $T^*\mathbb{V}^n$ at $z = (z, p)$. The form λ is called the "canonical 1-form" on $T^*\mathbb{V}^n$; its exterior derivative $\omega = d\lambda$ is called the "canonical 2-form" on $T^*\mathbb{V}^n$ and one easily checks that it indeed is a symplectic form (in local coordinates $\lambda = pdx$ and $\sigma = dp \wedge dx$). The symplectic manifold $(T^*\mathbb{V}^n, \omega)$ is in a sense the most straightforward non-linear version of the standard symplectic space (to which it reduces when $\mathbb{V}^n = \mathbb{R}^n_x$ since $T^*\mathbb{R}^n_x$ is just $\mathbb{R}^n_x \times (\mathbb{R}^n_x)^* \equiv \mathbb{R}^{2n}_z$). Observe that $T^*\mathbb{V}^n$ never is a compact manifold.

A symplectic manifold is always orientable: the non-degeneracy of ω namely implies that the $2n$-form

$$\omega^{\wedge n} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}}$$

never vanishes on M and is thus a volume form on M. We will call the exterior power $\omega^{\wedge n}$ the *symplectic volume form*. When M is the standard symplectic space then the usual volume form on $\mathbb{R}^{2n}_z$

$$\mathrm{Vol}_{2n} = (dp_1 \wedge \cdots \wedge dp_n) \wedge (dx_1 \wedge \cdots \wedge dx_n)$$

is related to the symplectic volume form by

$$\mathrm{Vol}_{2n} = (-1)^{n(n-1)/2} \frac{1}{n!} \sigma^{\wedge n}. \tag{1.7}$$

1.4 Skew-Orthogonality

All vectors in a symplectic space (E, ω) are skew-orthogonal (one also says "isotropic") in view of the antisymmetry of a symplectic form: $\sigma(z, z') = 0$ for all $z \in E$. The notion of length therefore does not make sense in symplectic geometry (whereas the notion of area does). The notion "skew orthogonality" is extremely interesting in the sense that it allows the definition of subspaces of a symplectic space having special properties. We begin by defining the notion of symplectic basis, which is the equivalent of orthonormal basis in Euclidean geometry.

Let M be an arbitrary subset of a symplectic space (E, ω). The *skew-orthogonal* set to M (one also says (or *annihilator*) is by definition the set

$$M^{\omega} = \{z \in E : \omega(z, z') = 0, \ \forall z' \in M\}.$$

Notice that we always have

$$M \subset N \Longrightarrow N^{\omega} \subset M^{\omega} \ \text{ and } \ (M^{\omega})^{\omega} \subset M.$$

It is traditional to classify subsets M of a symplectic space (E, ω) as follows:

$M \subset E$ is said to be:

- *Isotropic* if $M^{\omega} \supset M : \omega(z, z') = 0$ for all $z, z' \in M$;
- *Coisotropic* (or: *involutive*) if $M^{\omega} \subset M$;
- *Lagrangian* if M is both isotropic and co-isotropic: $M^{\omega} = M$;
- *Symplectic if* $M \cap M^{\omega} = 0$.

Notice that the non-degeneracy of a symplectic form is equivalent to saying that the only vector that of a symplectic space which is skew-orthogonal to all other vectors is 0.

Following proposition describes some straightforward but useful properties of the skew-orthogonal of a linear subspace of a symplectic space:

Proposition 6 *(i) If M is a linear subspace of E, then so is M^{ω} and*

$$\dim M + \dim M^{\omega} = \dim E \ \textit{and} \ (M^{\omega})^{\omega} = M. \tag{1.8}$$

(ii) If M_1, M_2 are linear subspaces of a symplectic space (E, ω), then

$$(M_1 + M_2)^{\omega} = M_1^{\omega} \cap M_2^{\omega} \quad , \quad (M_1 \cap M_2)^{\omega} = M_1^{\omega} + M_2^{\omega}. \tag{1.9}$$

Proof. *Proof of* (i). That M^{ω} is a linear subspace of E is clear. Let $\Phi : E \longrightarrow E^*$ be the linear mapping (1.3); since the dimension of E is finite the non-degeneracy of ω implies that Φ is an isomorphism. Let $\{e_1, ..., e_k\}$ be a basis of M; we have

$$M^{\omega} = \bigcap_{j=1}^{k} \ker(\Phi(e_j))$$

so that M^{ω} is defined by k independent linear equations, hence

$$\dim M^{\omega} = \dim E - k = \dim E - \dim M$$

which proves the first formula (1.8). Applying that formula to the subspace $(M^\omega)^\omega$ we get

$$\dim(M^\omega)^\omega = \dim E - \dim M^\omega = \dim M$$

and hence $M = (M^\omega)^\omega$ since $(M^\omega)^\omega \subset M$ whether M is linear or not. *Proof of (ii).* It is sufficient to prove the first equality (1.9) since the second follows by duality, replacing M_1 by M_1^ω and M_2 by M_2^ω and using the first formula (1.8). Assume that $z \in (M_1 + M_2)^\omega$; then $\omega(z, z_1 + z_2) = 0$ for all $z_1 \in M_1, z_2 \in M_2$. In particular $\omega(z, z_1) = \omega(z, z_2) = 0$ so that we have both $z \in M_1^\omega$ and $z \in M_2^\omega$, proving that $(M_1 + M_2)^\omega \subset M_1^\omega \cap M_2^\omega$. If conversely $z \in M_1^\omega \cap M_2^\omega$ then $\omega(z, z_1) = \omega(z, z_2) = 0$ for all $z_1 \in M_1, z_2 \in M_2$ and hence $\omega(z, z') = 0$ for all $z' \in M_1 + M_2$. Thus $z \in (M_1 + M_2)^\omega$ and $M_1^\omega \cap M_2^\omega \subset (M_1 + M_2)^\omega$. ■

Let M be a linear subspace of (E, ω) such that $M \cap M^\omega = \{0\}$; in the terminology introduced above M is a "symplectic subset of E".

Exercise 7 *If $M \cap M^\omega = \{0\}$, then $(M, \omega_{|M})$ and $(M^\omega, \omega_{|M^\omega})$ are complementary symplectic spaces of (E, ω):*

$$(E, \omega) = (M \oplus M^\omega, \omega_{|M} \oplus \omega_{|M^\omega}). \tag{1.10}$$

[Hint: M^ω is a linear subspace of E so it suffices to check that the restriction $\omega_{|M}$ is non-degenerate].

1.5 The Symplectic Gram–Schmidt theorem

The following result is a symplectic version of the Gram–Schmidt orthonormalization process of Euclidean geometry. Because of its importance and its many applications we give it the status of a theorem:

Theorem 8 *Let A and B be two (possibly empty) subsets of $\{1, ..., n\}$. For any two subsets $\mathcal{E} = \{e_i : i \in A\}$, $\mathcal{F} = \{f_j : j \in B\}$ of the symplectic space (E, ω) ($\dim E = 2n$), such that the elements of $\mathcal{E}$ and $\mathcal{F}$ satisfy the relations*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \text{ , } \omega(f_i, e_j) = \delta_{ij} \text{ for } (i, j) \in A \times B \tag{1.11}$$

there exists a symplectic basis $\mathcal{B}$ of (E, ω) containing $\mathcal{E} \cup \mathcal{F}$.

Proof. We will distinguish three cases. *(i) The case $A = B = \emptyset$.* Choose a vector $e_1 \neq 0$ in E and let f_1 be another vector with $\omega(f_1, e_1) \neq 0$ (the existence of f_1 follows from the non-degeneracy of ω). These vectors are linearly independent, which proves the theorem in the considered case when $n = 1$. Suppose $n > 1$ and let M be the subspace of E spanned by $\{e_1, f_1\}$ and set $E_1 = M^\omega$; in view of the first formula (1.8) we have $\dim M + \dim E_1 = 2n$. Since $\omega(f_1, e_1) \neq 0$ we have $E_1 \cap M = 0$ hence $E = E_1 \oplus M$, and the restriction ω_1 of ω to E_1 is non-degenerate (because if $z_1 \in E_1$ is such that $\omega_1(z_1, z) = 0$ for all $z \in E_1$ then $z_1 \in E_1^\omega = M$ and hence $z_1 = 0$); (E_1, ω_1) is thus a symplectic space of dimension $2(n-1)$. Repeating the construction above $n-1$ times we obtain a strictly decreasing sequence

$$(E,\omega) \supset (E_1,\omega_1) \supset \cdots \supset (E_{n-1},\omega_{n-1})$$

of symplectic spaces with $\dim E_k = 2(n-k)$ and also an increasing sequence

$$\{e_1,f_1\} \subset \{e_1,e_2;f_1;f_2\} \subset \cdots \subset \{e_1,...,e_n;f_1,...,f_n\}$$

of sets of linearly independent vectors in E, each set satisfying the relations (1.11). *(ii) The case* $A = B \neq \emptyset$. We may assume without restricting the argument that $A = B = \{1,2,...,k\}$. Let M be the subspace spanned by $\{e_1,...,e_k;f_1,...,f_k\}$. As in the first case we find that $E = M \oplus M^\omega$ and that the restrictions ω_M and ω_{M^ω} of ω to M and M^ω, respectively, are symplectic forms. Let $\{e_{k+1},...,e_n;f_{k+1},...,f_n\}$ be a symplectic basis of M^ω; then

$$\mathcal{B} = \{e_1,...,e_n;f_1,...,f_n\}$$

is a symplectic basis of E. *(iii) The case* $B\backslash A \neq \emptyset$ *(or* $B\backslash A \neq \emptyset$*)*. Suppose for instance $k \in B\backslash A$ and choose $e_k \in E$ such that $\omega(e_i,e_k) = 0$ for $i \in A$ and $\omega(f_j,e_k) = \delta_{jk}$ for $j \in B$. Then $\mathcal{E} \cup \mathcal{F} \cup \{e_k\}$ is a system of linearly independent vectors: the equality

$$\lambda_k e_k + \sum_{i\in A} \lambda_i e_i + \sum_{j\in B} \mu_j e_j = 0$$

implies that we have

$$\lambda_k \omega(f_k,e_k) + \sum_{i\in A} \lambda_i \omega(f_k,e_i) + \sum_{j\in B} \mu_j \omega(f_k,e_j) = \lambda_k = 0$$

and hence also $\lambda_i = \mu_j = 0$. Repeating this procedure as many times as necessary, we are led back to the case $A = B \neq \emptyset$. ■

Remark 9 *The proof above shows that we can construct symplectic subspaces of* (E,ω) *having any given even dimension* $2m < \dim E$ *containing any pair of vectors* e,f *such that* $\omega(f,e) = 1$*. In fact,* $M = \operatorname{Span}\{e,f\}$ *is a two-dimensional symplectic subspace ("symplectic plane") of* (E,ω)*. In the standard symplectic space* $(\mathbb{R}_z^{2n},\sigma)$ *every plane* x_j,p_j *of "conjugate coordinates" is a symplectic plane.*

It follows from the theorem above that if (E,ω) and (E',ω') are two symplectic spaces with same dimension $2n$ there always exists a symplectic isomorphism $\Phi : (E,\omega) \longrightarrow (E',\omega')$. Let in fact

$$\mathcal{B} = \{e_1,...,e_n\} \cup \{f_1,...,f_n\} \ \ , \mathcal{B}' = \{e_1',...,e_n'\} \cup \{f_1',...,f_n'\}$$

be symplectic bases of (E,ω) and (E',ω'), respectively. The linear mapping $\Phi : E \longrightarrow E'$ defined by $\Phi(e_j) = e_j'$ and $\Phi(f_j) = f_j'$ $(1 \leq j \leq n)$ is a symplectic isomorphism.

This result, together with the fact that any skew-product takes the standard form in a symplectic basis shows why it is no restriction to develop symplectic geometry from the standard symplectic space: all symplectic spaces of a given dimension are just isomorphic copies of $(\mathbb{R}^{2n}_z, \sigma)$.

We end this subsection by briefly discussing the restrictions of symplectic transformations to subspaces:

Proposition 10 *Let $(F, \omega_{|F})$ and $(F', \omega_{|F'})$ be two symplectic subspaces of (E, ω). If $\dim F = \dim F'$ there exists a symplectic automorphism of (E, ω) whose restriction $\varphi_{|F}$ is a symplectic isomorphism $\varphi_{|F} : (F, \omega_{|F}) \longrightarrow (F', \omega_{|F'})$.*

Proof. Assume that the common dimension of F and F' is $2k$ and let

$$\mathcal{B}_{(k)} = \{e_1, ..., e_k\} \cup \{f_1, ..., f_k\}$$
$$\mathcal{B}'_{(k)} = \{e'_1, ..., e'_k\} \cup \{f'_1, ..., f'_k\}$$

be symplectic bases of F and F', respectively. In view of Theorem 8 we may complete $\mathcal{B}_{(k)}$ and $\mathcal{B}_{(k')}$ into full symplectic bases $\mathcal{B}$ and $\mathcal{B}'$ of (E, ω). Define a symplectic automorphism Φ of E by requiring that $\Phi(e_i) = e'_i$ and $\Phi(f_j) = f'_j$. The restriction $\varphi = \Phi_{|F}$ is a symplectic isomorphism $F \longrightarrow F'$. ■

Let us now work in the standard symplectic space $(\mathbb{R}^{2n}_z, \sigma)$; everything can however be generalized to vector spaces with a symplectic form associated to a complex structure. We leave this generalization to the reader as an exercise.

Definition 11 *A basis of $(\mathbb{R}^{2n}_z, \sigma)$ which is both symplectic and orthogonal (for the scalar product $\langle z, z' \rangle = \sigma(Jz, z')$) is called an orthosymplectic basis.*

The canonical basis is trivially an orthosymplectic basis. It is easy to construct orthosymplectic bases starting from an arbitrary set of vectors $\{e'_1, ..., e'_n\}$ satisfying the conditions $\sigma(e'_i, e'_j) = 0$: let ℓ be the vector space (Lagrangian plane) spanned by these vectors; using the classical Gram–Schmidt orthonormalization process we can construct an orthonormal basis $\{e_1, ..., e_n\}$ of ℓ. Define now $f_1 = -Je_1, ..., f_n = -Je_n$. The vectors f_i are orthogonal to the vectors e_j and are mutually orthogonal because J is a rotation; in addition

$$\sigma(f_i, f_j) = \sigma(e_i, e_j) = 0 \quad , \quad \sigma(f_i, e_j) = \langle e_i, e_j \rangle = \delta_{ij}$$

hence the basis

$$\mathcal{B} = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$$

is both orthogonal and symplectic.

We leave it to the reader as an exercise to generalize this construction to any set

$$\{e_1, ..., e_k\} \cup \{f_1, ..., f_m\}$$

of normed pairwise orthogonal vectors satisfying in addition the symplectic conditions $\sigma(f_i, f_j) = \sigma(e_i, e_j) = 0$ and $\sigma(f_i, e_j) = \delta_{ij}$.

Chapter 2

The Symplectic Group

In this second Chapter we study in some detail the symplectic group of a symplectic space (E, ω), with a special emphasis on the standard symplectic group $\mathrm{Sp}(n)$, corresponding to the case $(E, \omega) = (\mathbb{R}_z^{2n}, \sigma)$.

2.1 The Standard Symplectic Group

Let us begin by working in the standard symplectic space $(\mathbb{R}_z^{2n}, \sigma)$.

Definition 12 *The group of all automorphisms s of $(\mathbb{R}_z^{2n}, \sigma)$ such that*

$$\sigma(sz, sz') = \sigma(z, z')$$

for all $z, z' \in \mathbb{R}_z^{2n}$ is is denoted by $\mathrm{Sp}(n)$ and called the "standard symplectic group" (one also frequently finds the notations $\mathrm{Sp}(2n)$ or $\mathrm{Sp}(2n, \mathbb{R})$ in the literature).

It follows from Proposition 5 that $\mathrm{Sp}(n)$ is isomorphic to the symplectic group $\mathrm{Sp}(E, \omega)$ of any $2n$-dimensional symplectic space.

The notion of linear symplectic transformation can be extended to diffeomorphisms:

Definition 13 *Let (E, ω), (E', ω') be two symplectic vector spaces. A diffeomorphism $f : (E, \omega) \longrightarrow (E', \omega')$ is called a "symplectomorphism[1]" if the differential $d_z f$ is a a linear symplectic mapping $E \longrightarrow E'$ for every $z \in E$. [In the physical literature one often says "canonical transformation" in place of "symplectomorphism"].*

It follows from the chain rule that the composition $g \circ f$ of two symplectomorphisms $f : (E, \omega) \longrightarrow (E', \omega')$ and $g : (E', \omega') \longrightarrow (E'', \omega'')$ is a symplectomorphism $(E, \omega) \longrightarrow (E'', \omega'')$. When

$$(E, \omega) = (E', \omega') = (\mathbb{R}_z^{2n}, \sigma)$$

[1] The word was reputedly coined by J.-M. Souriau.

a diffeomorphism f of $(\mathbb{R}_z^{2n}, \sigma)$ is a symplectomorphism if and only if its Jacobian matrix (calculated in any symplectic basis) is in $\mathrm{Sp}(n)$. Summarizing:

$$f \textit{ is a symplectomorphism of } (\mathbb{R}_z^{2n}, \sigma)$$
$$\Longleftrightarrow$$
$$Df(z) \in \mathrm{Sp}(n) \textit{ for every } \ z \in (\mathbb{R}_z^{2n}, \sigma).$$

It follows directly from the chain rule $D(g \circ f)(z) = Dg(f(z)Df(z)$ that the symplectomorphisms of the standard symplectic space $(\mathbb{R}_z^{2n}, \sigma)$ form a group. That group is denoted by $\mathrm{Symp}(n)$.

Definition 14 *Let (E, ω) be a symplectic space. The group of all linear symplectomorphisms of (E, ω) is denoted by $\mathrm{Sp}(E, \omega)$ and called the "symplectic group of (E, ω)".*

Following exercise produces infinitely many examples of linear symplectomorphisms:

The notion of symplectomorphism extends in the obvious way to symplectic manifold: if (M, ω) and (M', ω') are two such manifolds, then a diffeomorphism $f : M \longrightarrow M'$ is called a symplectomorphism if it preserves the symplectic structures on M and M', that is if $f^*\omega' = \omega$ where $f^*\omega'$ (the "pull-back of ω' by f) is defined by

$$f^*\omega'(z_0)(Z, Z') = \omega'(f(z_0))((d_{z_0}f)Z, (d_{z_0}f)Z')$$

for every $z_0 \in M$ and $Z, Z' \in T_{z_0}M$. If f and g are symplectomorphisms $(M, \omega) \longrightarrow (M', \omega')$ and $(M', \omega') \longrightarrow (M'', \omega'')$ then $g \circ f$ is a symplectomorphism $(M, \omega) \longrightarrow (M'', \omega'')$.

The symplectomorphisms $(M, \omega) \longrightarrow (M, \omega)$ obviously form a group, denoted by $\mathrm{Symp}(M, \omega)$.

2.2 Symplectic Matrices and Eigenvalues

For practical purposes it is often advantageous to work in coordinates and to represent the elements of $\mathrm{Sp}(n)$ by matrices.

Recall that definition (1.1) of the standard symplectic form can be rewritten in matrix form as

$$\sigma(z, z') = (z')^T Jz = \langle Jz, z' \rangle \tag{2.1}$$

where J is the standard symplectic matrix:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{2.2}$$

Notice that $J^T = -J$ and $J^2 = -I$.

Choose a symplectic basis in $(\mathbb{R}_z^{2n}, \sigma)$ we will identify a linear mapping $s : \mathbb{R}_z^{2n} \longrightarrow \mathbb{R}_z^{2n}$ with its matrix S in that basis. In view of (2.1) we have

$$S \in \mathrm{Sp}(n) \Longleftrightarrow S^T JS = J$$

where S^T is the transpose of S. Since

$$\det S^T JS = \det S^2 \det J = \det J$$

it follows that $\det S$ can, a priori, take any of the two values ± 1. It turns out, however, that

$$S \in \mathrm{Sp}(n) \Longrightarrow \det S = 1.$$

There are many ways of showing this; none of them is really totally trivial. Here is an algebraic proof making use of the notion of Pfaffian (we will give an alternative proof later on). Recall that to every antisymmetric matrix A one associates a polynomial $\mathrm{Pf}(A)$ ("the Pfaffian of A") in the entries of A, it has the following properties:

$$\mathrm{Pf}(S^T AS) = (\det S)\,\mathrm{Pf}(A) \quad , \quad \mathrm{Pf}(J) = 1.$$

Choose now $A = J$ and $S \in \mathrm{Sp}(n)$. Since $S^T JS = J$ we have

$$\mathrm{Pf}(S^T JS) = \det S = 1$$

which was to be proven.

Remark 15 *The group* $\mathrm{Sp}(n)$ *is stable under transposition: the condition* $S \in \mathrm{Sp}(n)$ *is equivalent to* $S^T JS = J$*; since* S^{-1} *also is in* $\mathrm{Sp}(n)$ *we have* $(S^{-1})^T JS^{-1} = J$*; taking the inverses of both sides of this equality we get* $SJ^{-1}S^T = J^{-1}$ *that is* $SJS^T = J$*, so that* $S^T \in \mathrm{Sp}(n)$*. It follows that we have the equivalences*

$$S \in \mathrm{Sp}(n) \Longleftrightarrow S^T JS = J \Longleftrightarrow SJS^T = J. \tag{2.3}$$

A symplectic basis of $(\mathbb{R}_z^{2n}, \sigma)$ being chosen, we can always write $S \in \mathrm{Sp}(n)$ in block-matrix form

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.4}$$

where the entries A, B, C, D are $n \times n$ matrices. The conditions (2.3) are then easily seen, by a direct calculation, equivalent to the two following sets of equivalent conditions[2]:

$$A^T C,\ B^T D \ \text{ symmetric, and } A^T D - C^T B = I \tag{2.5}$$

$$AB^T,\ CD^T \ \text{ symmetric, and } AD^T - BC^T = I. \tag{2.6}$$

It follows from the second of these sets of conditions that the inverse of S is

$$S^{-1} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}. \tag{2.7}$$

[2] These conditions are sometimes called the "Luneburg relations" in theoretical optics.

Example 16 *Here are three classes of symplectic matrices which are useful: if P and L are, respectively, a symmetric and an invertible $n \times n$ matrix we set*

$$V_P = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \ , \ U_P = \begin{bmatrix} -P & I \\ -I & 0 \end{bmatrix} \ , \ M_L = \begin{bmatrix} L^{-1} & 0 \\ 0 & L^T \end{bmatrix}. \tag{2.8}$$

The matrices V_P are sometimes called "symplectic shears".

It turns out – as we shall prove later on – that both sets

$$\mathcal{G} = \{J\} \cup \{V_P : P \in \mathrm{Sym}(n, \mathbb{R})\} \cup \{M_L : L \in \mathrm{GL}(n, \mathbb{R})\}$$

and

$$\mathcal{G}' = \{J\} \cup \{U_P : P \in \mathrm{Sym}(n, \mathbb{R})\} \cup \{M_L : L \in \mathrm{GL}(n, \mathbb{R})\}$$

generate the symplectic group $\mathrm{Sp}(n)$.

Example 17 *Let X and Y be two symmetric $n \times n$ matrices, X invertible. Then*

$$S = \begin{bmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{bmatrix}$$

is a symplectic matrix.

We can also form direct sums of symplectic groups. Consider for instance $(\mathbb{R}^{2n_1}, \sigma_1)$ and $(\mathbb{R}^{2n_2}, \sigma_2)$, the standard symplectic spaces of dimension $2n_1$ and $2n_2$; let $\mathrm{Sp}(n_1)$ and $\mathrm{Sp}(n_2)$ be the respective symplectic groups. The direct sum $\mathrm{Sp}(n_1) \oplus \mathrm{Sp}(n_2)$ is the group of automorphisms of

$$(\mathbb{R}^{2n}_z, \sigma) = (\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, \sigma_1 \oplus \sigma_2)$$

defined, for $z_1 \in \mathbb{R}^{2n_1}$ and $z_2 \in \mathbb{R}^{2n_2}$, by

$$(s_1 \oplus s_2)(z_1 \oplus z_2) = s_1 z_1 \oplus s_2 z_2.$$

It is evidently a subgroup of $\mathrm{Sp}(n)$:

$$\mathrm{Sp}(n_1) \oplus \mathrm{Sp}(n_2) \subset \mathrm{Sp}(n)$$

which can be expressed in terms of block-matrices as follows: let

$$S_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

be elements of $\mathrm{Sp}(n_1)$ and $\mathrm{Sp}(n_2)$, respectively. Then

$$S_1 \oplus S_2 = \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix} \in \mathrm{Sp}(n_1 + n_2). \tag{2.9}$$

The mapping $(S_1, S_2) \longmapsto S_1 \oplus S_2$ thus defined is a group monomorphism

$$\mathrm{Sp}(n_1) \oplus \mathrm{Sp}(n_2) \longrightarrow \mathrm{Sp}(n).$$

The elements of $\mathrm{Sp}(n)$ are linear isomorphisms; we will sometimes also consider affine symplectic isomorphisms. Let $S \in \mathrm{Sp}(n)$ and denote by $T(z_0)$ the translation $z \longmapsto z + z_0$ in $\mathbb{R}^{2n}_z$. The composed mappings

$$T(z_0)S = ST(S^{-1}z_0) \text{ and } ST(z_0) = T(Sz_0)S$$

are both symplectomorphisms, as is easily seen by calculating their Jacobians. These transformations form a group.

Definition 18 *The semi-direct product* $\mathrm{Sp}(n) \ltimes_s T(2n)$ *of the symplectic group and the group of translations in* $\mathbb{R}^{2n}_z$ *is called the affine (or: inhomogeneous) symplectic group, and is denoted by* $\mathrm{ISp}(n)$.

For practical calculations it is often useful to identify $\mathrm{ISp}(n)$ with a matrix group:

Exercise 19 *Show that the group of all matrices*

$$[S, z_0] \equiv \begin{bmatrix} S & z_0 \\ 0_{1\times 2n} & 1 \end{bmatrix}$$

is isomorphic to $\mathrm{ISp}(n)$ *(here* $0_{1\times 2n}$ *is the* $2n$*-column matrix with all entries equal to zero).*

Let us now briefly discuss the eigenvalues of a symplectic matrix. It has been known for a long time that the eigenvalues of symplectic matrices play an fundamental role in the study of Hamiltonian periodic orbits; this is because the stability of these orbits depend in a crucial way on the structure of the associated linearized system. It turns out that these eigenvalues also play an essential role in the understanding of symplectic squeezing theorems, which we study later in this book.

Let us first prove the following result:

Proposition 20 *Let* $S \in \mathrm{Sp}(n)$.

(i) If λ *is an eigenvalue of* S *then so are* $\bar{\lambda}$ *and* $1/\lambda$ *(and hence also* $1/\bar{\lambda}$*);*

(ii) if the eigenvalue λ *of* S *has multiplicity* k *then so has* $1/\lambda$.

(iii) S *and* S^{-1} *have the same eigenvalues.*

Proof. *Proof of (i).* We are going to show that the characteristic polynomial $P_S(\lambda) = \det(S - \lambda I)$ of S satisfies the reflexivity relation

$$P_S(\lambda) = \lambda^{2n} P_S(1/\lambda); \tag{2.10}$$

(i) will follow since for real matrices eigenvalues appear in conjugate pairs. Since $S^T JS = J$ we have $S = -J(S^T)^{-1}J$ and hence

$$\begin{aligned} P_S(\lambda) &= \det(-J(S^T)^{-1}J - \lambda I) \\ &= \det(-(S^T)^{-1}J + \lambda I) \\ &= \det(-J + \lambda S) \\ &= \lambda^{2n}\det(S - \lambda^{-1}I) \end{aligned}$$

which is precisely (2.10). *Proof of (ii).* Let $P_S^{(j)}$ be the j-th derivative of the polynomial P_S. If λ_0 has multiplicity k; then $P_S^{(j)}(\lambda_0) = 0$ for $0 \leq j \leq k-1$ and $P_S^{(k)}(\lambda) \neq 0$. In view of (2.10) we also have $P_S^{(j)}(1/\lambda) = 0$ for $0 \leq j \leq k-1$ and $P_S^{(k)}(1/\lambda) \neq 0$. Property *(iii)* immediately follows from *(ii)*. ■

Notice that as an immediate consequence of this result is that if ± 1 is an eigenvalue of $S \in \mathrm{Sp}(n)$ then its multiplicity is necessarily even.

We will see in next subsection (Proposition 22) that any positive-definite symmetric symplectic matrix can be diagonalized using an orthogonal transformation which is at the same time symplectic.

2.3 The Unitary Group $\mathrm{U}(n)$

The complex structure associated to the standard symplectic matrix J is very simple: it is defined by

$$(\alpha + i\beta)z = \alpha + \beta Jz$$

and corresponds to the trivial identification $z = (x, p) \equiv x + ip$. The unitary group $\mathrm{U}(n, \mathbb{C})$ acts in a natural way on $(\mathbb{R}_z^{2n}, \sigma)$:

Proposition 21 *The monomorphism $\mu : M(n, \mathbb{C}) \longrightarrow M(2n, \mathbb{R})$ defined by $u = A + iB \longmapsto \mu(u)$ with*

$$\mu(u) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

(A and B real) identifies the unitary group $\mathrm{U}(n, \mathbb{C})$ with the subgroup

$$\mathrm{U}(n) = \mathrm{Sp}(n) \cap O(2n, \mathbb{R}). \tag{2.11}$$

of $\mathrm{Sp}(n)$.

Proof. In view of (2.7) the inverse of $U = \mu(u)$, $u \in \mathrm{U}(n, \mathbb{C})$, is

$$U^{-1} = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = U^T$$

hence $U \in O(2n, \mathbb{R})$ which proves the inclusion $\mathrm{U}(n) \subset \mathrm{Sp}(n) \cap O(2n, \mathbb{R})$. Suppose conversely that $U \in \mathrm{Sp}(n) \cap O(2n, \mathbb{R})$. Then

$$JU = (U^T)^{-1}J = UJ$$

which implies that $U \in \mathrm{U}(n)$ so that $\mathrm{Sp}(n) \cap O(2n, \mathbb{R}) \subset \mathrm{U}(n)$. ■

We will loosely talk about $\mathrm{U}(n)$ as of the "unitary group" when there is no risk of confusion; notice that it immediately follows from conditions (2.5), (2.6) that we have the equivalences:

$$A + iB \in \mathrm{U}(n) \tag{2.12}$$

$$\Longleftrightarrow$$

$$A^T B \text{ symmetric and } A^T A + B^T B = I \tag{2.13}$$

$$\Longleftrightarrow$$

$$AB^T \text{ symmetric and } \; AA^T + BB^T = I; \tag{2.14}$$

of course these conditions are just the same thing as the conditions

$$(A + iB)^*(A + iB) = (A + iB)(A + iB)^* = I$$

for the matrix $A + iB$ to be unitary.

In particular, taking $B = 0$ we see the matrices

$$R = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \text{ with } \; AA^T = A^T A = I \tag{2.15}$$

also are symplectic, and form a subgroup $\mathrm{O}(n)$ of $\mathrm{U}(n)$ which we identify with the rotation group $\mathrm{O}(n, \mathbb{R})$. We thus have the chain of inclusions

$$\mathrm{O}(n) \subset \mathrm{U}(n) \subset \mathrm{Sp}(n).$$

Let us end this subsection by mentioning that it is sometimes useful to identify elements of $\mathrm{Sp}(n)$ with complex symplectic matrices. The group $\mathrm{Sp}(n, \mathbb{C})$ is defined, in analogy with $\mathrm{Sp}(n)$, by the condition

$$\mathrm{Sp}(n, \mathbb{C}) = \{M \in M(2n, \mathbb{C}) : M^T J M = J\}.$$

Let now K be the complex matrix

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix} \in \mathrm{U}(2n, \mathbb{C})$$

and consider the mapping

$$\mathrm{Sp}(n) \longrightarrow \mathrm{Sp}(n, \mathbb{C}) \quad , \quad S \longmapsto S_c = K^{-1} S K.$$

One verifies by a straightforward calculation left to the reader as an exercise that $S_c \in \mathrm{Sp}(n, \mathbb{C})$. Notice that if $U \in \mathrm{U}(n)$ then

$$U_c = \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix}.$$

We know from elementary linear algebra that one can diagonalize a symmetric matrix using orthogonal transformations. From the properties of the eigenvalues of a symplectic matrix follows that when this matrix is in addition symplectic and positive definite this diagonalization can be achieved using a symplectic rotation:

Proposition 22 *Let S be a positive definite and symmetric symplectic matrix. Let $\lambda_1 \leq \cdots \leq \lambda_n \leq 1$ be the n smallest eigenvalues of S and set*

$$\Lambda = \mathrm{diag}[\lambda_1, ..., \lambda_n; 1/\lambda_1, ..., 1/\lambda_n]. \tag{2.16}$$

There exists $U \in \mathrm{U}(n)$ such that $S = U^T \Lambda U$.

Proof. Since $S > 0$ its eigenvalues occur in pairs $(\lambda, 1/\lambda)$ of positive numbers (Proposition 20); if $\lambda_1 \leq \cdots \leq \lambda_n$ are n eigenvalues then $1/\lambda_1, ..., 1/\lambda_n$ are the other n eigenvalues. Let now U be an orthogonal matrix such that $S = U^T \Lambda U$ with, Λ being given by (2.16). We claim that $U \in \mathrm{U}(n)$. It suffices to show that we can write U in the form

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

with

$$AB^T = B^T A \ \ , \ \ AA^T + BB^T = I. \tag{2.17}$$

Let $e_1, ..., e_n$ be n orthonormal eigenvectors of U corresponding to the eigenvalues $\lambda_1, ..., \lambda_n$. Since $SJ = JS^{-1}$ (because S is both symplectic and symmetric) we have, for $1 \leq k \leq n$,

$$SJe_k = JS^{-1}e_k = \frac{1}{\lambda_j} Je_k$$

hence $\pm Je_1, ..., \pm Je_n$ are the orthonormal eigenvectors of U corresponding to the remaining n eigenvalues $1/\lambda_1, ..., 1/\lambda_n$. Write now the $2n \times n$ matrix $(e_1, ..., e_n)$ as

$$[e_1, ..., e_n] = \begin{bmatrix} A \\ B \end{bmatrix}$$

where A and B are $n \times n$ matrices; we have

$$[-Je_1, ..., -Je_n] = -J \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -B \\ A \end{bmatrix}$$

hence U is indeed of the type

$$U = [e_1, ..., e_n; -Je_1, ..., -Je_n] = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

The symplectic conditions (2.17) are automatically satisfied since $U^T U = I$. ■

An immediate consequence of Proposition 22 is that the square root of a positive-definite symmetric symplectic matrix is also symplectic. More generally:

Corollary 23 *(i) For every $\alpha \in \mathbb{R}$ there exists a unique $R \in \mathrm{Sp}(n)$, $R > 0$, $R = R^T$, such that $S = R^\alpha$.*

(ii) Conversely, if $R \in \mathrm{Sp}(n)$ is positive definite, then $R^\alpha \in \mathrm{Sp}(n)$ for every $\alpha \in \mathbb{R}$.

Proof. *Proof of (i).* Set $R = U^T \Lambda^{1/\alpha} U$; then $R^\alpha = U^T \Lambda U = S$. *Proof of (ii).* It suffices to note that we have

$$R^\alpha = (U^T \Lambda U)^\alpha = U^T \Lambda^\alpha U \in \mathrm{Sp}(n).$$

■

2.4 The Symplectic Lie Algebra

$\mathrm{Sp}(n)$ is a Lie group; we will call its Lie algebra the "symplectic algebra", and denote it by $\mathfrak{sp}(n)$. There is one-to-one correspondence between the elements of $\mathfrak{sp}(n)$ and the one-parameter groups in $\mathrm{Sp}(n)$. this correspondence is the starting point of linear Hamiltonian mechanics.

Let

$$\Phi : \mathrm{GL}(2n, \mathbb{R}) \longrightarrow \mathbb{R}^{4n^2}$$

be the continuous mapping defined by $\Phi(M) = M^T J M - J$. Since $S \in \mathrm{Sp}(n)$ if and only if $S^T J S = J$ we have $\mathrm{Sp}(n) = \Phi^{-1}(0)$ and $\mathrm{Sp}(n)$ is thus a closed subgroup of $\mathrm{GL}(2n, \mathbb{R})$, hence a "classical Lie group". The set of all real matrices X such that the exponential $\exp(tX)$ is in $\mathrm{Sp}(n)$ is the Lie algebra of $\mathrm{Sp}(n)$; we will call it the "symplectic algebra" and denote it by $\mathfrak{sp}(n)$:

$$X \in \mathfrak{sp}(n) \Longleftrightarrow S_t = \exp(tX) \in \mathrm{Sp}(n) \ \ \textit{for all} \ \ t \in \mathbb{R}. \tag{2.18}$$

The one-parameter family (S_t) thus defined is a group: $S_t S_{t'} = S_{t+t'}$ and $S_t^{-1} = S_{-t}$.

The following result gives an explicit description of the elements of the symplectic algebra:

Proposition 24 *Let X be a real $2n \times 2n$ matrix.*

(i) We have

$$X \in \mathfrak{sp}(n) \Longleftrightarrow XJ + JX^T = 0 \Longleftrightarrow X^T J + JX = 0. \tag{2.19}$$

(ii) Equivalently, $\mathfrak{sp}(n)$ consists of all block-matrices X such that

$$X = \begin{bmatrix} U & V \\ W & -U^T \end{bmatrix} \quad \textit{with } V = V^T \textit{ and } W = W^T. \tag{2.20}$$

Proof. Let (S_t) be a differentiable one-parameter subgroup of $\mathrm{Sp}(n)$ and a $2n \times 2n$ real matrix X such that $S_t = \exp(tX)$. Since S_t is symplectic we have $S_t J (S_t)^T = J$ that is

$$\exp(tX) J \exp(tX^T) = J.$$

Differentiating both sides of this equality with respect to t and then setting $t = 0$ we get $XJ + JX^T = 0$, and applying the same argument to the transpose S_t^T we get $X^T J + JX = 0$ as well. Suppose conversely that X is such that $XJ + JX^T = 0$ and let us show that $X \in \mathfrak{sp}(n)$. For this it suffices to prove that $S_t = \exp(tX)$ is in $\mathrm{Sp}(n)$ for every t. The condition $X^T J + JX = 0$ is equivalent to $X^T = JXJ$ hence $S_t^T = \exp(tJXJ)$; since $J^2 = -I$ we have $(JXJ)^k = (-1)^{k+1} JX^k J$ and hence

$$\exp(tJXJ) = -\sum_{k=0}^{\infty} \frac{(-t)^k}{k!}(JXJ)^k = -Je^{-tX}J.$$

It follows that

$$S_t^T J S_t = (-Je^{-tX}J)Je^{tX} = J$$

so that $S_t \in \mathrm{Sp}(n)$ as claimed. ∎

Remark 25 *The symmetric matrices of order n forming a $n(n+1)/2$-dimensional vector space (2.20) implies, by dimension count, that $\mathfrak{sp}(n)$ has dimension $n(2n+1)$. Since $\mathrm{Sp}(n)$ is connected we consequently have*

$$\dim \mathrm{Sp}(n) = \dim \mathfrak{sp}(n) = n(2n+1). \tag{2.21}$$

One should be careful to note that the exponential mapping

$$\exp : \mathfrak{sp}(n) \longrightarrow \mathrm{Sp}(n)$$

is neither surjective nor injective. This is easily seen in the case $n = 1$. We claim that

$$S = \exp X \quad \text{with } X \in \mathfrak{sp}(1) \Longrightarrow \mathrm{Tr}\, S \geq -2. \tag{2.22}$$

In view of (2.20) we have $X \in \mathfrak{sp}(1)$ if and only $\mathrm{Tr}\, X = 0$, so that Hamilton-Cayley's equation for X is just $X^2 + \lambda I = 0$ where $\lambda = \det X$. Expanding $\exp X$ in power series it is easy to see that

$$\exp X = \cos\sqrt{\lambda} I + \frac{1}{\sqrt{\lambda}} \sin\sqrt{\lambda} X \quad \text{if } \lambda > 0$$
$$\exp X = \cosh\sqrt{-\lambda} I + \frac{1}{\sqrt{-\lambda}} \sinh\sqrt{-\lambda} X \quad \text{if } \lambda < 0.$$

Since $\mathrm{Tr}\, X = 0$ we see that in the case $\lambda > 0$ we have

$$\mathrm{Tr}(\exp X) = 2\cos\sqrt{\lambda} \geq -2$$

and in the case $\lambda < 0$

$$\mathrm{Tr}(\exp X) = 2\cosh\sqrt{\lambda} \geq 1.$$

However:

Proposition 26 *A symplectic matrix S is symmetric positive definite if and only if $S = \exp X$ with $X \in \mathfrak{sp}(n)$ and $X = X^T$. The mapping* exp *is a diffeomorphism*

$$\mathfrak{sp}(n) \cap \mathrm{Sym}(2n, \mathbb{R}) \longrightarrow \mathrm{Sp}(n) \cap \mathrm{Sym}_+(2n, \mathbb{R})$$

($\mathrm{Sym}_+(2n, \mathbb{R})$ *is the set of positive definite symmetric matrices).*

Proof. If $X \in \mathfrak{sp}(n)$ and $X = X^T$ then S is both symplectic and symmetric positive definite. Assume conversely that S symplectic and symmetric positive definite. The exponential mapping is a diffeomorphism $\exp : \mathrm{Sym}(2n, \mathbb{R}) \longrightarrow \mathrm{Sym}_+(2n, \mathbb{R})$ (the positive definite symmetric matrices) hence there exits a unique $X \in \mathrm{Sym}(2n, \mathbb{R})$ such that $S = \exp X$. Let us show that $X \in \mathfrak{sp}(n)$. Since $S = S^T$ we have $SJS = J$ and hence $S = -JS^{-1}J$. Because $-J = J^{-1}$ it follows that

$$\exp X = J^{-1}(\exp(-X))J = \exp(-J^{-1}XJ)$$

and $J^{-1}XJ$ being symmetric, we conclude that $X = J^{-1}XJ$ that is $JX = -XJ$, showing that $X \in \mathfrak{sp}(n)$. ■

Chapter 3

Williamson's Theorem

The message of Williamson's theorem is that one can diagonalize any positive definite symmetric matrix M using a symplectic matrix, and that the diagonal matrix has the very simple form

$$D = \begin{bmatrix} \Lambda_\sigma & 0 \\ 0 & \Lambda_\sigma \end{bmatrix}$$

where the diagonal elements of Λ_σ are the moduli of the eigenvalues of JM. This is a truly remarkable result which will allow us to construct a precise phase space quantum mechanics in the ensuing Chapters. One can without exaggeration say that this theorem carries in germ the recent developments of symplectic topology; it leads immediately to a proof of Gromov's famous non-squeezing theorem in the linear case and has many applications both in mathematics and physics. Williamson proved this result in 1963 and it has been rediscovered several times since that – with different proofs.

3.1 Williamson normal form

Let M be a real $m \times m$ symmetric matrix: $M = M^T$. Elementary linear algebra tells us that all the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ of M are real, and that M can be diagonalized using an orthogonal transformation: $M = R^T DR$ with $R \in O(m)$ and $D = \mathrm{diag}[\lambda_1, \lambda_2, ..., \lambda_{nm}]$. Williamson's theorem provides us with the symplectic variant of this result. It says that every symmetric and positive definite matrix M can be diagonalized using *symplectic* matrices, and this in a very particular way. Because of its importance in everything that will follow, let us describe Williamson's diagonalization procedure in detail.

Theorem 27 *Let M be a positive-definite symmetric real $2n \times 2n$ matrix.*
(i) There exists $S \in \mathrm{Sp}(n)$ such that

$$S^T MS = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \quad , \quad \Lambda \textit{ diagonal,} \tag{3.1}$$

the diagonal entries λ_j of Λ being defined by the condition

$$\pm i\lambda_j \text{ is an eigenvalue of } JM. \tag{3.2}$$

(ii) The sequence $\lambda_1, ..., \lambda_n$ does not depend, up to a reordering of its terms, on the choice of S diagonalizing M.

Proof. (i) A quick examination of the simple case $M = I$ shows that the eigenvalues are $\pm i$, so that it is a good idea to work in the space $\mathbb{C}^{2n}$ and to look for complex eigenvalues and vectors for JM. Let us denote by $\langle\cdot,\cdot\rangle_M$ the scalar product associated with M, that is $\langle z, z'\rangle_M = \langle Mz, z\rangle$. Since both $\langle\cdot,\cdot\rangle_M$ and the symplectic form are non-degenerate we can find a unique invertible matrix K of order $2n$ such that

$$\langle z, Kz'\rangle_M = \sigma(z, z')$$

for all z, z'; that matrix satisfies

$$K^T M = J = -MK.$$

Since the skew-product is antisymmetric we must have $K = -K^M$ where $K^M = -M^{-1}K^T M$ is the transpose of K with respect to $\langle\cdot,\cdot\rangle_M$; it follows that the eigenvalues of $K = -M^{-1}J$ are of the type $\pm i\lambda_j$, $\lambda_j > 0$, and hence those of JM^{-1} are $\pm i\lambda_j^{-1}$. The corresponding eigenvectors occurring in conjugate pairs $e'_j \pm if'_j$ we thus obtain a $\langle\cdot,\cdot\rangle_M$-orthonormal basis $\{e'_i, f'_j\}_{1\leq i,j\leq n}$ of $\mathbb{R}^{2n}_z$ such that $Ke'_i = \lambda_i f'_i$ and $Kf'_j = -\lambda_j e'_j$. Notice that it follows from these relations that

$$K^2 e'_i = -\lambda_i^2 e'_i \quad , \quad K^2 f'_j = -\lambda_j^2 f'_j$$

and that the vectors of the basis $\{e'_i, f'_j\}_{1\leq i,j\leq n}$ satisfy the relations

$$\begin{aligned}
\sigma(e'_i, e'_j) &= \langle e'_i, Ke'_j\rangle_M = \lambda_j\langle e'_i, f'_j\rangle_M = 0\\
\sigma(f'_i, f'_j) &= \langle f'_i, Kf'_j\rangle_M = -\lambda_j\langle f'_i, e'_j\rangle_M = 0\\
\sigma(f'_i, e'_j) &= \langle f'_i, Ke'_j\rangle_M = \lambda_i\langle e'_i, f'_j\rangle_M = -\lambda_i\delta_{ij}.
\end{aligned}$$

Setting $e_i = \lambda_i^{-1/2}e'_i$ and $f_j = \lambda_j^{-1/2}f'_j$, the basis $\{e_i, f_j\}_{1\leq i,j\leq n}$ is symplectic. Let S be the element of $\mathrm{Sp}(n)$ mapping the canonical symplectic basis to $\{e_i, f_j\}_{1\leq i,j\leq n}$. The $\langle\cdot,\cdot\rangle_M$-orthogonality of $\{e_i, f_j\}_{1\leq i,j\leq n}$ implies (3.1) with $\Lambda = \mathrm{diag}[\lambda_1, ..., \lambda_n]$. To prove the uniqueness statement (ii) it suffices to show that if there exists $S \in \mathrm{Sp}(n)$ such that $S^T LS = L'$ with $L = \mathrm{diag}[\Lambda, \Lambda]$, $L' = \mathrm{diag}[\Lambda', \Lambda']$, then $\Lambda = \Lambda'$. Since S is symplectic we have $S^T JS = J$ and hence $S^T LS = L'$ is equivalent to $S^{-1}JLS = JL'$ from which follows that JL and JL' have the same eigenvalues. These eigenvalues are precisely the complex numbers $\pm i/\lambda_j$. ■

The diagonalizing matrix S in the Theorem above has no reason to be unique. However:

Proposition 28 *Assume that S and S' are two elements of* $\mathrm{Sp}(n)$ *such that*

$$M = (S')^T DS' = S^T DS$$

where D is the Williamson diagonal form of M. Then $S(S')^{-1} \in \mathrm{U}(n)$.

Proof. Set $U = S(S')^{-1}$; we have $U^T DU = D$. We are going to show that $UJ = JU$; the Lemma will follow. Setting $R = D^{1/2}UD^{-1/2}$ we have

$$R^T R = D^{-1/2}(U^T DU)D^{-1/2} = D^{-1/2}DD^{-1/2} = I$$

hence $R \in O(2n)$. Since J commutes with each power of D we have, since $JU = (U^T)^{-1}J$,

$$\begin{aligned} JR &= D^{1/2}JUD^{-1/2} = D^{1/2}(U^T)^{-1}JD^{-1/2} \\ &= D^{1/2}(U^T)^{-1}D^{-1/2}J = (R^T)^{-1}J \end{aligned}$$

hence $R \in \mathrm{Sp}(n) \cap O(2n)$ so that $JR = RJ$. Now $U = D^{-1/2}RD^{1/2}$ and therefore

$$\begin{aligned} JU &= JD^{-1/2}RD^{1/2} = D^{-1/2}JRD^{1/2} \\ &= D^{-1/2}RJD^{1/2} = D^{-1/2}RD^{1/2}J \\ &= UJ \end{aligned}$$

which was to be proven. ■

Let M be a positive-definite and symmetric real matrix: $M > 0$. We have seen above that the eigenvalues of JM are of the type $\pm i\lambda_{\sigma,j}$ with $\lambda_{\sigma,j} > 0$. We will always order the positive numbers $\lambda_{\sigma,j}$ as a decreasing sequence:

$$\lambda_{\sigma,1} \geq \lambda_{\sigma,2} \geq ... \geq \lambda_{\sigma,n} > 0. \tag{3.3}$$

Definition 29 *With the ordering convention above $(\lambda_{\sigma,1}, ..., \lambda_{\sigma,n})$ is called the "symplectic spectrum of M and is denoted by* $\mathrm{Spec}_\sigma(M)$*:*

$$\mathrm{Spec}_\sigma(M) = (\lambda_{\sigma,1}, ..., \lambda_{\sigma,n})$$

Here are two important properties of the symplectic spectrum:

Proposition 30 *Let* $\mathrm{Spec}_\sigma(M) = (\lambda_{\sigma,1}, ..., \lambda_{\sigma,n})$ *be the symplectic spectrum of M.*

(i) $\mathrm{Spec}_\sigma(M)$ *is a symplectic invariant:*

$$\mathrm{Spec}_\sigma(S^T MS) = \mathrm{Spec}_\sigma(M) \quad \textit{for every } S \in \mathrm{Sp}(n); \tag{3.4}$$

(ii) the sequence $(\lambda_{\sigma,n}^{-1}, ..., \lambda_{\sigma,1}^{-1})$ is the symplectic spectrum of M^{-1}:

$$\mathrm{Spec}_\sigma(M^{-1}) = (\mathrm{Spec}_\sigma(M))^{-1} \tag{3.5}$$

Proof. *(i)* is an immediate consequence of the definition of $\mathrm{Spec}_\sigma(M)$. *(ii)* The eigenvalues of JM are the same as those of $M^{1/2}JM^{1/2}$; the eigenvalues of JM^{-1} are those of $M^{-1/2}JM^{-1/2}$. Now

$$M^{-1/2}JM^{-1/2} = -(M^{1/2}JM^{1/2})^{-1}$$

hence the eigenvalues of JM and JM^{-1} are obtained from each other by the transformation $t \longmapsto -1/t$. The result follows since the symplectic spectra are obtained by taking the moduli of these eigenvalues. ■

Here is a result allowing us to compare the symplectic spectra of two positive definite symmetric matrices. It is important, because it is an algebraic version of Gromov's non-squeezing theorem in the linear case.

Theorem 31 *Let M and M' be two symmetric positive definite matrices of same dimension. We have*

$$M \leq M' \Longrightarrow \mathrm{Spec}_\sigma(M) \leq \mathrm{Spec}_\sigma(M'). \tag{3.6}$$

Proof. When two matrices A and B have the same eigenvalues we will write $A \simeq B$. When those of A are smaller than or equal to those of B (for a common ordering) we will write $A \leq B$. Notice that when A or B is invertible we have $AB \simeq BA$. With these notations, the statement of is equivalent to

$$M \leq M' \Longrightarrow (JM')^2 \leq (JM)^2$$

since the eigenvalues of JM and JM' occur in pairs $\pm i\lambda$, $\pm i\lambda'$ with λ and λ' real. The relation $M \leq M'$ is equivalent to $z^T Mz \leq z^T M'z$ for every $z \in \mathbb{R}^{2n}_z$. Replacing z by successively $(JM^{1/2})z$ and $(JM'^{1/2})z$ in $z^T Mz \leq z^T M'z$ we thus have, taking into account the fact that $J^T = -J$, that is, since $J^T = -J$,

$$M^{1/2}JM'JM^{1/2} \leq M^{1/2}JMJM^{1/2}. \tag{3.7}$$

$$M'^{1/2}JM'JM'^{1/2} \leq M'^{1/2}JMJM'^{1/2}. \tag{3.8}$$

Noting that we have

$$M^{1/2}JM'JM^{1/2} \simeq MJM'J$$
$$M'^{1/2}JMJM'^{1/2} \simeq M'JMJ \simeq MJM'J$$

we can rewrite the relations (3.7) and (3.8) as

$$MJM' \leq JM^{1/2}JM'JM^{1/2}$$
$$M'^{1/2}JM'JM'^{1/2} \leq MJM'J$$

and hence, by transitivity

$$M'^{1/2}JM'JM'^{1/2} \leq M^{1/2}JMJM^{1/2}. \tag{3.9}$$

Since we have

$$M^{1/2}JMJM^{1/2} \simeq (MJ)^2 \quad , \quad M'^{1/2}JM'JM'^{1/2} \simeq (M'J)^2$$

the relation (3.9) is equivalent to $(M'J)^2 \leq (MJ)^2$, which was to be proven. ■

Let M be a positive-definite and symmetric real matrix $2n \times 2n$; we denote by $\mathbb{M}$ the ellipsoid in $\mathbb{R}_z^{2n}$ defined by the condition $\langle Mz, z\rangle \leq 1$:

$$\mathbb{M} : \langle Mz, z\rangle \leq 1.$$

In view of Williamson's theorem there exist $S \in \mathrm{Sp}(n)$ such that $S^T MS = D$ with $D = \mathrm{diag}[\Lambda, \Lambda]$ and that $\Lambda = \mathrm{diag}[\lambda_{1,\sigma}, ..., \lambda_{n,\sigma}]$ where $(\lambda_{1,\sigma}, ..., \lambda_{n,\sigma})$ is the symplectic spectrum of M. It follows that

$$S^{-1}(\mathbb{M}) : \sum_{j=1}^{n} \lambda_{j,\sigma}(x_j^2 + p_j^2) \leq 1.$$

Definition 32 *The number $R_\sigma(\mathbb{M}) = 1/\sqrt{\lambda_{1,\sigma}}$ is called the symplectic radius of the phase-space ellipsoid $\mathbb{M}$; $c_\sigma(\mathbb{M}) = \pi R_\sigma^2 = \pi/\lambda_{1,\sigma}$ is its symplectic area.*

The properties of the symplectic area are summarized in the following result, whose "hard" part follows from Theorem 31:

Corollary 33 *Let $\mathbb{M}$ and $\mathbb{M}'$ be two ellipsoids in $(\mathbb{R}_z^{2n}, \sigma)$.*
(i) If $\mathbb{M} \subset \mathbb{M}'$ then $c_\sigma(\mathbb{M}) \leq c_\sigma(\mathbb{M}')$;
(ii) For every $S \in \mathrm{Sp}(n)$ we have $c_\sigma(S(\mathbb{M})) = c_\sigma(\mathbb{M})$;
(iii) For every $\lambda > 0$ we have $c_\sigma(\lambda\mathbb{M}) = \lambda^2 c_\sigma(\mathbb{M})$.

Proof. *(i)* Assume that $\mathbb{M} : \langle Mz, z\rangle \leq 1$ and $\mathbb{M}' : \langle M'z, z\rangle \leq 1$. If $\mathbb{M} \subset \mathbb{M}'$ then $M \geq M'$ and hence $\mathrm{Spec}_\sigma(M) \geq \mathrm{Spec}_\sigma(M')$ in view of the implication (3.6) in Theorem 31; in particular $\lambda_{1,\sigma} \leq \lambda'_{1,\sigma}$. Let us prove *(ii)*. We have $S(\mathbb{M}) : \langle M'z, z\rangle \leq 1$ with $S' = (S^{-1})^T MS^{-1}$ and M' thus have the same symplectic spectrum as M in view of Proposition 30, *(i)*. Property *(iii)* is obvious. ■

In next subsection we generalize the notion of symplectic radius and area to arbitrary subsets of phase space.

3.2 The notion of symplectic capacity

Let us now denote $B(R)$ the phase-space ball $|z| \leq R$ and by $Z_j(R)$ the phase-space cylinder with radius R based on the conjugate coordinate plane x_j, p_j:

$$Z_j(R) : x_j^2 + p_j^2 \leq R^2.$$

Since we have

$$Z_j(R) : \langle Mz, z\rangle \leq 1$$

where the matrix M is diagonal and only has two entries different from zero, we can view $Z_j(R)$ as a degenerate ellipsoid with symplectic radius R. This observation motivates following definition; recall that $\mathrm{Symp}(n)$ is the group of all symplectomorphisms of the standard symplectic space $(\mathbb{R}^{2n}_z, \sigma)$.

Definition 34 *A "symplectic capacity" on $(\mathbb{R}^{2n}_z, \sigma)$ is a mapping c which to every subset Ω of $\mathbb{R}^{2n}_z$ associates a number $c_{\mathrm{lin}}(\Omega) \geq 0$, or ∞, and having the following properties:*

(i) $c(\Omega) \leq c(\Omega')$ if $\Omega \subset \Omega'$;

(ii) $c(f(\Omega)) = c(\Omega)$ for every $f \in \mathrm{Symp}(n)$;

(iii) $c(\lambda\Omega) = \lambda^2 c(\Omega)$ for every $\lambda \in \mathbb{R}$;

(iv) $c(B(R)) = c(Z_j(R)) = \pi R^2$.

When the properties (i)–(iv) above only hold for affine symplectomorphisms $f \in \mathrm{ISp}(n)$ we say that c is a "linear symplectic capacity" and we write $c = c_{\mathrm{lin}}$.

While the construction of general symplectic capacities is very difficult (the existence of any symplectic capacity is equivalent to Gromov's non-squeezing theorem), it is reasonably easy to exhibit linear symplectic capacities:

Example 35 *For $\Omega \subset \mathbb{R}^{2n}_z$ set*

$$\underline{c}_{\mathrm{lin}}(\Omega) = \sup_{f \in \mathrm{ISp}(n)} \{\pi R^2 : f(B^{2n}(R)) \subset \Omega\} \tag{3.10}$$

$$\overline{c}_{\mathrm{lin}}(\Omega) = \inf_{f \in \mathrm{ISp}(n)} \{\pi R^2 : f(\Omega) \subset Z_j(R)\}. \tag{3.11}$$

Both $\underline{c}_{\mathrm{lin}}$ and $\overline{c}_{\mathrm{lin}}$ are linear symplectic capacities and we have

$$\underline{c}_{\mathrm{lin}}(\Omega) \leq c_{\mathrm{lin}}(\Omega) \leq \overline{c}_{\mathrm{lin}}(\Omega)$$

for every linear symplectic capacity c_{lin} on $(\mathbb{R}^{2n}_z, \sigma)$.

We have several times mentioned Gromov's non-squeezing theorem in this Chapter. It is time now to state it. Let us first give a definition:

Definition 36 *Let Ω be an arbitrary subset of $\mathbb{R}^{2n}_z$. Let R_σ be the supremum of the set*

$$\{R : \exists f \in \mathrm{Symp}(n) \text{ such that } f(B(R)) \subset \Omega\}.$$

The number $c_G(\Omega) = \pi R_\sigma^2$ is called "symplectic area" (or "Gromov width") of Ω.

For $r > 0$ let

$$Z_j(r) = \{z = (x, p) : x_j^2 + p_j^2 \leq r^2\}$$

be a cylinder with radius R based on the x_j, p_j plane.

Theorem 37 *We have $c_G(\Omega) = \pi R_\sigma^2$; equivalently: there exists a symplectomorphism f of $\mathbb{R}_z^{2n}$ such that $f(B^{2n}(z_0,R)) \subset Z_j(r)$ if and only if $R \leq r$.*

(The sufficiency of the condition $R \leq r$ is trivial since if $R \leq r$ then the translation $z \longmapsto z - z_0$ sends $B^{2n}(z_0,R)$ in any cylinder $Z_j(r)$.)

All known proofs of this theorem are notoriously difficult; Gromov used pseudo-holomorphic tools to establish it.

As pointed out above Gromov's theorem and the existence of one single symplectic capacity are equivalent. Let us prove that Gromov's theorem implies that the symplectic area c_{G} indeed is a symplectic capacity:

Corollary 38 *Let $\Omega \subset \mathbb{R}_z^{2n}$ and let R_σ be the supremum of the set*

$$\{R : \exists f \in \mathrm{Symp}(n) \text{ such that } f(B(R)) \subset \Omega\}.$$

The formulae $c_G(\Omega) = \pi R_\sigma^2$ if $R_\sigma < \infty$, $c_G(\Omega) = \infty$ if $R_\sigma = \infty$, define a symplectic capacity on $(\mathbb{R}_z^{2n}, \sigma)$.

Proof. Let us show that the axioms (i)–(iv) of Definition 34 are verified by c_{G}. Axiom (i) (that is $c_{\mathrm{G}}(\Omega) \leq c_{\mathrm{G}}(\Omega')$ if $\Omega \subset \Omega'$) is trivially verified since a symplectomorphism sending $B(R)$ in Ω' also sends $B(R)$ in any set Ω' containing Ω. Axiom (ii) requires that $c_{\mathrm{G}}(f(\Omega)) = c_{\mathrm{G}}(\Omega)$ for every symplectomorphism f; to prove that this is true, let $g \in \mathrm{Symp}(n)$ be such that $g(B(R)) \subset \Omega$; then $(f \circ g)(B(R)) \subset f(\Omega)$ for every $f \in \mathrm{Symp}(n)$ hence $c_{\mathrm{G}}(f(\Omega)) \geq c_{\mathrm{G}}(\Omega)$. To prove the opposite inequality we note that replacing Ω by $f^{-1}(\Omega)$ leads to $c_{\mathrm{G}}(\Omega)) \geq c_{\mathrm{G}}(f^{-1}(\Omega))$; since f is arbitrary we have in fact $c_{\mathrm{G}}(\Omega)) \geq c_{\mathrm{G}}(f(\Omega))$ for every $f \in \mathrm{Symp}(n)$. Axiom (iii), which says that one must have $c_{\mathrm{G}}(\lambda\Omega) = \lambda^2 c_{\mathrm{G}}(\Omega)$ for all $\lambda \in \mathbb{R}$ is trivially satisfied if f is linear. To prove it holds true in the general case as well, first note that it is no restriction to assume $\lambda \neq 0$ and define, for $f : \mathbb{R}_z^{2n} \longrightarrow \mathbb{R}_z^{2n}$ a mapping f_λ by $f_\lambda(z) = \lambda f(\lambda^{-1}z)$. It is clear that f_λ is a symplectomorphism if and only f is. The condition $f(B(R)) \subset \Omega$ being equivalent to $\lambda^{-1} f_\lambda(\lambda B(R)) \subset \Omega$, that is to $f_\lambda(B(\lambda R)) \subset \lambda\Omega$ it follows that $c_{\mathrm{G}}(\lambda\Omega) = \pi(\lambda R_\sigma)^2 = \lambda^2 c_{\mathrm{G}}(\Omega)$. Let us finally prove that Axiom (iv) is verified by $c_{\mathrm{G}}(\Omega)$. The equality $c_{\mathrm{G}}(B(R)) = \pi R^2$ is obvious: every ball $B(r)$ with $R' \leq R$ is sent into $B(R)$ by the identity and if $R' \geq R$ there exists no $f \in \mathrm{Symp}(n)$ such that $f(B(R')) \subset B(R)$ because symplectomorphisms are volume-preserving. There remains to show that $c_{\mathrm{G}}(Z_j(R)) = \pi R^2$; it is at this point – and only at this point! – we will use Gromov's theorem. If $R' \leq R$ then the identity sends $B(R')$ in $Z_j(R)$ hence $c_{\mathrm{G}}(Z_j(R)) \leq \pi R^2$. Assume that $c_{\mathrm{G}}(Z_j(R)) > \pi R^2$; then there exists a ball $B(R')$ with $R' > R$ and a symplectomorphism f such that $f(B(R')) \subset Z_j(R)$ and this would violate Gromov's theorem. ■

Remark 39 *Conversely, the existence of a symplectic capacity implies Gromov's theorem.*

The number R_σ defined by $c_{\mathrm{G}}(\Omega) = \pi R_\sigma^2$ is called the *symplectic radius* of Ω; that this terminology is consistent with that introduced in Definition 32 above follows from the fact that all symplectic capacities (linear or not) agree on ellipsoids. Let us prove this important property:

Proposition 40 *Let $\mathbb{M} : \langle Mz, z\rangle \leq 1$ be an ellipsoid in $\mathbb{R}_z^{2n}$ and c an arbitrary linear symplectic capacity on $(\mathbb{R}_z^{2n}, \sigma)$. Let $\lambda_{1,\sigma} \geq \lambda_{2,\sigma} \geq \cdots \geq \lambda_{n,\sigma}$ be the symplectic spectrum of the symmetric matrix M. We have*

$$c(\mathbb{M}) = \frac{\pi}{\lambda_{n,\sigma}} = c_{lin}(\mathbb{M}) \tag{3.12}$$

where c_{lin} is any linear symplectic capacity.

Proof. Let us choose $S \in \mathrm{Sp}(n)$ such that the matrix $S^T M S = D$ is in Williamson normal form; $S^{-1}(\mathbb{M})$ is thus the ellipsoid

$$\sum_{j=1}^{n} \lambda_{j,\sigma}(x_j^2 + p_j^2) \leq 1. \tag{3.13}$$

Since $c(S^{-1}(\mathbb{M})) = c(\mathbb{M})$ it is sufficient to assume that the ellipsoid $\mathbb{M}$ is represented by (3.13). In view of the double inequality

$$\lambda_{n,\sigma}(x_n^2 + p_n^2) \leq \sum_{j=1}^{n} \lambda_{j,\sigma}(x_j^2 + p_j^2) \leq \lambda_{n,\sigma} \sum_{j=1}^{n} (x_j^2 + p_j^2) \tag{3.14}$$

we have

$$B(\lambda_{n,\sigma}^{-1/2}) \subset \mathbb{M} \subset Z(\lambda_{n,\sigma}^{-1/2})$$

hence, using the monotonicity axiom (i) for symplectic capacities

$$c(B(\lambda_{n,\sigma}^{-1/2})) \subset c(\mathbb{M}) \subset (Z(\lambda_{n,\sigma}^{-1/2})).$$

The first equality in formula (3.12) now follows from Gromov's theorem; the second equality is obvious since we have put $\mathbb{M}$ in normal form using a linear symplectomorphism. ■

The image of an ellipsoid by an invertible linear transformation is still an ellipsoid. Particularly interesting are the ellipsoids obtained by deforming a ball in $(\mathbb{R}_z^{2n}, \sigma)$ using elements of $\mathrm{Sp}(n)$.

We will denote by $B^{2n}(z_0, R)$ the closed ball in $\mathbb{R}_z^{2n}$ with center z_0 and radius R; when the ball is centered at the origin, *i.e.* when $z_0 = 0$, we will simply write $B^{2n}(R)$.

Definition 41 *A "symplectic ball" $\mathbb{B}^{2n}$ in $(\mathbb{R}_z^{2n}, \sigma)$ is the image of a ball $B^{2n}(z_0, R)$ by some $S \in \mathrm{Sp}(n)$; we will say that R is the radius of $\mathbb{B}^{2n}$ and Sz_0 its center.*

We will drop any reference to the dimension when no confusion can arise, and drop the superscript $2n$ and write $B(z_0, R)$, $\mathbb{B}$, $\mathbb{Q}$ instead of $B(z_0, R)$, $\mathbb{B}^{2n}$, $\mathbb{Q}^{2n}$.

The definition of a symplectic ball can evidently be written as

$$\mathbb{B}^{2n} = S(B^{2n}(z_0, R)) = T(Sz_0)S(B^{2n}(R))$$

for some $S \in \mathrm{Sp}(n)$ and $z_0 \in \mathbb{R}^{2n}_z$; $T(Sz_0)$ is the translation $z \longmapsto z + Sz_0$. That is:

A symplectic ball with radius R in $(\mathbb{R}^{2n}, \sigma)$ is the image of $B^{2n}(R)$ by an element of the affine symplectic group $\mathrm{ISp}(n)$*.*

For instance, since the elements of $\mathrm{Sp}(1)$ are just the area preserving linear automorphisms of $\mathbb{R}^2$ a symplectic ball in the plane is just any phase plane ellipse with area πR^2 (respectively $\frac{1}{2}h$). For arbitrary n we note that since symplectomorphisms are volume-preserving (they have determinant equal to one) the volume of a symplectic ball is just

$$\mathrm{Vol}\, B^{2n}(R) = \frac{\pi^n}{n!} R^{2n}$$

Lemma 42 *An ellipsoid $\mathbb{M} : \langle Mz, z\rangle \leq 1$ in $\mathbb{R}^{2n}_z$ is a symplectic ball with radius one if and only $M \in \mathrm{Sp}(n)$ and we then have $\mathbb{M} = S(B(1))$ with $M = (SS^T)^{-1}$.*

Proof. Assume that $\mathbb{M} = S(B(1)$. Then $\mathbb{M}$ is the set of all $z \in \mathbb{R}^{2n}_z$ such that $\left\langle S^{-1}z, S^{-1}z\right\rangle \leq 1$ hence $M = (S^{-1})^T S^{-1}$ is a symmetric definite-positive symplectic matrix. Assume conversely that $M \in \mathrm{Sp}(n)$. Since $M > 0$ we also have $M^{-1} > 0$ and there exists $S \in \mathrm{Sp}(n)$ such that $M^{-1} = SS^T$. Hence $\mathbb{M} : \left\langle S^{-1}z, S^{-1}z\right\rangle \leq 1$ is just $S(B(1)$. ■

Another very useful observation is that we do not need all symplectic matrices to produce all symplectic balls:

Lemma 43 *For every centered symplectic ball $\mathbb{B}^{2n} = S(B^{2n}(R))$ there exist unique real symmetric $n \times n$ matrices L ($\det L \neq 0$) and Q such that $\mathbb{B}^{2n} = S_0(B^{2n}(R))$ and*

$$S_0 = \begin{bmatrix} L & 0 \\ Q & L^{-1} \end{bmatrix} \in \mathrm{Sp}(n). \tag{3.15}$$

Proof. We can factorize $S \in \mathrm{Sp}(n)$ as $S = S_0U$ where $U \in \mathrm{U}(n)$ and S_0 is of the type (3.15). The claim follows since $U(B^{2n}(R)) = B^{2n}(R)$ (that L and Q are uniquely defined is clear for if $S_0(B^{2n}(R)) = S_0'(B^{2n}(R))$ then $S_0(S_0')^{-1} \in \mathrm{U}(n)$ and S_0' can only be of the type (3.15) if it is identical to S_0). ■

The proof above shows that *every* symplectic ball can be obtained from a ball with same radius by first performing a symplectic rotation which takes it into another ball, and by thereafter applying two successive symplectic transformations of the simple types

$$M_{L^{-1}} = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix} \quad , \quad V_P = \begin{bmatrix} I & 0 \\ -P & 0 \end{bmatrix} \quad (L = L^T,\ P = P^T);$$

the first of these transformations is essentially a symplectic rescaling of the coordinates, and the second a "symplectic shear".

Proposition 44 *The intersection of $\mathbb{B} = S(B(R))$ with a symplectic plane $\mathbb{P}$ is an ellipse with area πR^2.*

Proof. We have $\mathbb{B} \cap \mathbb{P} = S_{|\mathbb{P}'}(B(R) \cap \mathbb{P}')$ where $S_{|\mathbb{P}'}$ is the restriction of S to the symplectic plane $\mathbb{P} = S^{-1}(\mathbb{P})$. The intersection $B(R) \cap \mathbb{P}'$ is a circle with area πR^2 and $S_{|\mathbb{P}'}$ is a symplectic isomorphism $\mathbb{P}' \longrightarrow \mathbb{P}$, and is hence area preserving. The area of the ellipse $\mathbb{B} \cap \mathbb{P}$ is thus πR^2 as claimed. ■

This property is actually a particular case of a more general result, which shows that the intersection of a symplectic ball with any symplectic subspace is a symplectic ball of this subspace. We will prove this in detail below (Theorem **??**).

We urge the reader to notice that the assumption that we are cutting $S(B^{2n}(R))$ with *symplectic* planes is essential. The following exercise provides a counterexample which shows that the conclusion of Proposition 44 is falsified if we intersect $S(B^{2n}(R))$ with a plane that is not symplectic.

Example 45 *Assume $n = 2$ and take $S = \operatorname{diag}[\lambda_1, \lambda_2, 1/\lambda_1, 1/\lambda_2]$ with $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1 \neq \lambda_2$. S is symplectic, but the intersection of $S(B^4(R))$ with the x_2, p_1 plane (which is not conjugate) does not have area πR^2.*

The assumption that S is symplectic is also essential in Proposition 44:

Example 46 *Assume that we swap the two last diagonal entries of the matrix S in the example above so that it becomes $S' = \operatorname{diag}[\lambda_1, \lambda_2, 1/\lambda_2, 1/\lambda_1]$. S' is not symplectic and the section $S'(B^{2n}(R))$ by the symplectic x_2, p_2 plane does not have area πR^2.*

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