The size Ramsey number of short subdivisions of bounded degree graphs *

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Abstract

For graphs G and F, write $G \to (F)_\ell$ if any coloring of the edges of G with ℓ colors yields a monochromatic copy of the graph F. Let positive integers h and d be given. Suppose $S^{(h)}$ is obtained from a graph S with s vertices and maximum degree d by subdividing its edges h times (that is, by replacing the edges of S by paths of length $h+1$). We prove that there exits a graph G with no more than $(\log s)^{20h} s^{1+1/(h+1)}$ edges for which $G \to (S^{(h)})_\ell$ holds, provided that $s \geq s_0(h, d, \ell)$, where $s_0(h, d, \ell)$ is some constant that depends only on h, d, and ℓ. We also extend this result to the case in which Q is a graph with maximum degree d on q vertices with the property that every pair of vertices of degree greater than 2 are distance at least $h+1$ apart. This complements work of Pak regarding the size Ramsey number of 'long subdivisions' of bounded degree graphs.

1 Introduction

For graphs H and G and an integer ℓ, we write $H \to (G)_\ell$ if every coloring of the edges of H with ℓ colors contains a monochromatic copy of G. In the two-color case ($\ell = 2$), we omit the subscript and simply write $H \to G$. For a graph G, the study of which graphs H have the property $H \to G$ is a major area of research in extremal combinatorics. One of the most well-known questions of this nature is to determine the *Ramsey number* $r(G)$, which is the minimum

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number of vertices in a graph H with the property $H \to G$ (naturally, in this definition, H can be restricted to be a complete graph). Analogously, the ℓ-color *Ramsey number* is

$$r_\ell(G) := \min\Big\{|V(H)| : H \to (G)_\ell\Big\}.$$

A variation of this problem, introduced by Erdős, Faudree, Rousseau, and Schelp [8] in 1978, asks for the minimum number of edges in a graph H with the property $H \to G$. This is the *size Ramsey number* of G and is often denoted by $\widehat{r}(G)$. Similarly, the *ℓ-color size Ramsey number* of G is

$$\widehat{r}_\ell(G) := \min\Big\{|E(H)| : H \to (G)_\ell\Big\}.$$

Trivially, $\widehat{r}(G) \le \binom{r(G)}{2}$ holds and a simple argument, attributed to Chvátal in [8], shows that equality holds for the case when G is the complete graph K_n: $\widehat{r}(K_n) = \binom{r(K_n)}{2}$. For many sparse graphs G, as we will see, the bound $\widehat{r}(G) \le \binom{r(G)}{2}$ is far from optimal.

One of the first problems investigated regarding the size Ramsey number was to determine the behavior of the function $\widehat{r}(P_n)$, where P_n is the path on n vertices. Erdős asked the following version of this question in [7]: Is it true that

$$\widehat{r}(P_n)/n \to \infty \qquad \text{and} \qquad \widehat{r}(P_n)/n^2 \to 0?$$

This was answered in the negative by Beck [2], who, using probabilistic methods, proved that $\widehat{r}(P_n) \le 900n$. This result was extended in [14], where it was established that cycles also have linear size Ramsey numbers (in fact, it was shown this even holds for the induced version of the size Ramsey number). Another extension by Friedman and Pippenger [10] established the linearity of the size Ramsey number for trees with bounded degree. More recently, Dellamonica [6] was able to determine asymptotically the size Ramsey number of general trees, confirming a conjecture of Beck. Other related results include [13, 16].

A significant open problem is to determine the largest possible size Ramsey number of a graph of a given order and a given maximum degree. Letting $\Delta(G)$ denote the maximum degree of G, we define this function of interest by

$$\widehat{r}(n,d) := \max\Big\{\widehat{r}(G) : |V(G)| = n,\ \Delta(G) \le d\Big\}.$$

In [3], Beck asked if $\widehat{r}(n,d)$ is always linear in n for fixed d. This was settled in the negative by Rödl and Szemerédi [24], who established that

$$\widehat{r}(n,3) = \Omega(n(\log n)^{1/60}).$$

Indeed, they constructed graphs G_n of order n and maximum degree 3 and argued that if H is any graph with fewer than $10^{-1}n(\log n)^{1/60}$ edges, then H does not have the property $H \to G_n$. In the same paper, it was conjectured that for all d there exists $\varepsilon_d > 0$ such that

$$n^{1+\varepsilon_d} \le \widehat{r}(n,d) \le n^{2-\varepsilon_d}. \tag{1}$$

The upper bound in (1) was subsequently proved by Kohayakawa, Rödl, Schacht, and Szemerédi in [19]. The lower bound in (1), however, remains open and closing the rather large remaining gap between the upper and lower bounds for $\widehat{r}(n,d)$ is of considerable interest. For further results on size Ramsey numbers, see [9, 21, 22, 23], or the more general recent survey on graph Ramsey theory [5].

Subdivisions of Graphs

For a graph S and positive integer h, the *h-subdivision* of S, denoted $S^{(h)}$, is the graph obtained by replacing each edge of S with a path on h internal vertices as demonstrated in Figure 1 for the case $h = 2$. Having in mind that the size Ramsey number of trees is quite well-understood and that much regarding the size Ramsey number of bounded degree graphs remains open, we believe it is of interest to investigate the size Ramsey number of subdivisions.

(a) A graph S (b) The subdivided graph $S^{(2)}$

Figure 1: A graph and its subdivision

The size Ramsey number of 'long' subdivisions of bounded degree graphs, which are subdivided graphs $S^{(h)}$ where $h > c\log|S^{(h)}|$ and the maximum degree of S is bounded, were studied by Pak [20] in 2002. Pak conjectured that $\widehat{r}(S^{(h)})$ is linear in terms of $|S^{(h)}|$ for such subdivisions and, by using results on mixing times of random walks on expanders, proved this conjecture up to a polylogarithmic factor.

Our main result relates to the size Ramsey number of 'short' subdivisions of bounded degree graphs, which are subdivided graphs $S^{(h)}$ where h and the maximum degree of S are fixed and the number of vertices $|V(S)|$ is relatively large. To state a more general form of this result, we introduce the following definition.

Definition 1 (Universal Size Ramsey Number)**.** *For $h, d, \ell, s \in \mathbb{Z}^+$, define the universal size Ramsey number* $\mathrm{USR}(h,d,\ell,s)$ *to be the smallest number of edges in a graph H that has the following universal Ramsey property:*

$$H \to (S^{(h)})_\ell \textit{ for every graph } S \textit{ on } s \textit{ vertices with maximum degree } d.$$

Theorem 2. *For any $h, d, \ell \in \mathbb{Z}^+$, there exists s_0 such that for all $s \geq s_0$,*

$$\mathrm{USR}(h, d, \ell, s) \leq (\log s)^{20h} s^{1+1/(h+1)}. \tag{2}$$

A corollary of Theorem 2 is that for any $h \geq 1$ and $d \geq 1$, there exists s_0 such that if S is any graph on $s \geq s_0$ vertices with maximum degree d,

$$\widehat{r}(S^{(h)}) \leq (\log s)^{20h} s^{1+1/(h+1)}.$$

A short counting argument yields the following lower bound.

Theorem 3. *For all $h, d, \ell, s \in \mathbb{Z}^+$ with $d \geq 3$,*

$$\mathrm{USR}(h, d, \ell, s) \geq \mathrm{USR}(h, d, 1, s) \geq s^{1+1/(h+1)-2/d(h+1)+o(1)}, \tag{3}$$

where $o(1) \to 0$ as $s \to \infty$.

The first inequality in (3) is trivial. The second inequality gives a lower bound for the number of edges in any graph H that contains $S^{(h)}$ as a subgraph for every graph S of maximum degree d on s vertices. Observe that for large d, the exponent in (2) is close to the exponent in (3).

We will also show that the proof of Theorem 2 can be extended to give the following more general theorem.

Theorem 4. *For any $h, d, \ell \in \mathbb{Z}^+$, there exists a constant q_0 such that the following holds. If Q is a graph with maximum degree at most d on $q \geq q_0$ vertices with the property that every pair of vertices of degree greater than 2 are distance at least $h+1$ apart, then*

$$\widehat{r}_\ell(Q) \leq (\log q)^{20h} q^{1+1/(h+1)}.$$

We believe that the power of the logarithm in both Theorems 2 and 4 could be substantially reduced, although our method does not allow for the dependency of the power of the logarithm on h to be removed. For the sake of celerity of presentation, we have opted not to make any attempt to optimize this power. We do believe, however, that removing the dependency on h or removing the logarithm entirely would be of interest. We also ask the following.

Question 5. *For every integer d, does there exist a constant c_d such that*

$$\widehat{r}(S^{(h)}) \leq c_d h s^{1+1/(h+1)}$$

for every integer h and for every graph S on s vertices with maximum degree d?

Notation

We use fairly standard notation, including the following. For a graph H and vertex subsets X_1 and X_2, we let $E_H(X_1, X_2)$ be the the set of edges between X_1 and X_2 and $e_H(X_1, X_2) = |E_H(X_1, X_2)|$. When unambiguous, we omit the subscript. Unless explicitly noted otherwise, a subgraph need not be induced. Also, as is standard, we omit floors and ceilings that do not affect the asymptotic nature of our calculations.

Organization

The rest of this paper is organized as follows. Section 2 introduces an *Existence Lemma* (Lemma 12), a *Coloring Lemma* (Lemma 9), and an *Embedding Lemma* (Lemma 14), and then establish Theorem 2 based upon these lemmas. The proofs of these lemmas are deferred to Sections 4, 3, and 5 respectively. Section 6 addresses Theorem 3. Section 7 addresses Theorem 4.

2 Proof of Theorem 2

The proof of Theorem 2 is based on an *Existence Lemma* (Lemma 12), a *Coloring Lemma* (Lemma 9), and an *Embedding Lemma* (Lemma 14). The Existence Lemma will establish the existence of a sparse graph G that has several properties including being a member of a class of graphs called $\mathcal{I}(N, p)$ (Definition 8). The Coloring Lemma will establish that, since $G \in \mathcal{I}(N, p)$, any ℓ-coloring of the edges of G yields a monochromatic subgraph H that is a member of a class of graphs called $\mathcal{H}(h, n, \varepsilon, q)$ (Definition 7). For appropriate parameters, we will have that the graph $H \in \mathcal{H}(h, n, \varepsilon, q)$ is also in a class of graphs called $\mathcal{J}(h, n, \delta)$ (Definition 13). For any graph S on s vertices that has maximum degree d, the Embedding Lemma will then establish that, since H is in $\mathcal{J}(h, n, \delta)$, the graph $S^{(h)}$ can be embedded into H. These lemmas together will be used to establish that $G \to (S^{(h)})_\ell$ for any graph S on s vertices with maximum degree d, as desired. The objective of this section is to introduce the terminology required to state these three lemmas and then to prove Theorem 2.

The following class describes graphs obtained from blowing up the cycle C_{h+1} by replacing each vertex by an independent set of size n and each edge by an arbitrary bipartite graph. In this definition and elsewhere, we say that H is a graph on $\bigsqcup_{i=1}^{h+1} X_i$ if $X_1, X_2, \ldots, X_{h+1}$ are pairwise disjoint sets and $V(H) = \bigcup_{i=1}^{h+1} X_i$. For notational convenience, we will index the sets X_i modulo $h+1$; in particular, we set $X_{h+2} := X_1$ and $X_0 := X_{h+1}$.

Definition 6. *Let $\mathcal{H}(h, n)$ be the set of all graphs on $\bigsqcup_{i=1}^{h+1} X_i$ such that both the following hold:*

(*i*) $|X_i| = n$ *for all* $i \in [h+1]$.

(*ii*) $E(H) = \bigsqcup_{i=1}^{h+1} E_H(X_i, X_{i+1})$.

The following subclass of $\mathcal{H}(h,n)$ describes graphs where the bipartite graphs induced on (X_i, X_{i+1}) have density q and uniformly distributed edges.

Definition 7. *Let $\mathcal{H}(h,n,\varepsilon,q)$ be the set of all graphs H on $\bigsqcup_{i=1}^{h+1} X_i$ that are in $\mathcal{H}(h,n)$ and satisfy the following additional two properties:*

(iii) $e(X_i, X_{i+1}) = qn^2$ *for all* $i \in [h+1]$.

(iv) For any integer $i \in [h+1]$ and vertex subsets $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ each of size $|\widehat{X}_i|, |\widehat{X}_{i+1}| \geq \varepsilon n$,

$$(1-\varepsilon)q|\widehat{X}_i||\widehat{X}_{i+1}| \leq e(\widehat{X}_i, \widehat{X}_{i+1}) \leq (1+\varepsilon)q|\widehat{X}_i||\widehat{X}_{i+1}|.$$

In the context of the random graph $G(N,p)$, the next definition introduces a class of graphs having neither 'dense bipartite patches' nor 'large bipartite holes'.

Definition 8. *Let $\mathcal{I}(N,p)$ be the set of N-vertex graphs G that have both the following properties:*

(i) For all disjoint sets $V_1, V_2 \subset V(G)$ with $1 \leq |V_1| \leq |V_2| \leq pN|V_1|$,

$$e(V_1,V_2) \leq p|V_1||V_2| + e^2\sqrt{6}\cdot\sqrt{pN|V_1||V_2|}.$$

(ii) For all disjoint sets $V_1, V_2 \subset V(G)$ with $|V_1|, |V_2| \geq N(\log N)^{-1}$,

$$(1/2)\cdot p|V_1||V_2| \leq e(V_1,V_2) \leq 2\cdot p|V_1||V_2|.$$

The following lemma is a deterministic statement about the previous two classes of graphs.

Lemma 9 (Coloring Lemma)**.** *For any $\varepsilon \in \mathbb{R}^+$ and $h, \ell \in \mathbb{Z}^+$, there exist $t, n_1 \in \mathbb{Z}^+$ such that, for all $n \geq n_1$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \textit{and} \qquad p := 4\ell q,$$

every graph $G \in \mathcal{I}(N,p)$ has the following property. Any ℓ-coloring of the edges of G yields disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H \in \mathcal{H}(h,n,\varepsilon,q)$.

The Existence Lemma, which we state next, establishes that there exists a graph G on N vertices that exhibits several properties including being in $\mathcal{I}(N,p)$. Combined with the Coloring Lemma, this gives that, for appropriate parameters, any ℓ-coloring of such a graph G will not only contain a monochromatic copy of some $H \in \mathcal{H}(h,n,\varepsilon,q)$, but one that inherits certain additional desirable properties which will be used to embed $S^{(h)}$. We now describe these additional properties.

Definition 10 (Path Abundance)**.** *Let H be a graph on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n)$.*

- *For vertices $u, v \in X_1$, a* transversal path *between u and v is an (undirected) path with endpoints u and v that has exactly $h+2$ vertices and exactly one vertex from each X_i for all $i \in [h+1] \setminus \{1\}$.*

- *H is* $(1-\delta, \log n)$-path abundant *if for at least $(1-\delta)\binom{n}{2}$ pairs of vertices $\{u, v\} \in \binom{X_1}{2}$, there are at least $\log n$ transversal paths between u and v that are pairwise edge-disjoint.*

Definition 11 (Cluster-Free). *Let F be a graph and $\mathcal{L} \subset \binom{V(F)}{2}$ be a set of pairs of vertices in F (which need not correspond to edges). Let $V(\mathcal{L}) := \bigcup_{\{u,v\}\in\mathcal{L}}\{u, v\}$ and $Z \subset V(F)$ be a subset of vertices with $Z \cap V(\mathcal{L}) = \emptyset$.*

- *An* $(\mathcal{L}, Z, h, \log n)$-cluster *is a set of paths $\mathcal{P}_{\mathcal{L}}$ such that:*
  - *For every $P \in \mathcal{P}_{\mathcal{L}}$, the path P has exactly $h+2$ vertices.*
  - *For every path $P \in \mathcal{P}_{\mathcal{L}}$, the endpoints u and v of P are such that $\{u, v\} \in \mathcal{L}$.*
  - *For every $P \in \mathcal{P}_{\mathcal{L}}$, the path P does not have an internal vertex in $V(\mathcal{L})$.*
  - *For every $\{u, v\} \in \mathcal{L}$, exactly $\log n$ paths in $\mathcal{P}_{\mathcal{L}}$ have endpoints u and v.*
  - *For every pair of paths P and $\widehat{P}$ in $\mathcal{P}_{\mathcal{L}}$, the paths P and $\widehat{P}$ are edge-disjoint.*
  - *For every $P \in \mathcal{P}_{\mathcal{L}}$, the path P has exactly one internal vertex in Z.*

- *We say that F is* (h, n)-cluster free *if F does not contain an $(\mathcal{L}, Z, h, \log n)$-cluster for every $\mathcal{L} \subset \binom{V(F)}{2}$ and $Z \subset V(F)$ with $|\mathcal{L}| \leq n(\log n)^{-6h}$ and $|Z| = h^2|\mathcal{L}|$.*

It follows from this definition that the graph obtained by taking the union of the paths in an $(\mathcal{L}, Z, h, \log n)$-cluster has at most $2|\mathcal{L}|+|Z|+|\mathcal{L}|(\log n)(h-1)$ vertices and exactly $|\mathcal{L}|(\log n)(h+1)$ edges, as well as a very specific structure. Also, observe that if F is (h, n)-cluster free, then any subgraph $\widehat{F}$ of F will be (h, n)-cluster free as well.

Lemma 12 (Existence Lemma). *For all $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$, there exists $\varepsilon \in \mathbb{R}^+$ such that, for any $t \in \mathbb{Z}^+$, there exists $n_2 \in \mathbb{Z}$ for which the following holds. For any $n \geq n_2$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

there exists a graph G on N vertices satisfying all of the following properties:

(i) *Every vertex in G has degree at most $(\log n)^3 n^{1/(h+1)}$.*

(ii) *G is (h, n)-cluster free.*

(iii) *$G \in \mathcal{I}(N, p)$.*

(iv) *For all disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$, every (not necessarily induced) subgraphs H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$ is $(1-\delta, \log n)$-path abundant.*

Observe that if G is any graph satisfying property (iii) in the Existence Lemma then, by the Coloring Lemma, any ℓ-coloring of G yields a monochromatic copy of some $H \in \mathcal{H}(h, n, \varepsilon, q)$. Moreover, if G also satisfies property (iv) in the Existence Lemma, then the monochromatic copy of H must be path abundant. Additionally, if G satisfies properties (i) and (ii) in the Existence Lemma, then the path abundant monochromatic H must also satisfy properties (i) and (ii) in the Existence Lemma. Such a graph H is described by the following definition. Note that this definition has no dependency on ε.

Definition 13. *Let $\mathcal{J}(h, n, \delta)$ be the set of all graphs H on $\bigsqcup_{i=1}^{h+1} X_i$ that are in $\mathcal{H}(h, n)$ and satisfy all the following:*

(i) Every vertex in H has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) H is (n, h)-cluster free.

(iii) H is $(1 - \delta, \log n)$-path abundant.

Our final lemma establishes that every $H \in \mathcal{J}(h, n, \delta)$ has the desired universal property to slightly smaller graphs provided δ is sufficiently small.

Lemma 14 (Embedding Lemma)**.** *For all $h, d \in \mathbb{Z}^+$, there exist $\delta \in \mathbb{R}^+$ and $n_3 \in \mathbb{Z}^+$ such that, for all $n \geq n_3$, the following holds. Every graph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{J}(h, n, \delta)$ is universal to the set of graphs*

$$\left\{ S^{(h)} : |V(S)| = \frac{n}{(\log n)^{7h}} \text{ and } \Delta(S) \leq d \right\}.$$

Proof of Theorem 2

We will now prove our main result based upon the three lemmas we have stated.

Proof of Theorem 2. Consider any $h, d, \ell \in \mathbb{Z}^+$. Recall that Lemmas 14, 12, and 9 are quantified as follows.

$$\begin{aligned} L14: &\quad \forall h, d \quad \exists \delta, n_3 \\ L12: &\quad \forall h, \ell, \delta \;\; \exists \varepsilon \;\; \forall t \;\; \exists n_2 \\ L9: &\quad \forall h, \ell, \varepsilon, \;\; \exists t, n_1 \end{aligned}$$

A sequential application of Lemmas 14, 12, 9, and 12 yields

$$\delta := \delta^{L14}(h, d), \quad n_3 := n_3^{L14}(h, d),$$

$$\varepsilon := \varepsilon^{L12}(h, \ell, \delta),$$

$$t := t^{L9}(h, \ell, \varepsilon), \quad n_1 := n_1^{L9}(h, \ell, \varepsilon),$$

$$n_2 := n_2^{L12}(h, \ell, \delta, \varepsilon, t).$$

Set $s_0 := \max\{n_1, n_2, n_3, e^t\}$ and consider any $s \geq s_0$. Take

$$n := (\log s)^{8h} s, \qquad N := nt, \qquad q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad \text{and} \qquad p := 4\ell q.$$

Observe that $n \geq s \geq s_0$. From the Existence Lemma (Lemma 12), we obtain a graph G on N vertices that satisfies the properties (i)–(iv) in the Existence Lemma. We will now show that G has the desired universal Ramsey property. That is, consider any ℓ-coloring of the edges of G. We will show that G contains a monochromatic copy of $S^{(h)}$ for every graph S with $|V(S)| = s$ and $\Delta(S) \leq d$.

Since $G \in \mathcal{I}(N, p)$, by the Coloring Lemma (Lemma 9), this coloring of G yields disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$. Since G also exhibits properties (i)–(iv) in the Existence Lemma, the monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ must be a member of the class $\mathcal{J}(h, n, \delta)$. By the Embedding Lemma (Lemma 14), the monochromatic subgraph H is universal to the family of graphs $\{S^{(h)} : |V(S)| = n(\log n)^{-7h} \text{ and } \Delta(S) \leq d\}$. Since $n = (\log s)^{8h} s$ was chosen so that $s \leq n(\log n)^{-7h}$, this gives that H is also universal to $\{S^{(h)} : |V(S)| = s \text{ and } \Delta(S) \leq d\}$, as desired.

Having established that G has the desired universal Ramsey property, we will now count the number of edges in G. Based upon the maximum degree in G being at most $(\log n)^3 n^{1/(h+1)}$ (and using $\log n \leq (\log s)^2$, $1 + 1/(h+1) \leq 3/2$, and $n \geq 2^t$), the number of edges in G is at most

$$(\log n)^3 n^{1/(h+1)} N \leq (\log n)^4 n^{1+1/(h+1)} \leq ((\log s)^2)^4 ((\log s)^{8h})^{3/2} s^{1+1/(h+1)} \leq (\log s)^{20h} s^{1+1/(h+1)}.$$

This completes the proof of Theorem 2. □

3 Proof of the Coloring Lemma

This section is devoted to proving Lemma 9. For the remainder of this section, fix $\varepsilon \in \mathbb{R}^+$ and $h, \ell \in \mathbb{Z}^+$ and set

$$q(n) := 4(\log n)^2 n^{-1+1/(h+1)} \qquad \text{and} \qquad p(n) := 4\ell q.$$

We must show there exists an integer t so that for sufficiently large n and $N := tn$, any ℓ-coloring of any graph $G \in \mathcal{I}(N, p)$ yields disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$ (see Definitions 8 and 7).

Our approach to finding a monochromatic subgraph $H \in \mathcal{H}(h, n, \varepsilon, q)$ will be to first find several intermediate classes of graphs. The main idea will be to first find a monochromatic subgraph H_2 (in the class $\mathcal{H}_2$ defined below) in which the number of vertices and edges are controlled but not yet exactly correct. We then transition to a subgraph $H_1 \subset H_2$ (in the class $\mathcal{H}_1$ defined below) in which the number of vertices is precisely as desired and the number of edges is still controlled. Finally, we will obtain a subgraph $H \subset H_1$ with $H \in \mathcal{H}(h,n,\varepsilon,q)$ in which both the number of vertices and the number of edges are exactly as desired.

To define the intermediate classes of graphs, we need the following pair of definitions.

Definition 15 ((η)-regular). *For $\eta \in \mathbb{R}^+$, we say that the bipartite graph $E(X_i, X_{i+1})$ is (η)-regular if, for every $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ with $|\widehat{X}_i| \geq \eta|X_i|$ and $|\widehat{X}_{i+1}| \geq \eta|X_{i+1}|$,*

$$(1-\eta)\frac{e(X_1, X_{i+1})}{|X_i||X_{i+1}|} \leq \frac{e(\widehat{X}_i, \widehat{X}_{i+1})}{|\widehat{X}_i||\widehat{X}_{i+1}|} \leq (1+\eta)\frac{e(X_1, X_{i+1})}{|X_i||X_{i+1}|}.$$

Definition 16 (Density). *We say that the bipartite graph $E(X_i, X_{i+1})$ has density*

$$d_i := \frac{e(X_i, X_{i+1})}{|X_i||X_{i+1}|}.$$

Definition 17 (Intermediate Graph Classes).

- *$\mathcal{H}_2(h,n,\varepsilon_2,q)$: A graph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ is in $\mathcal{H}_2(h,n,\varepsilon_2,q)$ if, for some integer m satisfying $4n \leq m \leq n\log n$, all the following hold:*
  - *(i) $|W_i| = m$ for all $i \in [h+1]$.*
  - *(ii) $E(H_2) = \bigsqcup_{i=1}^{h+1} E_{H_2}(W_i, W_{i+1})$.*
  - *(iii) For each $i \in [h+1]$, the bipartite graph $E_{H_2}(W_i, W_{i+1})$ is (ε_2)-regular.*
  - *(iv) For each $i \in [h+1]$, the bipartite graph $E_{H_2}(W_i, W_{i+1})$ has density d_i satisfying $2q \leq d_i \leq 8\ell q$.*
- *$\mathcal{H}_1(h,n,\varepsilon_1,q)$: A graph H_1 on $\bigsqcup_{i=1}^{h+1} X_i$ is in $\mathcal{H}_1(h,n,\varepsilon_1,q)$ if all the following hold:*
  - *(i) $|X_i| = n$ for all $i \in [h+1]$.*
  - *(ii) $E(H_1) = \bigsqcup_{i=1}^{h+1} E_{H_1}(X_i, X_{i+1})$.*
  - *(iii) For each $i \in [h+1]$, the bipartite graph $E_{H_1}(X_i, X_{i+1})$ is (ε_1)-regular.*
  - *(iv) For each $i \in [h+1]$, the bipartite graph $E_{H_1}(X_i, X_{i+1})$ has density d_i satisfying $(3/2)q \leq d_i \leq 12\ell q$.*
- *$\mathcal{H}(h,n,\varepsilon,q)$: Recall that $\mathcal{H}(h,n,\varepsilon,q)$ was introduced in Definition 7. It follows from this definition that a graph H on $\bigsqcup_{i=1}^{h+1} X_i$ is in $\mathcal{H}(h,n,\varepsilon,q)$ if all the following hold:*
  - *(i) $|X_i| = n$ for all $i \in [h+1]$.*
  - *(ii) $E(H) = \bigsqcup_{i=1}^{h+1} E_H(X_i, X_{i+1})$.*
  - *(iii) For each $i \in [h+1]$, the bipartite graph $E(X_i, X_{i+1})$ is (ε)-regular.*

(*iv*) *For each $i \in [h+1]$, the bipartite graph $E(X_i, X_{i+1})$ has density d_i satisfying $d_i = q$.*

We will now state three claims. The first claim (Claim 18) will establish that, for appropriate parameters, any ℓ-coloring of any graph $G \in \mathcal{I}(N,p)$ contains a monochromatic subgraph $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$. The next claim (Claim 19) will establish that, for appropriate parameters, any graph $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$ contains a subgraph $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$. The final claim (Claim 20) will establish that, for appropriate parameters, any graph $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$ contains a subgraph in $H \in \mathcal{H}(h,n,\varepsilon_{,}q)$. These claims will then be used to prove the Coloring Lemma.

Claim 18. *For any $\varepsilon_2 \in \mathbb{R}^+$, there exists $t \in \mathbb{Z}^+$ such that, for every sufficiently large integer n and $N := tn$, every graph $G \in \mathcal{I}(N,p)$ has the following property. Any ℓ-coloring of the edges of G yields disjoint vertex subsets $W_1, W_2, \ldots, W_{h+1} \subset V(G)$ and a monochromatic subgraph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$.*

Claim 19. *For any $\varepsilon_1 \in \mathbb{R}^+$, there exist $\varepsilon_2 \in \mathbb{R}^+$ such that, for every sufficiently large integer n the following holds. Every graph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$ contains vertex subsets $X_i \subset W_i$ and a subgraph $H_1 \subset H_2$ on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$.*

Claim 20. *For any $\varepsilon \in \mathbb{R}^+$, there exist $\varepsilon_1 \in \mathbb{R}^+$ such that, for all sufficiently large n, the following holds. Every graph H_1 on $\bigsqcup_{i=1}^{h+1} X_i$ with $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$ has a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H \in \mathcal{H}(h,n,\varepsilon,q)$.*

The proofs of Claims 18, 19, and 20 will be provided in Subsections 3.1, 3.2, and 3.3 respectively. We will now show how these claims establish the Coloring Lemma. Recall that we have already fixed ε, h, and ℓ and defined $q(n)$ and $p(n)$ at the beginning of this section. Fix

$$\varepsilon_1 := \varepsilon_1^{C20}(\varepsilon), \qquad \varepsilon_2 := \varepsilon_2^{C19}(\varepsilon_1), \qquad \text{and} \qquad t^{C18} := t(\varepsilon_2).$$

Let n be any sufficiently large integer and define $N := tn$. Consider any ℓ-coloring of any graph $G \in \mathcal{I}(N,p)$. Claim 18 yields disjoint vertex subsets $W_1, W_2, \ldots, W_{h+1} \subset V(G)$ and a monochromatic subgraph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h,n,\varepsilon_2,q)$. Claim 19 gives vertex subsets $X_i \subset W_i$ and a subgraph $H_1 \subset H_2$ on $\bigsqcup_{i=1}^{h+1} X_i$ such that $H_1 \in \mathcal{H}_1(h,n,\varepsilon_1,q)$. Claim 20 gives that the graph H_1 on $\bigsqcup_{i=1}^{h+1} \widehat{X}_i$ contains a subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n,\varepsilon,q)$. This completes the proof of the Coloring Lemma.

3.1 Proof of Claim 18

This whole subsection is devoted to the proof of Claim 18. Consider any $\varepsilon_2 \in \mathbb{R}^+$. We must show that there exists $t \in \mathbb{Z}^+$ such that, for every sufficiently large integer n and $N := tn$, every graph $G \in \mathcal{I}(N,p)$ has the following property. Any ℓ-coloring of the edges of G yields a monochromatic subgraph in $\mathcal{H}_2(h,n,\varepsilon_2,q)$.

Let $r_\ell(K_{h+1})$ denote the ℓ-color Ramsey number for K_{h+1}, i.e., the least integer j such that every ℓ-coloring of the edges of the complete graph K_j yields a monochromatic copy of K_{h+1}. Set

$$r := r_\ell(K_{h+1}), \qquad \varepsilon_{\text{reg}} := \min\{1/r^2, \varepsilon_2/2\ell\}, \qquad \text{and} \qquad k_{\min} := r.$$

Observe that every graph on $k \geq k_{\min}$ vertices with at least $(1-\varepsilon_{\text{reg}})\binom{k}{2}$ edges contains a copy of K_r. Having defined ε_{reg} and $k_{\min}$ and having fixed the integer ℓ at the beginning of this section, we will procure the integers $k_{\max}$, N_0, and D_0 from the sparse regularity lemma. Its statement requires the following definition.

Definition 21 ((η,ρ)-regular)**.** *We say that the bipartite graph $E(X_i, X_{i+1})$ is (η,ρ)-regular if, for every $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ with $|\widehat{X}_i| \geq \eta|X_i|$ and $|\widehat{X}_{i+1}| \geq \eta|X_{i+1}|$,*

$$\left|\frac{e(X_1, X_{i+1})}{|X_i||X_{i+1}|} - \frac{e(\widehat{X}_i, \widehat{X}_{i+1})}{|\widehat{X}_i||\widehat{X}_{i+1}|}\right| \leq \eta\rho. \tag{4}$$

The following is a suitable variant of Szemerédi's regularity lemma for sparse graphs [17, 18] (see also [12, 25]).

Fact 22 (Sparse Regularity Lemma)**.** *For every $\varepsilon_{\text{reg}} \in \mathbb{R}^+$ and integers $k_{\min}, \ell \in \mathbb{Z}^+$, there exist $k_{\max}, N_0, D_0 \in \mathbb{Z}^+$ such that the following holds. Consider any integer $N \geq N_0$ and real number p with $pN \geq D_0$, and any set of graphs $G_1, G_2, \ldots, G_\ell$ on the same vertex set $[N]$ that each satisfy property (i) in the definition of $\mathcal{I}(N,p)$ (Definition 8). Then there exists an integer k satisfying $k_{\min} \leq k \leq k_{\max}$ and a vertex partition $[N] = V_1 \cup V_2 \cdots \cup V_k$ that has the following properties.*

- *For all $i \in [k]$, we have $|V_i| = N/k$.*
- *For at least $(1-\varepsilon_{\text{reg}})\binom{k}{2}$ of the pairs $\{i,j\} \in \binom{[k]}{2}$, all the bipartite graphs $E_{G_{\ell'}}(V_i, V_j)$, where $\ell' \in [\ell]$, are $(\varepsilon_{\text{reg}}, p)$-regular.*

Having obtained $k_{\max}$, N_0, and D_0 from the above lemma, set

$$t := 4k_{\max}.$$

Let n be any integer large enough so that $N = nt \geq N_0$ and $pN = 4t(\log n)^2 n^{1/(h+1)} \geq D_0$. Consider any graph $G \in \mathcal{I}(N,p)$ and any ℓ-coloring of G. Our goal is to show that this arbitrary edge coloring of G yields a monochromatic subgraph in $\mathcal{H}_2(h,n,\varepsilon_2,q)$.

Observe that this coloring corresponds to a partition of $E(G)$ into subgraphs $G_1, G_2, \ldots, G_\ell$ which each inherit property (i) in the definition of $\mathcal{I}(N,p)$. Hence, by the Sparse Regularity Lemma, there exists an integer k satisfying $k_{\min} \leq k \leq k_{\max}$ and a vertex partition $V(G) = V_1 \cup V_2 \cdots \cup V_k$ into classes of size $m := N/k$ such that for at least $(1-\varepsilon_{\text{reg}})\binom{k}{2}$ of the pairs $\{i,j\} \in \binom{[k]}{2}$, the bipartite graph $E(V_i, V_j)$ is $(\varepsilon_{\text{reg}}, p)$-regular with respect to every color class.

Define an auxiliary *cluster graph* on $[k]$ by joining vertex i to vertex j if the bipartite graph $E(V_i, V_j)$ is $(\varepsilon_{\text{reg}}, p)$-regular with respect to every color class. The cluster graph has $k \geq k_{\min}$ vertices and at least $(1-\varepsilon_{\text{reg}})\binom{k}{2}$ edges, implying that the cluster graph contains a copy of K_r.

Define a coloring of this copy of K_r in the cluster graph with the color set $[\ell]$ as follows. Color the edge ij with color $\ell' \in [\ell]$ if the bipartite graph $E(V_i, V_j)$ has density at least $2q$ in color ℓ'. Edges may be colored with multiple colors, but every edge will receive at least one color because condition *(ii)* in the definition of $\mathcal{I}(n,p)$ guarantees that the bipartite graph $E(V_i, V_j)$ has density at least $(1/2)p = 2\ell q$. By the definition of the Ramsey number r, this ℓ-coloring of K_r contains a monochromatic copy of K_{h+1}, and hence a monochromatic copy of the cycle C_{h+1} in some color ℓ'. This corresponds to sets $W_1, W_2, \ldots, W_{h+1}$ of size $m = N/k$ so that, for each $i \in [h+1]$, the bipartite graph $E_{G_{\ell'}}(W_i, W_{i+1})$ is $(\varepsilon_{\text{reg}}, p)$-regular with density d_i satisfying $2q \le d_i \le 8\ell q$, where the upper bound on d_i follows from condition *(ii)* in the definition of $\mathcal{I}(n,p)$. Observe that $m = N/k \ge N/k_{\max} = 4n$ and that $m \le N < n \log n$. To complete the proof, we must only demonstrate that every $(\varepsilon_{\text{reg}}, p)$-regular graph $E(W_i, W_{i+1})$ having density d_i satisfying $2q \le d_i \le 8\ell q$ is also (ε_2)-regular. To this end, consider any subsets $\widehat{W}_i \subset W_i$ and $\widehat{W}_{i+1} \subset W_{i+1}$ with $|\widehat{W}_i|, |\widehat{W}_{i+1}| \ge \varepsilon_2 m$. Since $E(W_i, W_{i+1})$ is $(\varepsilon_{\text{reg}}, p)$-regular and $|\widehat{W}_i|, |\widehat{W}_{i+1}| \ge \varepsilon_2 m \ge \varepsilon_{\text{reg}} m$, it follows from Definition 21 that

$$\left| \frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|} - \frac{e(\widehat{W}_i, \widehat{W}_{i+1})}{|\widehat{W}_i||\widehat{W}_{i+1}|} \right| \le \varepsilon_{\text{reg}} p.$$

Furthermore, since $d_i \ge 2q = p/2\ell$ and $\varepsilon_{\text{reg}} \le \varepsilon_2/2\ell$, this gives that

$$\left| \frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|} - \frac{e(\widehat{W}_i, \widehat{W}_{i+1})}{|\widehat{W}_i||\widehat{W}_{i+1}|} \right| \le \varepsilon_{\text{reg}} p \le \frac{\varepsilon_2}{2\ell}(2\ell d_i) = \varepsilon_2 \frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|},$$

which implies

$$(1-\varepsilon_2)\frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|} \le \frac{e(\widehat{W}_i, \widehat{W}_{i+1})}{|\widehat{W}_i||\widehat{W}_{i+1}|} \le (1+\varepsilon_2)\frac{e(W_1, W_{i+1})}{|W_i||W_{i+1}|}.$$

This concludes the proof of Claim 18.

3.2 Proof of Claim 19

In this subsection we give the proof of Claim 19. Consider any $\varepsilon_1 \in \mathbb{R}^+$. We must show that there exist $\varepsilon_2 \in \mathbb{R}^+$ such that, for every sufficiently large integer n, every graph in $\mathcal{H}_2(h, n, \varepsilon_2, q)$ contains a subgraph in $\mathcal{H}_1(h, n, \varepsilon_1, q)$.

Set $\beta := 1/2$ and $\widehat{\varepsilon}_1 := \varepsilon_1/2$. We obtain the positive real number ε_2 and the constant c from the following lemma. Roughly speaking, the lemma asserts that most induced subgraphs of a (ε_2)-regular bipartite graph can be made (ε_1)-regular by the deletion of only a few vertices provided that $\varepsilon_2 \ll \varepsilon_1$. This basic idea of the lemma is shown in Figure 2.

Fact 23 (Corollary 3.9 in [11])**.** *For all $0 < \beta < 1$ and $\widehat{\varepsilon}_1 > 0$, there exists $\varepsilon_2, c > 0$ such that the following holds for any (ε_2)-regular bipartite graph $E_i = E(W_i, W_{i+1})$ with density d_i satisfying $2n \ge c d_i^{-1}$.*

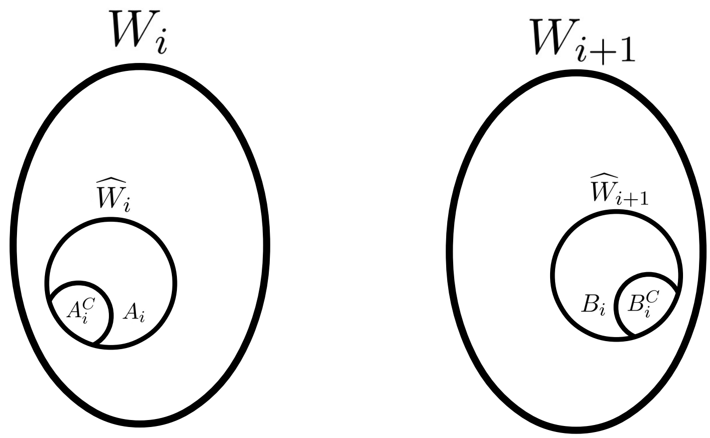

Figure 2: Given an (ε_2)-regular bipartite graph $E_i = E(W_i, W_{i+1})$, the induced bipartite graph $E_{E_i}(\widehat{W}_i, \widehat{W}_{i+1})$ is in $\mathcal{G}$ if there exists small subsets $A_i^C \subseteq \widehat{W}_i$ and $B_i^C \subseteq \widehat{W}_{i+1}$ such that, for $A_i := \widehat{W}_{i+1} \setminus A_i^C$ and $B_i := \widehat{W}_{i+1} \setminus B_i^C$, the induced bipartite graph $E_{E_i}(A_i, B_i)$ is $(\widehat{\varepsilon}_1)$-regular with appropriate density.

- *Let $\mathcal{G}$ be the set of induced subgraphs $E_{E_i}(\widehat{W}_i, \widehat{W}_{i+1}) \subset E(W_i, W_{i+1})$ which have the following property: There exist $A_i \subset \widehat{W}_i$ and $B_i \subset \widehat{W}_{i+1}$ with $|A_i| \geq (1-\widehat{\varepsilon}_1)|\widehat{W}_i|$ and $|B_i| \geq (1-\widehat{\varepsilon}_1)|\widehat{W}_{i+1}|$ such that the induced bipartite graph $E_{E_i}(A_i, B_i)$ is $(\widehat{\varepsilon}_1)$-regular with density $\widehat{d}_i$ satisfying $(1-\widehat{\varepsilon}_1)d_i \leq \widehat{d}_i \leq (1+\widehat{\varepsilon}_1)d_i$.*

Then the number of induced subgraphs $E_{E_i}(\widehat{W}_i, \widehat{W}_{i+1})$ with $\widehat{W}_i \in \binom{W_i}{2n}$ and $\widehat{W}_{i+1} \in \binom{W_{i+1}}{2n}$ that are not in $\mathcal{G}$ is at most $\beta^{2n}\binom{|W_i|}{2n}\binom{|W_{i+1}|}{2n}$.

Having obtained ε_2 and c from the above lemma, let n by any integer large enough so that $2n \geq cq^{-1}$. Now consider any graph H_2 on $\bigsqcup_{i=1}^{h+1} W_i$ with $H_2 \in \mathcal{H}_2(h, n, \varepsilon_2, q)$. For some fixed integer m satisfying $4n \leq m \leq n \log n$, we have that $|W_i| = m$ for all $i \in [h+1]$. Recall that our aim is to show that there exist a collection of n element subsets $\{X_i \subset W_i : i \in [h+1]\}$ so that, for each $i \in [h+1]$, the induced bipartite graph $E(X_i, X_{i+1})$ is (ε_1)-regular with density between $(3/2)q$ and $12\ell q$.

To this end, we first consider a random selection of $2n$ element subsets $\{\widehat{W}_i \subset W_i : i \in [h+1]\}$. By the union bound and Fact 23 (applied with $|W_i| = |W_{i+1}| = m$ and having $\beta = 1/2$), with probability at least $1-(h+1)(1/2)^{2n} > 0$, this random selection of subsets will have the property that, for each $i \in [h+1]$, the bipartite graph $E_i := E(\widehat{W}_i, \widehat{W}_{i+1})$ is in $\mathcal{G}$ (as defined in Fact 23). Hence, we may fix such a selection $\{\widehat{W}_i \subset W_i : i \in [h+1]\}$ of $2n$ element subsets such that each of the bipartite graphs $E_i = E(\widehat{W}_i, \widehat{W}_{i+1})$ are in $\mathcal{G}$. Now, for each $i \in [h+1]$ and associated bipartite graph $E_i = E(\widehat{W}_i, \widehat{W}_{i+1})$, we may find subsets $A_i \subset \widehat{W}_i$ and $B_i \subset \widehat{W}_{i+1}$ with $|A_i|, |B_i| \geq (1-\widehat{\varepsilon}_1)|2n|$ such that $E_{E_i}(A_i, B_i)$ is $(\widehat{\varepsilon}_1)$-regular with density $\widehat{d}_i$ satisfying $(1-\widehat{\varepsilon}_1)d_i \leq \widehat{d}_i \leq (1+\widehat{\varepsilon}_1)d_i$. Thus for the set $\widehat{W}_i$, we have selected subsets $A_i \subset \widehat{W}_i$ and $B_{i-1} \subset \widehat{W}_i$ with respect to the bipartite graphs $E_i = E(\widehat{W}_i, \widehat{W}_{i+1})$ and $E_{i-1} = E(\widehat{W}_{i-1}, \widehat{W}_i)$ respectively. For each $\widehat{W}_i$, let X_i be any subset of $A_i \cap B_{i-1}$ of size n.

For each $i \in [h+1]$, the bipartite graph $E(X_i, X_{i+1})$ is (ε_1)-regular as desired since:

- $E(X_i, X_{i+1})$ is a subgraph of the $(\widehat{\varepsilon}_1)$-regular bipartite graph $E(A_i, B_i)$.

- $(1-\widehat{\varepsilon}_1)2n \le |A_i| \le 2n$ and $(1-\widehat{\varepsilon}_1)2n \le |B_i| \le 2n$.
- $|X_i| = |X_{i+1}| = n$.
- $\widehat{\varepsilon}_1 = \varepsilon_1/2$.

Also, $E(X_i, X_{i+1})$ has density between $(3/2)q$ and $12\ell q$ since:

- $E(X_i, X_{i+1})$ is a subgraph of the $(\widehat{\varepsilon}_1)$-regular bipartite graph $E(A_i, B_i)$ of density $\widehat{d_i}$ satisfying $(1-\widehat{\varepsilon}_2)2q \le \widehat{d_i} \le (1+\widehat{\varepsilon}_2)8\ell q$.
- $|X_i| \ge \widehat{\varepsilon}_1 |A_i|$ and $|X_{i+1}| \ge \widehat{\varepsilon}_1 |B_i|$.

The proof of Claim 19 is complete.

3.3 Proof of Claim 20

This short section is devoted to the proof of Claim 20. Consider any $\varepsilon \in \mathbb{R}^+$. Take $\varepsilon_1 := \varepsilon/2$ and let n be any sufficiently large integer. Consider any graph H_1 on $\bigsqcup_{i=1}^{h+1} X_i$ with $H_1 \in \mathcal{H}_1(h, n, \varepsilon_1, q)$. We must show that H_1 has a monochromatic subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$.

For each $i \in [h+1]$, consider a random selection $R_i \subset E(X_i, X_{i+1})$ of qn^2 edges. We claim that the random subgraph $R := \bigcup_{i \in [h+1]} R_i$ will have the desired property $R \in \mathcal{H}(h, n, \varepsilon, q)$ with positive probability. Indeed, this probability can be easily bounded using the hypergometric distribution (See Lemma 34), keeping in mind that $(3/2)qn^2 \le e(X_i, X_{i+1}) \le 12\ell qn^2$. This establishes the existence of the desired subgraph $H \in \mathcal{H}(h, n, \varepsilon, q)$, and the proof of Claim 20 is complete.

4 Proof of the Existence Lemma

This section of the paper proves Lemma 12, which asserts the existence of a sparse graph G with certain properties. It suffices to prove the following lemma.

Lemma 24. *For all constants $h, \ell \in \mathbb{Z}^+$ and any constant $\delta \in \mathbb{R}^+$, there exists a constant $\varepsilon \in \mathbb{R}^+$ such that, for any constant $t \in \mathbb{Z}^+$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

an instance G of the random graph $G(N,p)$ asymptotically almost surely has each of the following properties:

(i) Every vertex in G has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) G is (h,n)-cluster free (see Definition 11).

(iii) $G \in \mathcal{I}(N,p)$ (see Definition 8).

(*iv*) *For all disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$, every (not necessarily induced) subgraphs H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h, n, \varepsilon, q)$ is $(1-\delta, \log n)$-path abundant (see Definitions 7 and 10).*

In the statement of the previous lemma and elsewhere in this section, we say that a number is a *constant* if it does not depend on n and that a statement holds *asymptotically almost surely* (*a.a.s.*) if the probability the statement is true approaches 1 as $n \to \infty$.

The first subsection contains Claims 25, 26, and 29, which respectively establish that properties (*i*), (*ii*), and (*iii*) in Lemma 24 each hold a.a.s. Notice that these properties do not depend upon ε. The second and most substantial subsection will establish a lemma (Lemma 31) derived from a result in [11]. In Subsection 4.3, Claim 42 will then use this lemma to establish the existence of an ε for which the property (*iv*) in Lemma 24 holds a.a.s. These claims together constitute a proof of Lemma 24.

4.1 Properties (*i*), (*ii*), and (*iii*) in Lemma 24

In this subsection, we prove Claims 25, 26, and 29, which correspond to properties (*i*), (*ii*), and (*iii*) in Lemma 24.

Claim 25. *For any constants $h, t, \ell \in \mathbb{Z}^+$, let $N := tn$ and $p := 4\ell (\log n)^2 n^{-1+1/(h+1)}$. Then a.a.s. the random graph $G(N,p)$ has maximum degree less than $(\log n)^3 n^{1/(h+1)}$.*

Proof. It is a well-known fact that the random graph $G(N,p)$ a.a.s. has maximum degree less than $2pN$ for all $p \gg (\log n)/n$, say. Moreover,

$$2pN = 2 \cdot 4\ell(\log n)^2 n^{-1+1/(h+1)} \cdot tn < (\log n)^3 n^{1/(h+1)},$$

and the proof is complete. □

Claim 26. *For any constants $h, t, \ell \in \mathbb{Z}^+$, let $N := tn$ and $p := 4\ell(\log n)^2 n^{-1+1/(h+1)}$. Then a.a.s. the random graph $G(N,p)$ is (h,n)-cluster free.*

Proof. Recall the definition of an $(\mathcal{L}, Z, h, \log n)$-cluster given in Definition 11. It follows that in the complete graph on N vertices, each $(\mathcal{L}, Z, h, \log n)$-cluster is defined by:

- Specifying a size of L for $\mathcal{L}$.
- Picking a set $\mathcal{L}$ of L pairs of vertices.
- Picking a set Z of vertices.
- For each $\{u, v\} \in \mathcal{L}$, picking a set of $\log n$ paths, each of which can be specified by:
    - Picking a vertex in Z to appear in the interior of the path.

– Picking $h-1$ other vertices to appear in the interior of the path.

– Ordering the h internal vertices on the path.

It follows that in $G(N,p)$, the expected number of $(\mathcal{L}, Z, h, \log n)$-clusters for $\mathcal{L} \subset \binom{[N]}{2}$ and $Z \subset [N]$ with $|\mathcal{L}| \le n(\log n)^{-6h}$ and $|Z| = h^2|\mathcal{L}|$ is bounded above for sufficiently large n by

$$\begin{aligned}
&\sum_{L=1}^{n(\log n)^{-6h}} \binom{N^2}{L}\binom{N}{h^2L} \cdot \left(h^2L \cdot \binom{N}{h-1} \cdot h!\right)^{(\log n)L} p^{(\log n)(h+1)L} \\
&\le \sum_{L=1}^{n(\log n)^{-6h}} N^{3h^2L} \cdot \left(h^{h+2}LN^{h-1}p^{h+1}\right)^{(\log n)L} \\
&\le \sum_{L=1}^{n(\log n)^{-6h}} N^{3h^2L} \cdot \left(h^{h+2} \cdot n(\log n)^{-6h} \cdot (nt)^{h-1} \cdot (4\ell(\log n)^2)^{h+1} n^{-h}\right)^{(\log n)L} \\
&= \sum_{L=1}^{n(\log n)^{-6h}} N^{3h^2L} \cdot \left(h^{h+2}t^{h-1}(4\ell)^{h+1}(\log n)^{2-4h}\right)^{(\log n)L} \\
&\le \sum_{L=1}^{n} N^{3h^2L} \cdot \left(\frac{1}{\log n}\right)^{(\log n)L} \le \sum_{L=1}^{n} \left(\frac{(nt)^{3h^2}}{(\log n)^{\log n}}\right)^{L} \le n \cdot \frac{(nt)^{3h^2}}{(\log n)^{\log n}},
\end{aligned}$$

which goes to 0 as $n \to \infty$. Since the expected number of forbidden $(\mathcal{L}, Z, h, \log n)$-clusters that $G(N,p)$ contains goes to 0, a.a.s. $G(N,p)$ is (h,n)-cluster free. ☐

Before we state the next claim, we introduce a definition and an external lemma that are needed in its proof.

Definition 27. *We say that a graph G is (p,a)-uniform if*

$$\left|e(V_1,V_2) - p|V_1||V_2|\right| \le a\sqrt{p|V(G)||V_1||V_2|}$$

for all disjoint sets $V_1, V_2 \subset V(G)$ such that $1 \le |V_1| \le |V_2| \le p|V(G)||V_1|$.

Fact 28 (Lemma 3.8 in [14])**.** *For every $p = p(N)$, $0 < p \le 1$, a.a.s. the random graph $G(N,p)$ is $(p, e^2\sqrt{6})$-uniform.*

Claim 29. *For any constants $h, t, \ell \in \mathbb{Z}^+$, $N := tn$ and $p := 4\ell(\log n)^2 n^{-1+1/(h+1)}$, a.a.s. the random graph $G(N,p)$ is in $\mathcal{I}(N,p)$.*

Proof. By Fact 28 stated above, a.a.s. we have that

$$e(V_1,V_2) \le p|V_1||V_2| + e^2\sqrt{6} \cdot \sqrt{pN|V_1||V_2|},$$

for all disjoint sets $V_1, V_2 \subset V(G(N,p))$ with $1 \le |V_1| \le |V_2| \le pN|V_1|$. This is exactly the first condition given in the definition of $\mathcal{I}(N,p)$. The other condition given in the definition of $\mathcal{I}(N,p)$ states that a.a.s.

$$(1/2) \cdot p|V_1||V_2| \le e(V_1, V_2) \le 2 \cdot p|V_1||V_2|$$

for all disjoint sets $V_1, V_2 \subset V(G(N,p))$ with $|V_1|, |V_2| \ge N(\log N)^{-1}$. This can easily be established by the union bound. □

4.2 Proof of Lemma 24

For the remainder of this subsection, let $X_1, X_2, \ldots, X_{h+1}$ be fixed (labeled) sets each of size n. The following class of graphs describes the graphs on $\bigsqcup_{i=1}^{h+1} X_i$ that do not have the desired path abundance property.

Definition 30. *Let $\mathcal{B}(h,n,\varepsilon,q,\delta)$ be the set of all graphs B on $\bigsqcup_{i=1}^{h+1} X_i$ such that $B \in \mathcal{H}(h,n,\varepsilon,q)$ and B is not $(1-\delta, \log n)$-path abundant.*

Lemma 31. *For any constant $h \in \mathbb{Z}^+$ and any constants $\delta, \beta \in \mathbb{R}^+$, there exist constants $\varepsilon, n_4 \in \mathbb{R}^+$ such that, for any $n \ge n_4$ and $q := 4(\log n)^2 n^{-1+1/(h+1)}$, we have that*

$$|\mathcal{B}(h,n,\varepsilon,q,\delta)| \le \beta^{qn^2} \binom{n^2}{qn^2}^{h+1}.$$

In Subsection 4.3, Lemma 31 will be used to establish Claim 42, which states that the random graph $G(N,p)$ a.a.s. has the property that it does not contain any selection of disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1}$ and subgraph B on $\bigsqcup_{i=1}^{h+1} X_i$ with $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$. In other words, Claim 42 implies that a.a.s. $G(N,p)$ has the property that for every section of disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1}$ and subgraph H on $\bigsqcup_{i=1}^{h+1} X_i$, the graph H is $(1-\delta, \log n)$-path abundant if $H \in \mathcal{H}(h,n,\varepsilon,q)$, which is exactly property (iv) in Lemma 24. Keep in mind that although Claim 42 concerns any selection of disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1}$ in $G(N,p)$, for the time being in this section we are only counting the graphs in $\mathcal{B}(h,n,\varepsilon,q,\delta)$ on already determined vertex sets $X_1, X_2, \ldots, X_{h+1}$.

Essentially, we are trying to show that all but exponentially few graphs on $\bigsqcup_{i=1}^{h+1} X_i$ in $\mathcal{H}(h,n,\varepsilon,q)$ (see Definition 7) have the property that almost all pairs of vertices in X_1 are joined by $\log n$ transversal paths. The key external lemma we will use establishes that all but exponentially few graphs in $\mathcal{H}(h,n,\widehat{\varepsilon},q/4\log n)$ (again see Definition 7) have the property that most pairs of vertices in X_1 are connected by at least one path. This lemma will be related to the result we are trying to prove by a double counting argument in which a set $\mathcal{F}$ of 'bad families' of graphs (see Definition 35) is considered. We now introduce not only the key external lemma and a related definition, but also the standard Hypergeometic Bound. This will be followed by a proof of Lemma 31.

Definition 32 (Path Dense)**.** *A graph H on on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n)$ is* $(1-\eta)$-path dense *if at least $(1-\eta)\binom{n}{2}$ pairs of vertices $\{u,v\} \in \binom{X_1}{2}$ are joined by at least one transversal path (transversal paths are defined in Definition 10).*

The next lemma is a corollary of Lemma 5.9 in [11]. (To obtain Fact 33 below, one sets the parameters in Lemma 5.9 as follows: $\ell = h+2$, $\beta = \widehat{\beta}$, $\delta = \delta/4$, $\gamma = \delta/4$, $\nu = \delta/2$, $q = 4(\log n)^2 n^{-1+1/(h+1)}$, $m = qn^2/(4\log n)$ and noticing that $n^{h+2} \ll m^{h+1}$.)

Fact 33. *For any $\widehat{\beta}, \delta \in \mathbb{R}^+$, there exists $\widehat{\varepsilon} \in \mathbb{R}^+$ so that, for $q = 4(\log n)^2 n^{-1+1/(h+1)}$, $m := qn^2/(4\log n)$, and sufficiently large n, the total number of graphs E on $\bigsqcup_{i=1}^{h+1} X_i$ with $E \in \mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$ that are not $(1-\delta/2)$-path dense is at most*

$$\widehat{\beta}^m \binom{n^2}{m}^{h+1}. \tag{5}$$

The following is a well-known bound on the hypergeometic distribution (see, e.g., Theorem 2.10 and Equation (2.12) in [15]).

Fact 34 (Hypergeometic Bound)**.** *Let Y be a set and $\widehat{Y}$ be a subset of Y. Suppose that $M \subset Y$ is a subset of size m chosen at random from Y and let the random variable X denote the number of elements in $M \cap \widehat{Y}$. Then*

$$\Pr\left(\left|X - \frac{m|\widehat{Y}|}{|Y|}\right| \le t\right) \ge 1 - 2\exp\left\{-\frac{2t^2}{|Y|}\right\}.$$

We will now prove Lemma 31.

Proof of Lemma 31. Consider any $h \in \mathbb{Z}^+$ and $\beta, \delta \in \mathbb{R}^+$. Let $q = 4(\log n)^2 n^{-1+1/(h+1)}$. We must show that there exists an $\varepsilon \in \mathbb{R}^+$ such that for sufficiently large n we have

$$|\mathcal{B}(h,n,\varepsilon,q,\delta)| \le \beta^{qn^2} \binom{n^2}{qn^2}^{h+1}.$$

Making use of Fact 33, set

$$\widehat{\beta} := \tfrac{\beta^2}{9^{2(h+1)}}, \qquad \widehat{\varepsilon} := \varepsilon^{F33}(\widehat{\beta},\delta), \qquad \varepsilon := \widehat{\varepsilon}/2, \qquad \text{and} \qquad m := \tfrac{qn^2}{4\log n}.$$

As mentioned before, the fundamental idea in our proof is to relate the bound in Fact 33 to $|\mathcal{B}(h,n,\varepsilon,q,\delta)|$ by counting the number of 'bad families,' which are defined as follows.

Definition 35 (Bad Family)**.** *We call a set of graphs $F = \{E_1, E_2, \ldots, E_{4\log n}\}$* a bad family *if both the following hold:*

- *Every $E \in F$ is a graph on $\bigsqcup_{i=1}^{h+1} X_i$ with $E \in \mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$.*
- *Fewer than half of the graphs $E \in F$ are $(1-\delta/2)$-path dense.*

Let $\mathcal{F}$ be the set of all bad families of graphs.

Proposition 36. *We have*

$$|\mathcal{F}| \leq \left(\widehat{\beta}^m \binom{n^2}{m}^{h+1}\right)^{2\log n} \left(\binom{n^2}{m}^{h+1}\right)^{2\log n}.$$

Proof. To verify Proposition 36, we use that for each $F \in \mathcal{F}$, there are $2\log n$ graphs $E \in F$ in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$ that are not $(1-\delta/2)$-path dense. By Fact 33, the number of graphs of this type is at most as in (5). This readily yields the bound in Proposition 36. $\square$

The next definition refers to $\mathcal{H}(h,n,1,m/n^2)$, which is the set of graphs in $\mathcal{H}(h,n)$ on $\bigsqcup_{i=1}^{h+1} X_i$ in which all of the bipartite graph (X_i, X_{i+1}) have m/n^2 edges (i.e., the choice of $\varepsilon = 1$ in Definition 7 imposes no uniformity restriction).

Definition 37 (Associated Family)**.** *For each graph $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$, we call the set of edge-disjoint graphs $A = \{E_1, E_2, \ldots, E_{4\log n}\}$* an associated family *to B if both the following hold:*

- *Every $E \in A$ is a graph o on $\bigsqcup_{i=1}^{h+1} X_i$ with $E \in \mathcal{H}(h,n,1,m/n^2)$.*
- *$B = \bigcup_{i=1}^{4\log n} E_i$.*

Since for each $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ an associated family A is obtained by partitioning the qn^2 edges in each of the $h+1$ bipartite graphs into $4\log n$ classes of size m, it follows that each B is associated to

$$\binom{qn^2}{m,m,\ldots,m}^{h+1}((4\log n)!)^{-1}$$

associated families. Moreover, no two distinct graphs $B_1, B_2 \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ will yield a common associated family. The next claim gives a lower bound for the size of $\mathcal{F}$ and will be proved by establishing that, for each $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$, half of its associated families are bad families.

Proposition 38. *We have*

$$|\mathcal{F}| \geq |\mathcal{B}(h,n,\varepsilon,q,\delta)|\frac{1}{2}\binom{qn^2}{m,m,\ldots,m}^{h+1}((4\log n)!)^{-1}.$$

Proof. As discussed before the proposition, it suffices to show that at least half the associated families for any $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ are bad families. Hence, to prove Proposition 38, it suffices to show the following two subpropositions.

Subproposition 39. *For every $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ and every associated family $A = \{E_1, E_2, \ldots, E_{4\log n}\}$, fewer than half of the graphs $E \in A$ are $(1-\delta/2)$-path dense.*

Subproposition 40. *For every $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$, at least half the associated families $A = \{E_1, E_2, \ldots, E_{4\log n}\}$ have the property that all $E \in A$ are in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$.*

Proof of Subproposition 39. We prove the contrapositive by arguing that if at least $2\log n$ of the graphs $E \in A$ are $(1-\delta/2)$-path dense, then B is $(1-\delta, \log n)$-path abundant. To this end, fix a set of $2\log n$ graphs $E \in A$ that are $(1-\delta/2)$-path dense. For each of theses graphs, fix one transversal path for each of the $(1-\delta/2)\binom{n}{2}$ pairs of vertices $\{u, v\} \in \binom{X_1}{2}$ that are joined by traversal paths. Let $\mathcal{P}$ be the set of paths obtained by this process, so that

$$|\mathcal{P}| = (2\log n)(1-\delta/2)\binom{n}{2}. \tag{6}$$

Also, observe that each pair of vertices $\{u, v\} \in \binom{X_1}{2}$ is joined by at most $2\log n$ paths in $\mathcal{P}$. Now suppose that exactly $\alpha\binom{n}{2}$ pairs of vertices in $\binom{X_1}{2}$ are joined by at least $\log n$ transversal paths in $\mathcal{P}$. It follows that

$$|\mathcal{P}| \le \alpha\binom{n}{2} 2\log n + (1-\alpha)\binom{n}{2}\log n. \tag{7}$$

From (6) and (7),

$$(2\log n)(1-\delta/2)\binom{n}{2} \le \alpha\binom{n}{2} 2\log n + (1-\alpha)\binom{n}{2}\log n,$$

which implies

$$2-\delta \le 2\alpha + (1-\alpha),$$

giving that $\alpha \ge 1-\delta$. This establishes that B is $(1-\delta, \log n)$-path abundant, completing the proof of Subproposition 39. □

Proof of Subproposition 40. Consider any $B \in \mathcal{B}(h, n, \varepsilon, q, \delta)$. For any $\widehat{X}_i \subset X_i$ and $\widehat{X}_{i+1} \subset X_{i+1}$ each of size $|\widehat{X}_i|, |\widehat{X}_{i+1}| \ge \widehat{\varepsilon} n \ge \varepsilon n$, by definition of $\mathcal{B}(h, n, \varepsilon, q, \delta)$ we have that

$$\left| e_B(\widehat{X}_i, \widehat{X}_{i+1}) - q|\widehat{X}_i||\widehat{X}_{i+1}| \right| \le \varepsilon q |\widehat{X}_i||\widehat{X}_{i+1}|,$$

or equivalently

$$\left| \frac{e_B(\widehat{X}_i, \widehat{X}_{i+1})}{4\log n} - \frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}| \right| \le \varepsilon \frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}|. \tag{8}$$

Now if M is a random subgraph on $m = qn^2/(4\log n)$ edges of the bipartite graph $E_B(X_i, X_{i+1})$ on qn^2 edges, then the hypergeometric bound stated in Lemma 34 (applied with $Y = E_B(X_i, X_{i+1})$ and $\widehat{Y} = E_B(\widehat{X}_i, \widehat{X}_{i+1})$) gives that

$$\left| e_M(\widehat{X}_i, \widehat{X}_{i+1}) - \frac{e_B(\widehat{X}_i, \widehat{X}_{i+1})}{4\log n} \right| \le \varepsilon \frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}| \tag{9}$$

holds with probability at least

$$1-2\exp\left\{-\frac{2(\varepsilon q|\widehat{X}_i||\widehat{X}_{i+1}|/4\log n)^2}{qn^2}\right\} \geq 1-2\exp\left\{-\frac{\varepsilon^6 qn^2}{8(\log n)^2}\right\} \geq 1-2\exp\left\{-2^{-1}\varepsilon^6 n^{1+\frac{1}{h+1}}\right\}.$$

From the triangle equality applied to (8) and (9) (and fact that $\varepsilon+\varepsilon=\widehat{\varepsilon}$), this gives

$$\left|e_M(\widehat{X}_i,\widehat{X}_{i+1})-\frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}|\right| \leq \widehat{\varepsilon}\frac{q}{4\log n}|\widehat{X}_i||\widehat{X}_{i+1}| \tag{10}$$

with probability at least

$$1-2\exp\left\{-2^{-1}\varepsilon^6 n^{1+1/(h+1)}\right\}. \tag{11}$$

Now consider a random partition of B into an associated family $A=\{E_1,E_2,\dots,E_{4\log n}\}$. The associated family A will have the desired property that all of the graphs $E\in A$ are in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)=\mathcal{H}(h,n,\widehat{\varepsilon},q/(4\log n))$ if inequality (10) is satisfied for every choice of $M=E_j$ for $j\in[4\log n]$, every choice of $i\in[h+1]$, and every choice of $\widehat{X}_i\subset X_i$ and $\widehat{X}_{i+1}\subset X_{i+1}$. It follows from (11) and the union bound that this will occur with probability at least

$$1-(4\log n)\cdot(h+1)\cdot 2^n\cdot 2^n\cdot 2\exp\left\{-2^{-1}\varepsilon^6 n^{1+1/(h+1)}\right\},$$

which tends to 1 as $n\to\infty$. This establishes that a random partition of B into an associated family $A=\{E_1,E_2,\dots,E_{4\log n}\}$ will have the property that all of the graphs $E\in F$ are in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$ with probability at least $1/2$ for sufficiently large n. It follows that at least half of the associated families $A=\{E_1,E_2,\dots,E_{4\log n}\}$ to any $B\in\mathcal{B}(h,n,\varepsilon,q,\delta)$ have the property that all of the graphs $E\in F$ are in $\mathcal{H}(h,n,\widehat{\varepsilon},m/n^2)$, which completes the proof of Subproposition 40. □

Hence, we have proved Proposition 38. □

We now return to the proof of Lemma 31, recalling that we would like to show

$$|\mathcal{B}(h,n,\varepsilon,q,\delta)| \leq \beta^{qn^2}\binom{n^2}{qn^2}^{h+1}.$$

Propositions 38 and 36, which we have already established, together give that

$$|\mathcal{B}(h,n,\varepsilon,q,\delta)| \leq \left(\widehat{\beta}^m\binom{n^2}{m}^{h+1}\right)^{2\log n}\left(\binom{n^2}{m}^{h+1}\right)^{2\log n}\cdot 2\binom{qn^2}{m,m,\dots,m}^{-(h+1)}(4\log n)!.$$

Thus to establish Lemma 31, it suffices to prove the following.

Proposition 41. *We have*

$$\widehat{\beta}^{2m\log n}\binom{n^2}{m}^{4(h+1)\log n}\cdot 2\binom{qn^2}{m,m,\ldots,m}^{-(h+1)}(4\log n)! \le \beta^{qn^2}\binom{n^2}{qn^2}^{h+1}.$$

Proof. Keeping in mind that $qn^2 = 4(\log n)m$, $\widehat{\beta} = \beta^2 9^{-2(h+1)}$, $\binom{a}{b} \le (\frac{ea}{b})^b$, $\binom{a}{b,b,\ldots,b} \ge \left(\frac{a}{be}\right)^a$, and $\left(\frac{a}{b}\right)^b \le \binom{a}{b}$, we see that

$$\begin{aligned}
&\widehat{\beta}^{2m\log n}\binom{n^2}{m}^{4(h+1)\log n}\cdot 2\binom{qn^2}{m,m,\ldots,m}^{-(h+1)}\cdot(4\log n)!\\
&\le \left(\frac{\beta^2}{9^{2(h+1)}}\right)^{2m\log n}\left(\frac{n^2e}{m}\right)^{m4(h+1)\log n}\cdot 2\left(\frac{qn^2}{me}\right)^{-qn^2(h+1)}\cdot(4\log n)!\\
&= \beta^{qn^2}\left(\frac{n^2e}{9m}\right)^{4(h+1)m\log n}\cdot 2\left(\frac{me}{qn^2}\right)^{qn^2(h+1)}\cdot(4\log n)!\\
&= 2\beta^{qn^2}\left(\frac{n^2e}{9m}\cdot\frac{me}{qn^2}\right)^{qn^2(h+1)}\cdot(4\log n)!\\
&\le \beta^{qn^2}\left(\frac{n^2}{qn^2}\right)^{qn^2(h+1)}\cdot\left(\frac{e^2}{9}\right)^{qn^2(h+1)}2(4\log n)!\\
&\le \beta^{qn^2}\left(\frac{n^2}{qn^2}\right)^{qn^2(h+1)} \le \beta^{qn^2}\binom{n^2}{qn^2}^{h+1},
\end{aligned}$$

which establishes Proposition 41. ☐

This completes the proof of Lemma 31. ☐

4.3 Property (*iv*) in Lemma 24

In this subsection, we will prove Claim 42, which correspond to property (*iv*) in Lemma 24.

Claim 42. *For all constants $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$, there exists a constant $\varepsilon \in \mathbb{R}^+$ such that the following holds. For any constant $t \in \mathbb{Z}^+$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \textit{and} \qquad p := 4\ell q,$$

the random graph $G(N,p)$ a.a.s. has the following property. For any selection of disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$, every (not necessarily induced) subgraphs H on $\bigsqcup_{i=1}^{h+1} X_i$ with $H \in \mathcal{H}(h,n,\varepsilon,q)$ is $(1-\delta, \log n)$-path abundant.

Proof. Consider any $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$. Let

$$\beta := (24\ell)^{-(h+1)}.$$

By Lemma 31, we may now fix

$$\varepsilon := \varepsilon^{L31}(h,\delta,\beta) \qquad \text{and} \qquad n_4 := n_4^{L31}(h,\delta,\beta),$$

and without loss of generality assume that $\varepsilon < 1/2$. Now consider any integer $t \in \mathbb{Z}^+$.

To show that a.a.s. every subgraph $H \in \mathcal{H}(h,n,\varepsilon,q)$ appearing in $G(N,p)$ is $(1-\delta, \log n)$-path abundant, as we previously remarked, it suffices to show that a.a.s. $G(N,p)$ does not contain disjoint subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a subgraph B on $\bigsqcup_{i=1}^{h+1} X_i$ with $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$. By Lemma 31, for all $n \geq n_4$, the expected total number of subgraphs $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ appearing in $G(N,p)$ over all choices of subsets is bounded above by

$$\begin{aligned}
&\binom{N}{(h+1)n}((h+1)n)! \cdot \beta^{qn^2}\binom{n^2}{qn^2}^{(h+1)} \cdot p^{qn^2(h+1)} \\
&\qquad \leq N^{(h+1)n} \cdot \beta^{qn^2}\left(\frac{en^2}{qn^2(h+1)}\right)^{qn^2(h+1)} p^{qn^2(h+1)} \\
&\qquad \leq 2^{(h+1)n\log N} \cdot \left(\frac{\beta^{1/(h+1)}e(4\ell q)}{q(h+1)}\right)^{qn^2(h+1)} \\
&\qquad \leq 2^{(h+1)n\log N} \cdot \left(\beta^{1/(h+1)}e4\ell\right)^{qn^2(h+1)} \\
&\qquad \leq 2^{(h+1)n\log tn} \cdot \left(\frac{1}{2}\right)^{qn^2},
\end{aligned}$$

which tends to 0 as $n \to \infty$. Therefore the probability that $G(N,p)$ contains disjoint vertex subsets $X_1, X_2, \ldots, X_{h+1} \subset V(G)$ and a subgraph B on $\bigsqcup_{i=1}^{h+1} X_i$ with $B \in \mathcal{B}(h,n,\varepsilon,q,\delta)$ also tends to 0 as $n \to \infty$, completing the proof of Claim 26. $\square$

5 Proof of the Embedding Lemma

In this section, we prove Lemma 14, which states that for certain parameters every $J \in \mathcal{J}(h,n,\delta)$ (see Definition 13) is universal to the set of graphs $\{S^{(h)} : |V(S)| = n(\log n)^{-7h} \text{ and } \Delta(S) \leq d\}$. The proof will be divided into two subsections, which are preceded by the following sketch of the proof.

Consider any graph $J \in \mathcal{J}(h,n,\delta)$ on $\bigsqcup_{i=1}^{h+1} X_i$ and any graph S with $|V(S)| = n(\log n)^{-7h}$ and $\Delta(S) \leq d$. Our aim will be to find a mapping $\phi : V(S) \to X_1$ such that each edge $uv \in E(S)$ can be paired with a transversal path (see Definition 10) between $\phi(u)$ and $\phi(v)$. Observe that if the set of transversal paths selected are internally vertex-disjoint, this will correspond to an embedding of the subdivided graph $S^{(h)}$ into J. Roughly speaking, this will be accomplished by first finding an embedding $\phi : V(S) \to X_1$ and associating each edge $uv \in E(S)$ with not one associated transversal path, but a family of many transversal paths between $\phi(u)$ and $\phi(v)$. This will be done so that all the paths in all the associated families are edge-disjoint. We then will select one path from each associated family to obtain the desired collection of internally vertex-disjoint paths.

We will now elaborate upon this sketch. For the graph J, we say that two vertices $u, v \in X_1$ are $(\log n)$*-path connected* if u and v are joined by at least $\log n$ pairwise edge-disjoint transversal paths in J. Since J is $(1-\delta, \log n)$-path abundant (see Definition 10), at least $(1-\delta)\binom{n}{2}$ pairs of vertices in X_1 are $(\log n)$-path connected (see Definition 10). Define an auxiliary graph A by

$$V(A) := X_1 \quad \text{and} \quad E(A) := \{uv : u \text{ and } v \text{ are } (\log n)\text{-path connected in } J\}.$$

For each $uv \in E(A)$, let Π_{uv} be a fixed set of $\log n$ pairwise edge-disjoint transversal paths in J with endpoints u and v. We say the distinct edges $e_1, e_2 \in E(A)$ are *incompatible* if there exist paths $\pi_{e_1} \in \Pi_{e_1}$ and $\pi_{e_2} \in \Pi_{e_2}$ such that π_{e_1} and π_{e_2} have an edge in common. Define the *incompatibility function* $f\colon E(A) \to \mathcal{P}(E(A))$ by

$$f(e_1) := \{e_2 : e_1 \text{ and } e_2 \text{ are incompatible}\}.$$

Given this set-up, the proof has two steps:

- Find a graph embedding $\phi\colon S \to A$ such that $\phi(e_1) \notin f(\phi(e_2))$ for every $e_1, e_2 \in E(S)$.
- For each edge $e \in E(S)$, select a path $\pi_{\phi(e)} \in \Pi_{\phi(e)}$ so that for all pairs of edges $e_1, e_2 \in E(S)$, the paths $\pi_{\phi(e_1)}$ and $\pi_{\phi(e_2)}$ are internally vertex-disjoint.

The key to the first of these two steps is the following lemma. Although stated in a general context, when we apply the lemma the function f will be the incompatibility function defined above.

Lemma 43. *Let d and n be positive integers. Let A be a graph such that:*

(*i*) $|V(A)| = n$.

(*ii*) *Every vertex in A has degree at least $(1 - 1/6d)n$.*

Let S be a graph such that:

(*iii*) $|V(S)| \leq n/6$.

(*iv*) *Every vertex in S has degree at most d.*

Let $f : E(A) \to \mathcal{P}(E(A))$ be a function that maps each edge $e \in E(A)$ to a set of edges $f(e) \subset E(A)$ such that:

(*v*) $|f(e)| \leq n/6^3d^4$ *for all $e \in E(A)$.*

(*vi*) $e_1 \in f(e_2)$ *if and only if* $e_2 \in f(e_1)$.

(vii) $e \notin f(e)$ *for all* $e \in E(A)$*.*

Then there is an embedding $\phi\colon S \to A$ *such that*

$$\phi(E(S)) \cap f(\phi(E(S))) = \emptyset, \tag{12}$$

where $f(\phi(E(S))) := \bigcup_{e\in\phi(E(S))} f(e)$*.*

To select a system of internally vertex-disjoint paths $\pi_{\phi(e)} \in \Pi_{\phi(e)}$ for the edges $e \in S$, we will make use of J being (h, n)-cluster free, that for distinct edges $e_1, e_2 \in S$ the families $\pi_{\phi(e_1)}$ and $\pi_{\phi(e_2)}$ consist of pairwise edge-disjoint paths, and the following result of Aharoni and Haxell.

Fact 44 ([1])**.** *Let* X *be a finite set and let* $\widehat{\Pi}_1, \ldots, \widehat{\Pi}_m \subset \binom{X}{h}$ *be families of* h*-subsets of* X*. Suppose that, for every non-empty* $L \subset [m]$*, there are more than* $h(|L| - 1)$ *pairwise disjoint* h*-sets in* $\bigcup_{l\in L} \widehat{\Pi}_l$*. Then there exist* $\widehat{\pi}_1, \ldots, \widehat{\pi}_m$ *with* $\widehat{\pi}_i \in \widehat{\Pi}_i$ *for every* $i \in [m]$ *such that* $\widehat{\pi}_i \cap \widehat{\pi}_j = \emptyset$ *for every distinct* $i, j \in [m]$*. We call* $\{\widehat{\pi}_i : i \in [m]\}$ *a* system of disjoint representatives *for* $\{\widehat{\Pi}_i : i \in [m]\}$*.*

The remaining part of this section is divided into two subsections. The first subsection contains a proof of Lemma 43 and the second subsection contains a proof of Lemma 14 based upon Lemma 43 and Fact 44.

5.1 Proof of Lemma 43

This whole subsection is devoted to the proof of Lemma 43. Let n, d, A, S, and f be as in the statement of Lemma 43. To prove the lemma, we introduce some terminology and then present an embedding algorithm.

Definition 45 (Dangerous Vertex)**.**

- *We call edges* e_1 *and* e_2 *in* $E(A)$ incompatible *if* $e_1 \in f(e_2)$*.*
- *We call a pair of incident edges* $xy, yz \in E(A)$ *that are incompatible a* useless P_3*. We call* y *the* center vertex *of the useless* P_3 *and the pair* x, z *the* end vertices *of the useless* P_3*.*
- *We call a pair of vertices* $\{u, v\} \in \binom{V(A)}{2}$ a dangerous pair *if* u, v *are the end vertices of at least* $n/6\binom{d}{2}$ *useless* P_3*.*
- *We call a vertex* $v \in V(A)$ a dangerous vertex *if it is in at least* $n/6d^2$ *dangerous pairs.*

We now work to obtain an upper bound for the number of dangerous vertices in A. Recalling that each edge is incompatible with at most $n/6^3d^4$ other edges, the number of useless P_3 is at most

$$\frac{n}{6^3d^4} \cdot \binom{n}{2} \leq \frac{n^3}{2^1 6^3 d^4}.$$

It follows that the number of dangerous pairs of vertices is at most

$$\frac{n^3}{2^1 6^3 d^4} \cdot \frac{6\binom{d}{2}}{n} \leq \frac{n^2}{2^2 6^2 d^2}.$$

Finally, the number of dangerous vertices is at most

$$2 \cdot \frac{n^2}{2^2 6^2 d^2} \cdot \frac{6d^2}{n} \leq \frac{n}{12}. \tag{13}$$

Set J_0 to be the set of dangerous vertices in A.

Definition 46 (Guilty Vertex)**.** *Suppose S' is an induced subgraph of S, A' is an induced subgraph of A, and ϕ' is an embedding of the graph S' into A'.*

- *We call $e \in E(A')$ a* forbidden edge *if $e \in f(\phi'(E(S')))$.*
- *We will call a vertex $v \in \phi'(V(S'))$* guilty by association*, or simply* guilty*, if v is incident to at least $n/6d$ forbidden edges.*

That is, a forbidden edge in A is incompatible with an edge that has already been used in the embedding, and a vertex is guilty by association if it is incident to too many forbidden edges.

Definition 47 (Safe and Legal Embeddings)**.** *Suppose S' is an induced subgraph of S, the graph A' is an induced subgraph of the graph A, and ϕ' is an embedding of the graph S' into the graph A'.*

- *We say that the embedding ϕ' is* legal *if $\phi'(E(S')) \cap f(\phi'(E(S'))) = \emptyset$.*
- *We say vertices s_1, s_2 in S are* P_3-connected *if $s_1v, s_2v \in E(S)$ for some $v \in V(S)$.*
- *We say that the embedding ϕ' is* safe *if none of the pairs $\{\phi'(s_1), \phi'(s_2)\}$ of vertices in A is dangerous for vertices $s_1, s_2 \in V(S')$ that are P_3-connected in S.*

That is, an embedding is legal if it has not used any pair of incompatible edges, and an embedding is safe if for each $s \in S$ and any pair of vertices $s_1, s_2 \in N(s)$, the embedding ϕ' has not mapped s_1 and s_2 onto a dangerous pair of vertices.

Before formally stating our embedding algorithm, we present the main idea, which is as follows. We keep a set J of 'jailed' vertices. We initially send all the dangerous vertices to jail. We then construct a legal and safe partial embedding ϕ' of an induced subgraph $S' \subset S$ into $A \setminus J$ by sequentially embedding vertices. As edges are added to the embedding, however, the number of forbidden edges may increase and already embedded vertices may become guilty by association. This is problematic because guilty vertices may prevent the embedding from being extended in a legal manner later. To resolve this, whenever guilty vertices appear in A', we send them to jail and remove them from the embedding. (Therefore, the domain S' of the partial embedding ϕ' may decrease in size as the algorithm progresses.) We will show that not too many vertices end up in jail and that when no guilty vertices are present, a legal and safe embedding can always be augmented to form a larger legal and safe embedding.

Algorithm: Initially take

$$S' := \emptyset, \qquad J := J_0, \qquad A' := A \setminus J,$$

and set $\phi' \colon S' \to A'$ to be the empty function. As we proceed through the algorithm, we will update these sets and this function.

STEP 1: If there exists a vertex $v \in \phi'(V(S'))$ that is guilty in the current embedding, replace J by $J \cup \{v\}$, replace S' by $S' \setminus \{\phi'^{-1}(v)\}$, update the function ϕ' by removing the pair $(\phi'^{-1}(v), v)$, update A' to $A \setminus J$, and repeat STEP 1. Otherwise, go to STEP 2.

STEP 2: Arbitrarily pick a vertex $s \in V(S) \setminus V(S')$ and extend ϕ' to s by mapping s to some vertex $v \in V(A') \setminus \phi(V(S'))$ so that the new embedding is both legal and safe. Also, replace S' by $S' \cup \{s\}$ and add (s, v) to ϕ'. If $S' = S$, terminate the algorithm; otherwise, go to STEP 1.

We make the following observations about this algorithm:

- Once a vertex is placed into J, it will always remain in J.
- The set of dangerous pairs and the set of dangerous vertices are both fixed from the beginning and do not change.
- Extending an embedding by adding a new vertex (and up to d edges) may make a vertex $v \in \phi'(V(S'))$ guilty.
- At the start of STEP 2, there are no guilty vertices and the current embedding is both legal and safe.

It remains to show that STEP 2 is always possible and that the algorithm will successfully terminate. This will be accomplished by the following two facts.

Proposition 48. *The size of the set J will never reach $n/6$.*

Proof. Towards contradiction, consider the first moment in the execution of the algorithm at which $|J| = n/6$. Let B be the set of edges that were forbidden at any point in time up to this stopping point. That is, B is the set of edges that appeared in $f(\phi'(E(S')))$ for any partial embedding ϕ' the algorithm considered over its run time. We will reach a contradiction by considering the size of B.

To obtain an upper bound for the size of B, notice that whenever a vertex was added to the embedding, up to d edges were added to the embedding as well, and thus at most $d \cdot n/(6^3 d^4)$

forbidden edges were added to B for each vertex embedded. Since the number of vertices added to the embedding is at most

$$|J| - |J_0| + |S| \le \frac{n}{6} + \frac{n}{6} \le \frac{n}{3},$$

it follows that

$$|B| \le \frac{n}{3} \cdot d \cdot \frac{n}{6^3 d^4} \le \frac{n^2}{6^3 d^3}. \tag{14}$$

We now obtain a lower bound for the size of B. Notice that each guilty vertex that was added to J was incident to at least $n/6d$ forbidden edges in A'. Moreover, since vertices in J remain in J, this set of $n/6d$ forbidden edges will never again appear in A'. This gives

$$|B| \ge (|J| - |J_0|) \cdot \frac{n}{6d} \ge \left(\frac{n}{6} - \frac{n}{12}\right) \cdot \frac{n}{6d} = \frac{n^2}{72d}. \tag{15}$$

Equalities (14) and (15) yield the contradiction

$$\frac{n^2}{72d} \le |B| \le \frac{n^2}{6^3 d^3},$$

completing the proof of Proposition 48. □

Proposition 49. *STEP 2 is always possible.*

Proof. Arbitrarily pick a vertex $s \in V(S) \setminus V(S')$ to extend the embedding to. We must find a vertex $v \in A'$ so that extending ϕ' to include the pair (s, v) will produce an embedding that is both legal and safe. We will now list six cases in which such a vertex $v \in A$ will not produce an embedding that is both legal and safe. Cases 1, 2, and 3 correspond to the map not being an embedding into A'; Case 4 corresponds to the embedding using an edge incompatible with an edge already used (and thus not being legal); Case 5 corresponds to the embedding using two new edges that are incompatible with each other (and thus not being legal); and Case 6 corresponds to the embedding not being safe.

1. The vertex v belongs to $\phi'(S')$.
2. The vertex v belongs to J.
3. For some $s' \in S'$ with $ss' \in E(S)$, the edge $\phi(s')v$ is not in $E(A)$.
4. For some $s' \in S'$ with $ss' \in E(S)$ and $e' \in E(S')$, the edge $\phi(s')v$ is in $f(\phi(e'))$.
5. For some $s_1, s_2 \in S'$ with $ss_1, ss_2 \in E(S)$, the edges $\phi(s_1)v$ and $\phi(s_2)v$ are incompatible.
6. For some $s' \in S'$ that is P_3-connected in S to s, the pair $\{\phi'(s'), v\}$ is dangerous.

Observe that if none of (1)–(6) holds, then extending ϕ to include (s, v) will produce an embedding that is both legal and safe.

The number of vertices in A in Cases 1 and 2 is at most

$$|S| + |J| \leq \left(\frac{n}{6} - 1\right) + \frac{n}{6} \leq \frac{2n}{6} - 1.$$

To count the number of vertices in A in Case 3, observe that s has at most d neighbors in S'. Hence, there are at most d choices for s'. Also, from hypothesis each s' is not adjacent to at most $n/6d$ vertices. Hence, the number of vertices in Case 3 at most

$$d \cdot \frac{n}{6d} \leq \frac{n}{6}.$$

Similarly, to count the number of vertices in A in Case 4, again recall that s has at most d neighbors in S'. Also for each such neighbor s', it follows from the fact that $\phi'(s')$ is not guilty by association that $\phi(s')$ is incident to at most $n/6d$ forbidden edges. Hence, the total number of vertices in Case 4 is at most

$$d \cdot \frac{n}{6d} = \frac{n}{6}.$$

To count the number of vertices in A in Case 5, observe that there are are at most $\binom{d}{2}$ choices for s_1 and s_2, and for any choice of s_1, s_2, since the embedding is safe, there are at most $n/6\binom{d}{2}$ vertices v that are part of a useless P_3 with $\phi'(s_1)$ and $\phi'(s_2)$. Hence the total number of vertices in Case 5 is at most

$$\binom{d}{2} \cdot \frac{n}{6\binom{d}{2}} \leq \frac{n}{6}.$$

Finally, to count the number of vertices that are in Case 6, observe that in the graph S, the vertex s is distance two away from at most d^2 other vertices. Since each of the images of these vertices is not dangerous, the images are each in at most $n/6d^2$ dangerous pairs. Hence, the total number of vertices $v \in A$ that are in Case 6 is at most

$$d^2 \cdot \frac{n}{6d^2} = \frac{n}{6}.$$

In conclusion, there must be at least

$$n - \left(\frac{2n}{6} - 1\right) - 4 \cdot \frac{n}{6} \geq 1$$

vertices $v \in A$ such that the map obtained by extending ϕ' to include (s, v) will produce both a legal and safe embedding. Proposition 49 is proved. □

This concludes the proof of Lemma 43.

5.2 Proof of Lemma 14

Consider any pair of positive integers h and d. We will make use of the following simple fact.

Fact 50. *For every $\nu > 0$ there exist $\delta > 0$ and n_6 such that for every integer $n \geq n_6$ the following holds. If A is a graph on n vertices with at least $(1-\delta)\binom{n}{2}$ edges, then there exists a subgraph $\widehat{A}$ with $|V(\widehat{A})| \geq (1-\nu)n$ and with minimum degree at most $(1-\nu)|V(\widehat{A})|$.*

With $\nu := 1/6d$, choose δ and n_6 in accordance with the previous fact. Choose $n_3 \geq n_6$ so that the second inequality in (17) below is satisfied for all $n \geq n_3$. Now consider any $n \geq n_3$, any $J \in \mathcal{J}(h, n, \delta)$, and any graph S with $|V(S)| = n(\log n)^{-7h}$ and $\Delta(S) \leq d$. We must show that $S^{(h)} \subseteq J$.

As at the beginning of Section 5, define the auxiliary graph A by

$$V(A) := X_1 \quad \text{and} \quad E(A) := \{uv : u \text{ and } v \text{ are } (\log n)\text{-path connected in } J\}.$$

Let $\widehat{A}$ be a subgraph of A on $\widehat{n}$ vertices such that $\widehat{n} \geq n/2$ and every vertex in $\widehat{A}$ has degree at least $(1 - 1/6d)\widehat{n}$, guaranteed by Fact 50. Also, for each $uv \in E(\widehat{A})$, let Π_{uv} be a fixed set of $\log n$ transversal paths between u and v in J that are pairwise edge-disjoint. As before, we say that a pair of distinct edges $e_1, e_2 \in E(A)$ are *incompatible* if there exist paths $\pi_{e_1} \in \Pi_{e_1}$ and $\pi_{e_2} \in \Pi_{e_2}$ such that π_{e_1} and π_{e_2} have an edge in common and define

$$f(e_1) := \{e_2 : e_1 \text{ and } e_2 \text{ are incompatable}\}.$$

We will use Lemma 43 to embed S into $\widehat{A}$. With the set-up above, all the hypotheses other than (v) in Lemma 43 are clearly satisfied. To verify (v), observe that, since J has maximum degree $(\log n)^3 n^{1/(h+1)}$, the number of transversal paths any edge $e \in E(J)$ can be in is at most

$$\left((\log n)^3 n^{1/(h+1)}\right)^h \leq (\log n)^{3h} n^{h/(h+1)}. \tag{16}$$

Moreover, since for every $e \in E(A)$ the family Π_e has exactly $\log n$ edge-disjoint paths,

$$f(e) \leq (\log n) \cdot (h+1) \cdot (\log n)^{3h} n^{h/(h+1)} < \frac{n/2}{6^3 d^4}, \tag{17}$$

where the second inequality follows from $n \geq n_3$. Thus, by Lemma 43, there exists an embedding ϕ of S into $\widehat{A}$ such that the image of $E(S)$ under ϕ contains no pair of incompatible edges.

Finally, to select a system of internally pairwise vertex-disjoint paths from the families $\{\Pi_{\phi(e)} : e \in E(S)\}$, the result of Aharoni and Haxell (Fact 44) will be used. Take $X := \bigcup_{i=2}^{h+1} X_i$, and set

$$\widehat{\Pi}_e := \{V(\pi) \cap X : \pi \in \Pi_e\},$$

so that each element in $\widehat{\Pi}_e$ is a set of vertices in X that corresponds to the interior of a path in Π_e. Thus a system of disjoint representatives for the set of families $\{\widehat{\Pi}_{\phi(e)} : e \in E(S)\}$ corresponds to an embedding of $S^{(h)}$ into J. Clearly,

$$|\{\widehat{\Pi}_{\phi(e)} : e \in E(S)\}| = |E(S)| \leq dn(\log n)^{-7} \leq n(\log n)^{-6}. \tag{18}$$

We claim that the hypothesis of Fact 44 holds. Towards contradiction, assume that there exists a set $\mathcal{L}$ of $L \leq n(\log n)^{-6}$ edges in $\phi(E(S)) \subseteq A$ such that there are at most $h(L-1)$ pairwise disjoint h-sets in $\bigcup_{l\in\mathcal{L}} \widehat{\Pi}_l$. Let Γ be a maximum set of pairwise disjoint h-sets in $\bigcup_{l\in\mathcal{L}} \widehat{\Pi}_l$. Let Z be the vertices in Γ. Observe

$$|Z| \leq h(L-1) \cdot h \leq h^2 L.$$

However, one may check that $\bigcup_{l\in\mathcal{L}} \Pi_l$ is an $(\mathcal{L}, Z, h, \log n)$-cluster of paths in the graph J. This contradicts the fact that J is (h, n)-cluster free (property (iv) in Definition 13). This contradiction establishes that the hypothesis of the Aharoni–Haxell theorem holds, and therefore the set of families $\{\widehat{\Pi}_{\phi(e)} : e \in E(S)\}$ has a set of disjoint representatives, yielding an embedding of $S^{(h)}$ into J. This completes the proof of Lemma 14.

6 Proof of Theorem 3

For brevity, we shall refer to graphs on n vertices that have maximum degree at most d as *(n, d)-graphs*. In this section, we show that if H is a graph that contains a copy of $S^{(h)}$ for every (n, d)-graph S, then H has at least $n^{1+1/(h+1)-2/d(h+1)+o(1)}$ edges. Hence, for fixed integers $h \geq 1$ and $d \geq 2$,

$$\mathrm{USR}(h, d, 1, n) \geq n^{1+1/(h+1)-2/d(h+1)+o(1)},$$

which is the statement in Theorem 3.

The proof is based upon the following external lemma.

Fact 51 ([4], Corollary II.4.17, p. 52)**.** *Let $d \geq 2$ be a fixed integer and suppose that dn is even. The number $L_d(n)$ of d-regular graphs on n labeled vertices satisfies*

$$L_d(n) = (1 + o(1))\sqrt{2}e^{-(d^2-1)/4} \left(\frac{d^{d/2}}{e^{d/2}d!}\right)^n n^{dn/2}.$$

Proof of Theorem 3. Let $L_{\leq d}(n)$ be the number of labeled (n, d)-graphs (recall that (n, d)-graphs have maximum degree at most d). Fact 51 gives that, for any fixed $d \geq 2$,

$$L_{\leq d}(n) \geq 2^{(d/2+o(1))n \log n}. \tag{19}$$

We now let $U_{\leq d}(n)$ be the number of *unlabeled* (n, d)-graphs, and let $U^{(h)}_{\leq d}(n)$ be the number of unlabeled h-subdivisions of such graphs.

We claim that

$$U_{\le d}^{(h)}(n) \ge 2^{(d/2-1+o(1))n\log n}. \tag{20}$$

Indeed, first observe that, from (19), we have

$$U_{\le d}(n) \ge \frac{1}{n!}\cdot 2^{(d/2+o(1))n\log n} \ge \frac{1}{n^n}\cdot 2^{(d/2+o(1))n\log n} \ge 2^{(d/2-1+o(1))n\log n}.$$

Second, observe that if two distinct unlabeled (n,d)-graphs S_1 and S_2 both have each edge subdivided h times, then the resulting graphs $S_1^{(h)}$ and $S_2^{(h)}$ are distinct unlabeled graphs. Together, these observations establish (20).

To complete the proof of Theorem 3, we use the fact that if H is a graph on m edges that contains a copy of every unlabeled h-subdivision of (n,d)-graphs, then it must be the case that

$$\sum_{i=0}^{nd(h+1)/2}\binom{m}{i} \ge U_{\le d}^{(h)}(n) \ge 2^{(d/2-1+o(1))n\log n}. \tag{21}$$

If $m \le nd(h+1)$, then the left hand side of (21) is at most $2^{nd(h+1)}$, which yields a contradiction to the inequality in (21). We therefore suppose that $m \ge nd(h+1)$. Then, using that every binomial coefficient in (21) is at most $\binom{m}{nd(h+1)/2}$ and that $\binom{n}{a} \le (en/a)^n$, we have

$$\sum_{i=0}^{nd(h+1)/2}\binom{m}{i} \le \frac{1}{2}nd(h+1)\cdot\left(\frac{em}{nd(h+1)/2}\right)^{nd(h+1)/2}. \tag{22}$$

From equations (21) and (22), we have

$$\frac{1}{2}nd(h+1)\cdot\left(\frac{em}{nd(h+1)/2}\right)^{nd(h+1)/2} \ge 2^{(d/2-1+o(1))n\log n},$$

or, equivalently,

$$\left(\frac{m}{n}\right)^{nd(h+1)/2} \ge 2^{(d/2-1+o(1))n\log n}.$$

This implies that

$$\frac{m}{n} \ge 2^{(1/(h+1)-2/((h+1)d)+o(1))\log n},$$

giving the desired bound of

$$m \ge n^{1+1/(h+1)-2/((h+1)d)+o(1)}.$$

□

7 Proof Sketch of Theorem 4

To prove Theorem 4, we must show that for any integers h, d and ℓ, there exists a constant q_0 such that the following holds. If Q is a graph of maximum degree at most d on $q \geq q_0$ vertices with the property that every pair of vertices of degree greater than 2 are distance at least $h+1$ apart, then $\widehat{r}_\ell(Q) \leq (\log q)^{20h} q^{1+1/(h+1)}$.

To accomplish this, we first define the 'super-subdivision' of a graph. We then show that for any graph Q as in Theorem 4, there exists a graph S such that the super-subdivision of S contains Q as a subgraph. It will then suffice to demonstrate how our main Theorem 2 concerning subdivisions can be extended to super-subdivisions.

Definition 52 (Super-subdivision $S^{(*)}$)**.** *Give a graph S and integers h and d, we define the (h,d)-super-subdivision $S^{(*)}$ of S to be the graph obtained by replacing each edge uv in S by a system of $d(h+1)$ mutually internally vertex-disjoint paths from u to v, of which exactly d paths have length k for each $k \in \{h+1, h+2, \ldots, 2h+1\}$.*

As the reader will have noticed, our notation $S^{(*)}$ for the (h,d)-super-subdivision of S does not contain the parameters h and d explicitly. This will not cause any confusion, as these two parameter will always be fixed in our discussion. In fact, we shall always use the simpler term *super-division* in lieu of (h,d)-super-subdivision. Also, notice that $|E(S^{(*)})| = d((3h+2)/2)|E(S)|$.

Proposition 53. *Let Q be any graph with $|V(Q)| = q$ and $\Delta(Q) \leq d$ with the property that every two vertices of degree greater than 2 are distance at least $h+1$ apart. Then there exists a graph S with $|V(S)| \leq q$ and $\Delta(S) \leq d$ such that $Q \subset S^{(*)}$.*

Proof. For vertices $x_1, x_2 \in Q$, let $\mathrm{dist}_Q(x_1,x_2)$ be the minimum number of edges in a path with endpoints x_1 and x_2. Let X be a maximal subset of vertices in Q that satisfies both of the following properties:

- All vertices of degree greater than 2 are contained in X.
- All pairs of vertices $x_1, x_2 \in X$ satisfy $\mathrm{dist}_Q(x_1,x_2) > h$.

Now construct a graph S by taking $V(S) = X$ and joining vertices $x_1, x_2 \in S$ if $\mathrm{dist}_Q(x_1,x_2) < 2h+2$. It follows that $\Delta(S) \leq \Delta(Q)$ and that $Q \subseteq S^{(*)}$. □

In view of Proposition 53, to establish Theorem 4 it suffices to establish the following lemma.

Lemma 54. *For any $h, d, \ell \in \mathbb{Z}^+$, there exists a constant s_0 such that for every graph S with $|V(S)| = s \geq s_0$ and $\Delta(S) \leq d$,*

$$\widehat{r}_\ell(S^{(*)}) \leq (\log s)^{20h} s^{1+1/(h+1)}.$$

To prove Lemma 54, we consider another way of obtaining the super-subdivision $S^{(*)}$ from the graph S. Begin by fixing a proper edge coloring $\chi : E(S) \to [d+1]$, which exists since $\Delta(S) \leq d$.

For integers $i \in [d+1]$, $j \in [d]$, and $k \in \{h+1, h+2, \ldots, 2h+1\}$, let $M_{i,j,k} := \chi^{-1}(i)$; it follows that $M_{i,j,k} = M_{i,j',k'}$ for all $j, j' \in [d]$ and $k, k' \in \{h+1, h+2, \ldots, 2h+1\}$. Define the multiset of matchings

$$\mathcal{M} := \big\{M_{i,j,k} : i \in [d+1],\ j \in [d],\ k \in \{h+1, h+2, \ldots, 2h+1\}\big\}.$$

We construct $S^{(*)}$ on $V(S)$ by the following procedure. For every $M_{i,j,k} \in \mathcal{M}$ and every $xy \in M_{i,j,k}$, add a path of length k between x and y. Consequently, for any $xy \in E(S)$, there are d paths of length k between x and y for each $k \in \{h+1, h+2, \ldots, 2h+1\}$. It follows that the resulting graph is the super-subdivision $S^{(*)}$ of S.

Since the full proof is notationally cumbersome, we first demonstrate the main ideas in the context of two propositions that allows for simpler notation. These propositions consider the simpler case where the multiset $\mathcal{M}$ of multiple matchings is replaced by a pair of matchings.

Definition 55 ($S^{(M_1,M_2,k_1,k_2)}$)**.** *Let S be a graph and $M_1, M_2 \subset E(S)$ be not necessarily disjoint matchings with $M_1 \cup M_2 = E(S)$. Let k_1 and k_2 be integers. Define $S^{(M_1,M_2,k_1,k_2)}$ to be the graph on $V(S)$ obtained by adding a path of length k_1 between x and y for every edge $xy \in M_1$ and a path of length k_2 between x and y for every edge $xy \in M_2$. (Since M_1 and M_2 need not be disjoint, some edges in $E(S)$ may be replaced by two paths.)*

Proposition 56. *For any $h, \ell \in \mathbb{Z}^+$, there exists a constant s_0 such that if S is a graph with $|V(S)| = s \geq s_0$ and M_1 and M_2 are matchings such that $M_1 \cup M_2 = E(S)$, then*

$$\widehat{r}_\ell(S^{(M_1,M_2,h+1,h+2)}) \leq (\log s)^{20h} s^{1+1/(h+1)}.$$

Proof. We will make three claims that are similar to the Coloring Lemma, Existence Lemma, and Embedding Lemma used in the proof of Theorem 2. Before stating the first of these claims, we introduce a couple definitions, the second of which is demonstrated in Figure 3.

Definition 57 ($C_{h+1,h+2}$)**.** *Let $C_{h+1,h+2}$ be the graph on $2h+2$ vertices obtained from a copy of the cycle C_{h+1} with cyclically ordered vertices $x_1^1, x_2^1, \ldots, x_{h+1}^1$ and a copy of the cycle C_{h+2} with cyclically ordered vertices $x_1^2, x_2^2, \ldots, x_{h+2}^2$ and with $x_1 := x_1^1 = x_1^2$.*

Definition 58 (Incomplete Blowup of $C_{h+1,h+2}$)**.** *An incomplete blowup H of $C_{h+1,h+2}$ is obtained by replacing each vertex x_j^i with a independent set X_j^i of n vertices and each edge by a (not necessarily complete) bipartite graph. Also, define $H^1 := H[\bigcup_{\alpha\in[h+1]} X_\alpha^1]$ and $H^2 := H[\bigcup_{\alpha\in[h+2]} X_\alpha^2]$.*

Recall that in the proof of Theorem 2 the class $\mathcal{H}(h, n, \varepsilon, q)$ was the set of incomplete blowups of C_{h+1} in which the bipartite graphs had exactly qn^2 edges and were (ε, q)-regular (as in Definition 21). We now define an analogous concept.

Definition 59. *Let $\mathcal{H}^*(h, n, \varepsilon, q)$ be the set of all graphs that are incomplete blowups of $C_{h+1,h+2}$ where every edge in $C_{h+1,h+2}$ corresponds to an (ε, q)-regular bipartite graph with exactly qn^2 edges.*

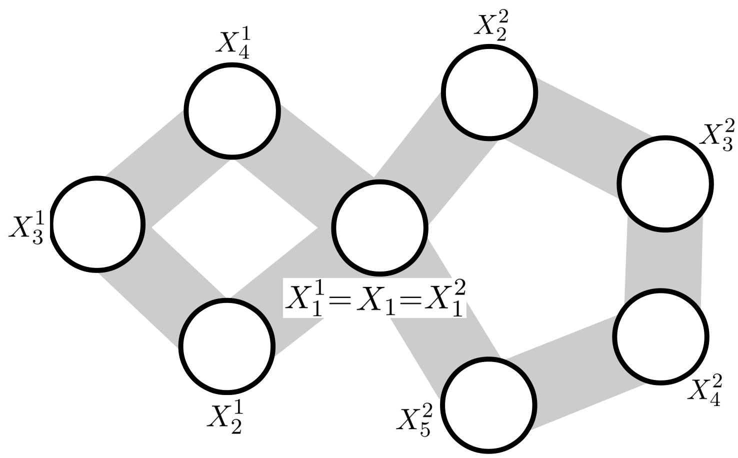

Figure 3: An incomplete blowup of $C_{h+1,h+2}$ for $h = 3$.

The next claim is analogous to the Coloring Lemma.

Claim 60. *For any $\varepsilon \in \mathbb{R}^+$ and $h, \ell \in \mathbb{Z}^+$, there exist $t, n_1 \in \mathbb{Z}^+$ such that for all $n \geq n_1$,*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

every graph $G \in \mathcal{I}(N, p)$ has the following property. Any ℓ-coloring of the edges of G yields a monochromatic subgraph $H \in \mathcal{H}^*(h, n, \varepsilon, q)$.

Proof. In the proof of the Coloring Lemma (Lemma 9), we defined a cluster graph that had vertices corresponding to the vertex classes obtained from an application of the Regularity Lemma and edges corresponding to pairs that exhibited regularity. The edges of the cluster graph were ℓ-colored by the majority color in the corresponding partition. We previously argued that the cluster graph contained a monochromatic clique of size $h+1$ (and hence a copy of C_{h+1}). By taking t sufficiently larger and an appropriate modification of the parameters in the proof, we can instead find a monochromatic clique of size $2h+2$, and hence a copy of $C_{h+1,h+2}$. This will yield a monochromatic $H \in \mathcal{H}^*(h, n, \varepsilon, q)$. □

Our next claim will be analogous to the Existence Lemma. To state it, we first need a modified notion of path abundance.

Definition 61 (Transversal Paths for $\mathcal{H}^*$)**.** *Let H be a partial blowup of $C_{h+1,h+2}$.*

- *For a pair of vertices $u, v \in X_1^1$,* a transversal path *between u and v in H^1 is the same as described in Definition 10.*
- *For a pair of vertices $u \in X_1^2$ and $v \in X_{h+2}^2$,* a transversal path between u and v *in H^2 is a path P of length $h+1$ with exactly one vertex in X_i^2 for each $i \in [h+2]$.*

Definition 62 (Path Abundance for $\mathcal{H}^*$)**.** *Let H be a partial blowup of $C_{h+1,h+2}$. We say that the graph H is $(1-\delta, \log n)$*-path abundant *if both of the following hold:*

- *The graph H^1 is path abundant (as defined in Definition 10).*
- *The graph H^2 has the property that for at least $(1-\delta)n^2$ pairs of vertices $u \in X_1^2$ and $v \in X_{h+2}^2$, there are at least $\log n$ transversal paths between u and v that are pairwise edge-disjoint (as defined in Definition 61).*

We now state the next claim that is analogous to the Existence Lemma.

Claim 63. *For all $h, \ell \in \mathbb{Z}^+$ and $\delta \in \mathbb{R}^+$, there exists $\varepsilon \in \mathbb{R}^+$ such that for any $t \in \mathbb{Z}^+$ there exists $n_2 \in \mathbb{Z}$ such that the following holds. For any $n \geq n_2$ and*

$$q := 4(\log n)^2 n^{-1+1/(h+1)}, \qquad N := tn, \qquad \text{and} \qquad p := 4\ell q,$$

there exists a graph G on N vertices satisfying all of the following properties:

(i) Every vertex in G has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) G is (h, n)-cluster free.

(iii) $G \in \mathcal{I}(N, p)$.

(iv) Every (not necessarily induced) subgraph $H \in \mathcal{H}^*(h, n, \varepsilon, q)$ of G is $(1-\delta, \log n)$-path abundant.

Proof. Properties (i)–(iii) are the same as in the Existence Lemma and the modified notion of path abundance in Property (iv) is proved analogously. □

After stating one more definition, we state a claim analogous to the Embedding Lemma.

Definition 64. *Let $\mathcal{J}^*(h, n, \delta)$ be the set of all graphs J that are partial blowups of $C_{h+1,h+2}$ such that:*

(i) Every vertex in J has degree at most $(\log n)^3 n^{1/(h+1)}$.

(ii) J is (n, h)-cluster free (as defined in Definition 11).

(iii) J is $(1-\delta, \log n)$-path abundant (as defined in Definition 62).

(iv) There is a matching of size $(1-\delta)n$ between X_{h+2}^2 and X_1^2.

As in the proof of Theorem 2, the Coloring Lemma and Existence Lemma together yield a monochromatic $H \in \mathcal{J}^*(h, n, \delta)$. Note that the additional Property (iv) follows from the fact that $H \in \mathcal{H}^*(h, n, \varepsilon, q)$ and hence the bipartite graph of H induced between X_{h+2}^2 and X_1^2 is (ε, p)-regular. The next claim is analogous to the Embedding Lemma.

Claim 65. *For all $h \in \mathbb{Z}^+$, there exist $\delta \in \mathbb{R}^+$ and $n_3 \in \mathbb{Z}^+$ such that for all $n \geq n_3$ the following holds. Every graph $H \in \mathcal{J}^*(h, n, \delta)$ is universal to the set of graphs*

$$\left\{S^{(M_1,M_2,h+1,h+2)} : |V(S)| = \frac{n}{(\log n)^{7h}}\right\}.$$

Proof. The proof of this claim follows the lines of the argument used to establish the Embedding Lemma where $S^{(h)}$ was embedded into $J \in \mathcal{J}(h, n, \delta)$. Recall that the main steps in this argument were:

- Considering an auxiliary graph A with vertex set X_1 where vertices $x, y \in X_1$ were joined if x and y were path connected (i.e., if there was a set Π_{xy} of $\log n$ edge-disjoint transversal paths between x and y).
- Defining an incompatibility function $f : E(A) \to \mathcal{P}(E(A))$ where each edge was incompatible with certain other edges.
- Finding an embedding ϕ of S into A such that $f(\phi(E(S))) \cap \phi(E(S)) = \emptyset$.
- Showing that for every edge $xy \in \phi(E(S))$, a path $\pi_{xy} \in \Pi_{xy}$ could be selected so that the set of paths selected $\{\pi_{xy} : xy \in \phi(E(S))\}$ were pairwise internally vertex-disjoint. This corresponded to embedding $S^{(h)}$ into J.

The proof of Claim 65 is similar, so we only mention where it differs. We begin by fixing a matching Γ between X^2_{h+2} and X_1 of size at least $(1-\delta)n$. For a vertex $v \in X_1$, denote the vertex it is matched to in X^2_{h+2} under Γ by $\widehat{v}$. Now fix an ordering $v_1, v_2, \ldots, v_n$ of the vertices in X_1. Given this setup, we introduce the following definition.

Definition 66 (Path Linked)**.** *For $i < j$, the vertices $v_i, v_j \in X_1$ are* path linked *in H^2 (see Definition 58) if v_i and $\widehat{v}_j$ are path connected (i.e., if there exists a set Π_{ij} of $\log n$ edge-disjoint transversal paths between v_i and $\widehat{v}_j$). If v_j is not incident to an edge in Γ, then $\widehat{v}_j$ is not defined and v_i and v_j are not path linked. This concept is illustrated in Figure 4.*

Observe that since most pairs of vertices $v_i \in X_1$ and $\widehat{v}_j \in X^2_{h+2}$ are path connected, most pairs of vertices $v_i, v_j \in X_1$ are path linked. Now, for all path linked pairs $v_i \in X_1$ and $v_j \in X_1$, fix a set Π^2_{ij} of $\log n$ edge-disjoint transversal paths between v_i and $\widehat{v}_j$ in H^2. Also, as in the original proof, fix a set Π^1_{ij} of edge-disjoint transversal paths in H^1 for all path linked pairs $v_i \in X_1$ and $v_j \in X_1$. The proof now continues to follow the lines of the argument used to establish the Embedding Lemma with the following modifications:

- Define A by joining two vertices if and only if they are path connected in H^1 and path linked in H^2. Observe that, as before, A will be an 'almost complete' graph.

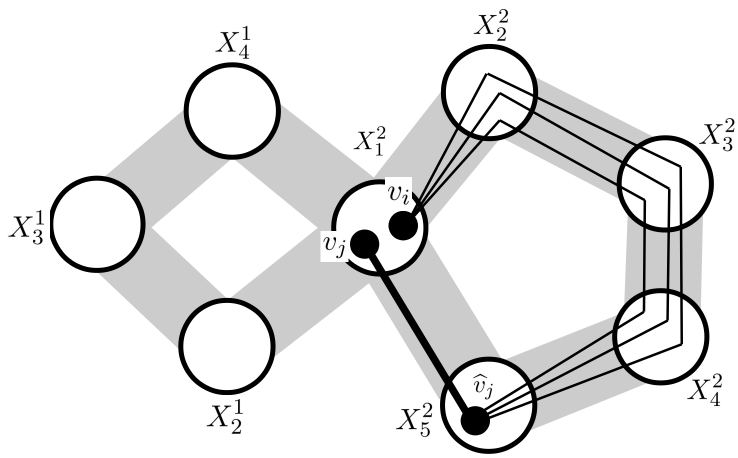

Figure 4: Vertices $v_i, v_j \in X_1^2$ are path linked in H^2 if there are many edge-disjoint paths between v_i and $\widehat{v}_j$.

- Define the edges v_iv_j and v_kv_l in A to be incompatible if either of the following two conditions are met:
  - There exist paths $\pi_{ij} \in \Pi_{ij}^1$ and $\pi_{kl} \in \Pi_{kl}^1$ such that π_{ij} and π_{kl} have an edge in common. (This is the same notion of incompatibility as used in the proof of the Embedding Lemma.)
  - There exist paths $\pi_{ij} \in \Pi_{ij}^2$ and $\pi_{kl} \in \Pi_{kl}^2$ such that π_{ij} and π_{kl} have an edge in common.
- As before, we find an embedding ϕ of S into A such that $f(\phi(E(S))) \cap \phi(E(S)) = \emptyset$. This is possible since S has bounded degree, the graph A is almost complete, and each edge is still incompatible with at most $o(n)$ other edges.
- Finally, for each edge $xy \in \phi(M_2)$, we select a path $\pi_{xy} \in \Pi_{xy}^2$ of length $h+1$ so that the sets of paths chosen $\{\pi_{xy} : xy \in \phi(M_2)\}$ are pairwise vertex-disjoint. Appending the appropriate matching edge in Γ to each path gives the desired set of paths of length $h+2$ in H^2. The paths of length $h+1$ are found in H^1 in the same manner as in our previous proof, considering M_1.

This completes the proof of Claim 65. □

We have now proved three claims analogous to the Coloring Lemma, Existence Lemma, and Embedding Lemma. The proof of Proposition 56 now follows the lines of the proof of Theorem 2. □

Our second proposition describes how the situation changes if the edges in the matching M are divided one additional time.

Proposition 67. *For any $h, \ell \in \mathbb{Z}^+$, there exists a constant s_0 such that every graph S with $|V(S)| = s \geq s_0$ satisfies*

$$\widehat{r}_\ell(S^{(M_1,M_2,h+1,h+3)}) \leq (\log s)^{20h} s^{1+1/(h+1)}.$$

Proof. The proof of this proposition differs from the previous proof as follows. In place of $C_{h+1,h+2}$, we take $C_{h+1,h+3}$, where the vertices are labeled $x_1^1, x_2^1, \ldots, x_{h+1}^1$ and $x_1^2, x_2^2, \ldots, x_{h+3}^2$ with $x_1 := x_1^1 = x_1^2$. We also require that 'almost perfect matchings' exist in both of the bipartite graphs (X_{h+1}^2, X_{h+2}^2) and (X_{h+2}^2, X_1^2).

We now begin the embedding process by fixing two such perfect matchings. These matchings together yield a collection of disjoint paths P_3 on three vertices that cover almost all vertices in $X_{h+1}^2 \cup X_{h+2}^2 \cup X_1^2$. For a vertex $v \in X_1$ which is covered by one of these paths of length two, define the vertex $\widehat{v} \in X_{h+1}^2$ to be the corresponding vertex it is joined to in X_1^2 under our fixed collection of P_3s. The remaining part of the proof is analogous to the proof of Claim 65. □

Having demonstrated the main idea of Lemma 54 in Propositions 56 and 67, we now briefly remark on how the proof of Lemma 54 differs.

Proof of Lemma 54. Previously in Propositions 56 and 67, the two matchings were accommodated by replacing C_{h+1} by $C_{h+1,h+2}$ and $C_{h+1,h+3}$ respectively. Here, we will 'append' a cycle of length k for each of the matchings $M_{i,j,k} \in \mathcal{M}$. More formally, let C_* be the graph obtained by the following process. Take $d(d+1)$ disjoint cycles of each of the lengths $k \in \{h+1, h+2, \ldots, 2h+1\}$, for a total of $d(d+1)(h+1)$ cycles. From these cycles, C_* results by identifying one common vertex from all the cycles.

Propositions 56 and 67 has already demonstrated the main ideas involved embedding matchings in two cycles simultaneously. These ideas easily generalize to $d(d+1)(h+1)$ matchings associated to finite lengths of at least $h+1$. □

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