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The Turán Theorem for Random Graphs

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The aim of this paper is to prove a Turán type theorem for random graphs. For $0 < \gamma$ and graphs G and H, write $G \to_\gamma H$ if any γ-proportion of the edges of G spans at least one copy of H in G. We show that for every graph H and every fixed real $\delta > 0$ almost every graph G in the binomial random graph model $\mathcal{G}(n,q)$, with $q = q(n) \gg ((\log n)^4/n)^{1/d(H)}$, satisfies $G \to_{(\chi(H)-2)/(\chi(H)-1)+\delta} H$, where as usual $\chi(H)$ denotes the chromatic number of H and $d(H)$ is the "degeneracy number" of H.

Since K_l, the complete graph on l vertices, is l-chromatic and $(l-1)$-degenerate we infer that for every $l \geq 2$ and every fixed real $\delta > 0$ almost every graph G in the binomial random graph model $\mathcal{G}(n,q)$, with $q = q(n) \gg ((\log n)^4/n)^{1/(l-1)}$, satisfies $G \to_{(l-2)/(l-1)+\delta} K_l$.

CONTENTS

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1. Introduction

A classical area of extremal graph theory investigates numerical and structural problems concerning H-free graphs, namely graphs that do not contain a copy of a given fixed graph H as a subgraph. Let $\mathrm{ex}(n, H)$ be the maximal number of edges that an H-free graph on n vertices may have. A basic question is then to determine or estimate $\mathrm{ex}(n, H)$ for any given H and large n. A solution to this problem is given by the celebrated Erdős–Stone–Simonovits theorem, which states that, as $n \to \infty$, we have

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \binom{n}{2}, \tag{1}$$

where as usual $\chi(H)$ is the chromatic number of H. Furthermore, as proved independently by Erdős and Simonovits, every H-free graph $G = G^n$ that has as many edges as in (1) is in fact 'very close' (in a certain precise sense) to the densest n-vertex $(\chi(H) - 1)$-partite graph. For these and related results, see, for instance, Bollobás [1].

Here we are interested in a variant of the function $\mathrm{ex}(n, H)$. Let G and H be graphs, and write $\mathrm{ex}(G, H)$ for the maximal number of edges that an H-free subgraph of G may have. Formally, $\mathrm{ex}(G, H) = \max\{|E(F)| \colon H \not\subset F \subset G\}$. For instance, if $G = K_n$, the complete graph on n vertices, then $\mathrm{ex}(K_n, H) = \mathrm{ex}(n, H)$ is the usual Turán number of H.

Our aim here is to study $\mathrm{ex}(G, H)$ when G is a *random graph*. Let $0 < q = q(n) \le 1$ be given. The binomial random graph G in $\mathcal{G}(n, q)$ has as its vertex set a fixed set $V(G)$ of cardinality n and two vertices are adjacent in G with probability q. All such adjacencies are independent. (For concepts and results concerning random graphs not given in detail below, see, *e.g.*, Bollobás [2].) Here we wish to investigate the random variables $\mathrm{ex}(\mathcal{G}(n, q), H)$, where $H = K_l$ $(l \ge 2)$ or H is a d-degenerate graph, a graph that may be reduced to the empty graph by the successive removal of vertices of degree less or equal d.

Let H be a graph of order $|H| = |V(H)| \ge 3$. Let us write $d_2(H)$ for the 2-density of H, that is,

$$d_2(H) = \max\left\{\frac{e(H') - 1}{|H'| - 2} \colon H' \subset H,\ |H'| \ge 3\right\}.$$

A general conjecture concerning $\mathrm{ex}(\mathcal{G}(n, q), H)$, first stated in [10], is as follows (as is usual in the theory of random graphs, we say that a property P holds *almost surely* or that *almost every* random graph G in $\mathcal{G}(n,q)$ satisfies P if P holds with probability tending to 1 as $n \to \infty$).

Conjecture 1.1. *Let H be a non-empty graph of order at least 3, and let $0 < q = q(n) \leq 1$ be such that $qn^{1/d_2(H)} \to \infty$ as $n \to \infty$. Then almost every G in $\mathcal{G}(n,q)$ satisfies*

$$\mathrm{ex}(G,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)|E(G)|.$$

In other words, for G in $\mathcal{G}(n,q)$ the Conjecture 1.1 claims that $G \to_\gamma H$ holds almost surely for any fixed $\gamma > 1 - 1/(\chi(H)-1)$. There are a few results in support of Conjecture 1.1.

Any result concerning the tree-universality of expanding graphs, or any simple application of Szemerédi's regularity lemma for sparse graphs (see Theorem 2.2 below), gives Conjecture 1.1 for H a forest. The cases in which $H = K_3$ and $H = C_4$ are essentially proved in Frankl and Rödl [3] and Füredi [4], respectively, in connection with problems concerning the existence of some graphs with certain extremal properties. The case for $H = K_4$ was proved by Kohayakawa, Łuczak, and Rödl [10] and the case in which H is a general cycle was settled by Haxell, Kohayakawa, and Łuczak [5, 6] (see also Kohayakawa, Kreuter, and Steger [9]).

Our main result relates to Conjecture 1.1 in the following way: we deal with the case in which $H = K_l$ and $q = q(n) \gg ((\log n)^4/n)^{1/(l-1)}$. More precisely we prove the following.

Theorem 1.2. *Let $l \geq 2$, $q = q(n) \gg \left((\log n)^4/n\right)^{1/(l-1)}$, and let $\mathcal{G}(n,q)$ be the binomial random graph model with edge probability q. Then for every $1/(l-1) > \delta > 0$ a graph G in $\mathcal{G}(n,q)$ satisfies the following property with probability $1-o(1)$: If F is an arbitrary, not necessarily induced subgraph of G with*

$$|E(F)| \geq \left(1 - \frac{1}{l-1} + \delta\right) q \binom{n}{2},$$

then F contains K_l, the complete graph on l vertices, as a subgraph. Moreover, there exists a constant $c = c(\delta, l)$ such that F contains at least $cq^{\binom{l}{2}}n^l$ copies of K_l.

In this paper we give a proof of Theorem 1.2.† In Section 5 we outline the proof of an extension of this result, Theorem 1.2′ (the detailed proof is given in [14]).

Recall that a graph H with $|V(H)| = h$ is d-degenerate if there exists an ordering of the vertices $v_1, \ldots, v_h$ such that each v_i ($1 \leq i \leq h$) has at most d neighbours in

† Very recently, Szabó and Vu [16] proved independently the same result under a slightly weaker assumption; in fact, they proved Theorem 1.2 for $q(n) \gg n^{-1/(l-3/2)}$. Their proof is elegant. To obtain the smaller lower bound for q, they make use of the fact that Conjecture 1.1 holds for $H = K_4$ [10] as the base of an induction; without using this result, their proof gives essentially the same condition on q as ours. Their approach extends to several infinite families of graphs H (see [16, Section 4]); the present proof extends to all graphs, and works for $q(n) \gg ((\log n)^4/n)^{1/d}$, where $d = d(H)$ is the "degeneracy number" of the graph H; see Theorem 1.2′.

$\{v_1, \ldots, v_{i-1}\}$ (for more details concerning d-degenerate graphs see [13, 15]). Since K_l is clearly $(l-1)$-degenerate and l-chromatic, the following result extends Theorem 1.2.

Theorem 1.2′. *Let d be a positive integer, H a d-degenerate graph on h vertices, $q = q(n) \gg \left((\log n)^4/n\right)^{1/d}$, and $\mathcal{G}(n,q)$ the binomial random graph model with edge probability q. Then for every $1/(\chi(H)-1) > \delta > 0$ a graph G in $\mathcal{G}(n,q)$ satisfies the following property with probability $1-o(1)$: If F is an arbitrary, not necessarily induced subgraph of G with*

$$|E(F)| \geq \left(1 - \frac{1}{\chi(H)-1} + \delta\right) q \binom{n}{2},$$

then F contains H as a subgraph. Moreover, there exists a constant $c = c(\delta, H)$ such that F contains at least $cq^{|E(H)|}n^h$ copies of H.

This paper is organized as follows. In Section 2 we describe a sparse version of Szemerédi's regularity lemma (Theorem 2.2) and we state the counting lemma (Lemma 2.3), which are crucial in our proof of Theorem 1.2. We prove Theorem 1.2 in Section 3. Section 4 is entirely devoted to the proof of Lemma 2.3. The proof of Lemma 2.3 relies on the 'Pick-Up Lemma' (Lemma 4.3) and on the 'k-tuple lemma' (Lemma 4.7). We give these preliminary results in Section 4.1–4.2. In Section 4.3 we outline the proof of Lemma 2.3 in the case $l = 4$. Finally, the proof is given in Section 4.4. We discuss the case when H is a d-degenerate graph and sketch the proof of Theorem 1.2′ in Section 5.

For a general remark about the notation we use throughout this paper see the remark in Section 2.3.

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2. Preliminary results

2.1. Preliminary definitions

Let a graph $G = G^n$ of order $|V(G)| = n$ be fixed. For U, $W \subset V = V(G)$, we write

$$E(U,W) = E_G(U,W) = \Big\{\{u,w\} \in E(G)\colon\; u \in U,\; w \in W\Big\}$$

for the set of edges of G that have one end-vertex in U and the other in W. Notice that each edge in $U \cap W$ occurs only once in $E(U,W)$. We set $e(U,W) = e_G(U,W) = |E(U,W)|$.

If G is a graph and $V_1, \ldots, V_t \subset V(G)$ are disjoint sets of vertices, we write $G[V_1, \ldots, V_t]$ for the t-partite graph naturally induced by $V_1, \ldots, V_t$.

2.2. The regularity lemma for sparse graphs

Our aim in this section is to state a variant of the regularity lemma of Szemerédi [17].

Let a graph $H = H^n = (V,E)$ of order $|V| = n$ be fixed. Suppose $\xi > 0$, $C > 1$, and $0 < q \leq 1$.

Definition 2.1 (**(ξ, C)-bounded**). *For $\xi > 0$ and $C > 1$ we say that $H = (V,E)$ is* *a (ξ, C)-bounded graph with respect to density q, if for all U, $W \subset V$, not necessarily disjoint, with $|U|$, $|W| \geq \xi|V|$, we have*

$$e_H(U, W) \leq Cq\left(|U||W| - \binom{|U \cap W|}{2}\right).$$

For any two *disjoint* non-empty sets U, $W \subset V$, let

$$d_{H,q}(U, W) = \frac{e_H(U, W)}{q|U||W|}. \tag{2}$$

We refer to $d_{H,q}(U, W)$ as the *q-density* of the pair (U, W) in H. When there is no danger of confusion, we drop H from the subscript and write $d_q(U, W)$.

Now suppose $\varepsilon > 0$, U, $W \subset V$, and $U \cap W = \emptyset$. We say that the pair (U, W) is *(ε, H, q)-regular*, or simply *(ε, q)-regular*, if for all $U' \subset U$, $W' \subset W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$ we have

$$|d_{H,q}(U', W') - d_{H,q}(U, W)| \leq \varepsilon. \tag{3}$$

Below, we shall sometimes use the expression *ε-regular with respect to density q* to mean that (U, W) is an (ε, q)-regular pair.

We say that a partition $P = (V_i)_0^t$ of $V = V(H)$ is *(ε, t)-equitable* if $|V_0| \leq \varepsilon n$, and $|V_1| = \cdots = |V_t|$. Also, we say that V_0 is the *exceptional* class of P. When the value of ε is not relevant, we refer to an (ε, t)-equitable partition as a *t-equitable* partition. Similarly, P is an *equitable* partition of V if it is a t-equitable partition for some t.

We say that an (ε, t)-equitable partition $P = (V_i)_0^t$ of V is *(ε, H, q)-regular*, or simply *(ε, q)-regular*, if at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j) with $1 \leq i < j \leq t$ are not (ε, q)-regular. We may now state a version of Szemerédi's regularity lemma for (ξ, C)-bounded graphs.

Theorem 2.2. *For any given $\varepsilon > 0$, $C > 1$, and $t_0 \geq 1$, there exist constants $\xi = \xi(\varepsilon, C, t_0)$ and $T_0 = T_0(\varepsilon, C, t_0) \geq t_0$ such that any sufficiently large graph H that is (ξ, C)-bounded with respect to density $0 < q \leq 1$ admits an (ε, H, q)-regular (ε, t)-equitable partition of its vertex set with $t_0 \leq t \leq T_0$.*

A simple modification of Szemerédi's proof of his lemma gives Theorem 2.2. For applications of this variant of the regularity lemma and its proof, see [8, 12].

2.3. The counting lemma for complete subgraphs of random graphs

Let $t \geq l \geq 2$ be fixed integers and n a sufficiently large integer. Let α and ε be constants greater than 0. Let $G \in \mathcal{G}(n, q)$ be the binomial random graph with edge probability $q = q(n)$, and suppose J is an l-partite subgraph of G with vertex classes $V_1, \ldots, V_l$. For all $1 \leq i < j \leq l$ we denote by J_{ij} the bipartite graph induced by V_i and V_j. Consider the following assertions for J.

(I) $|V_i| = m = n/t$
(II) $q^{l-1}n \gg (\log n)^4$
(III) J_{ij} has $T = pm^2$ edges where $1 > \alpha q = p \gg 1/n$, and
(IV) J_{ij} is (ε, q)-regular.

Remark. Strictly speaking, in (I) we should have, say, $\lfloor m/t \rfloor$, because m is an integer. However, throughout this paper we will omit the floor and ceiling signs $\lfloor \quad \rfloor$ and $\lceil \quad \rceil$, since they have no significant effect on the arguments. Moreover, let us make a few more comments about the notation that we shall use. For positive functions $f(n)$ and $g(n)$, we write $f(n) \gg g(n)$ to mean that $\lim_{n\to\infty} g(n)/f(n) = 0$. Unless otherwise stated, we understand by $o(1)$ a function approaching zero as the number of vertices of a given random graph goes to infinity.

Finally, we observe that our logarithms are natural logarithms.

We are interested in the number of copies of complete graphs on l vertices in such a subgraph J satisfying conditions (I)–(IV).

Lemma 2.3 (Counting lemma). *For every α, $\sigma > 0$ and integer $l \geq 2$ there exists $\varepsilon > 0$ such that for every fixed integer $t \geq l$ a random graph G in $\mathcal{G}(n, q)$ satisfies the following property with probability $1 - o(1)$: Every subgraph $J \subseteq G$ satisfying conditions (I)–(IV) contains at least*

$$(1-\sigma)p^{\binom{l}{2}}m^l$$

copies of the complete graph K_l.

We will prove Lemma 2.3 later in Section 4.

3. The main result

In this section we will prove the main result of this paper, Theorem 1.2. This section is organized as follows. First, we state two properties that hold for almost every $G \in \mathcal{G}(n, q)$. Then, in Section 3.2, we prove a deterministic statement about the regularity of certain subgraphs of an (ε, q)-regular α-dense t-partite graph. Finally, we prove Theorem 1.2.

3.1. Properties of almost all graphs

We start with a well known fact of random graph theory which follows easily from the properties of the binomial distribution.

Fact 3.1. *If G is a random graph in $\mathcal{G}(n, q)$, then*

$$|E(G)| = (1 + o(1))\, q \binom{n}{2}$$

holds with probability $1 - o(1)$.

The next property refers to Definition 2.1 and will enable us to apply Theorem 2.2.

Lemma 3.2. *For every $C > 1$, $\xi > 0$ and $q = q(n) \gg 1/n$ a random graph G in $\mathcal{G}(n, q)$ is (ξ, C)-bounded with probability $1 - o(1)$.*

We will apply the following one-sided estimate of a binomially distributed random variable.

Lemma 3.3. *Let X be a binomial distributed random variable in* $\mathrm{Bi}(N, q)$ *with expectation* $\mathbb{E}X = Nq$ *and let* $C > 1$ *be a constant. Then*

$$\mathbb{P}(X \geq C\mathbb{E}X) \leq \exp(-\tau C\mathbb{E}X),$$

where $\tau = \log C - 1 + 1/C > 0$ *for* $C > 1$ *(recall that all logarithms are to base* e*, see the remark in Section 2.3).*

Proof. The proof is given in [7] (see Corollary 2.4). □

Proof of Lemma 3.2. Let $G \in \mathcal{G}(n, q)$ and let U, $W \subseteq V(G)$ be two not necessarily disjoint sets such that $|U|$, $|W| \geq \xi n$. Clearly, $e(U, W)$ is a binomial random variable with

$$\mathbb{E}[e(U, W)] = q\left(|U||W| - \binom{|U \cap W|}{2}\right).$$

Observe that $\mathbb{E}[e(U, W)] \gg n$ since $q \gg 1/n$. Set $\tau = \log C - 1 + 1/C$. Then Lemma 3.3 implies

$$\mathbb{P}\left(e(U, W) > C\mathbb{E}[e(U, W)]\right) \leq \exp\left(-\tau C\mathbb{E}[e(U, W)]\right).$$

We now sum over all choices for U and W to deduce that

$$\begin{aligned}\mathbb{P}(G \text{ is not } (\xi, C)\text{-bounded}) \leq \\ &\sum_{|U| \geq \xi n} \sum_{|W| \geq \xi n} \binom{n}{|U|}\binom{n}{|W|} \exp\left(-\tau C\mathbb{E}[e(U, W)]\right) \\ &\leq 4^n \exp\left(-\tau C\mathbb{E}[e(U, W)]\right) = o(1),\end{aligned}$$

since $\tau C > 0$ and $\mathbb{E}[e(U, W)] \gg n$. □

3.2. A deterministic subgraph lemma

The next lemma states that every (ε, q)-regular, bipartite graph with at least $\alpha q m^2$ edges contains an $(3\varepsilon, q)$-regular subgraph with exactly $\alpha q m^2$ edges.

Lemma 3.4. *For every* $\varepsilon > 0$, $\alpha > 0$, *and* $C > 1$ *there exists* m_0 *such that if* $H = (U, W; F)$ *is a bipartite graph satisfying*

(i) $|U| = m_1$, $|W| = m_2 > m_0$,
(ii) $Cqm_1m_2 \geq e_H(U, W) \geq \alpha q m_1 m_2$ *for some function* $q = q(m_0) \gg 1/m_0$, *and*
(iii) H *is* (ε, q)*-regular,*

then there exists a subgraph $H' = (U, W; F') \subseteq H$ *such that*

(ii′) $e_{H'}(U, W) = \alpha q m_1 m_2$ *and*
(iii′) H' *is* $(3\varepsilon, q)$*-regular.*

Proof. We select a set D of

$$|D| = e_H(U,W) - \alpha q m_1 m_2$$

different edges in $E_H(U,W)$ uniformly at random and fix $H' = (U,W;F\setminus D)$. We naturally define the density in D with respect to q for sets $U' \subseteq U$ and $W' \subseteq W$ by

$$d_{D,q}(U',W') = \frac{|E_H(U',W') \cap D|}{q|U'||W'|}. \tag{4}$$

In order to check the $(3\varepsilon, H', q)$-regularity of (U,W), it is enough to verify the inequality corresponding to (3) for sets $U' \subseteq U$, $W' \subseteq W$ such that $|U'| = 3\varepsilon m_1$ and $|W'| = 3\varepsilon m_2$. Let (U',W') be such a pair. We distinguish three cases depending on $|D|$ and $e_H(U',W')$.

Case 1 ($|D| \le \varepsilon^3 q m_1 m_2$). The graph H is (ε, H, q)-regular and thus

$$d_{H,q}(U',W') \ge d_{H,q}(U,W) - \varepsilon.$$

Since $d_{H',q}(U',W') \ge d_{H,q}(U',W') - d_{D,q}(U',W')$, we have

$$d_{H',q}(U',W') \ge d_{H,q}(U',W') - \frac{|D|}{9\varepsilon^2 q m_1 m_2} \ge d_{H,q}(U,W) - \frac{10}{9}\varepsilon,$$

which implies that H' is $(3\varepsilon, q)$-regular.

Case 2 ($e_H(U',W') \le \varepsilon^3 q m_1 m_2$). Observe that $e_H(U',W') \le \varepsilon^3 q m_1 m_2$ implies

$$d_{H,q}(U',W') \le \frac{\varepsilon}{9}. \tag{5}$$

H is (ε, H, q)-regular and thus

$$d_{H,q}(U,W) \le \varepsilon + d_{H,q}(U',W') \le \frac{10}{9}\varepsilon. \tag{6}$$

On the other hand, $d_{H',q}(X,Y) \le d_{H,q}(X,Y)$ for arbitrary $X \subseteq U$ and $Y \subseteq W$, which combined with (5) and (6) yields

$$|d_{H',q}(U,W) - d_{H',q}(U',W')| \le \frac{10}{9}\varepsilon + \frac{\varepsilon}{9} \le 3\varepsilon.$$

Up to now, we have not used the fact that D is chosen at random. To deal with the case that we are left with (that is, the case in which $|D| > \varepsilon^3 q m_1 m_2$ and $e_H(U',W') > \varepsilon^3 q m_1 m_2$), we will make use of this randomness. Before we start, we state the following two-sided estimate for the hypergeometric distribution.

Lemma 3.5. *Let sets $B \subseteq U$ be fixed. Let $|U| = u$ and $|B| = b$. Suppose we select a d-set D uniformly at random from U. Then, for $3/2 \ge \lambda > 0$, we have*

$$\mathbb{P}\left(\left||D\cap B| - \frac{bd}{u}\right| \ge \lambda\frac{bd}{u}\right) \le 2\exp\left(-\frac{\lambda^2}{3}\frac{bd}{u}\right).$$

Proof. For the proof we refer to [7] (Theorem 2.10). ☐

We continue with the proof of Lemma 3.4.

Case 3 ($|D| > \varepsilon^3 q m_1 m_2$ **and** $e_H(U', W') > \varepsilon^3 q m_1 m_2$). Recall that $U' \subseteq U$ and $V' \subseteq V$ are such that $|U'| = 3\varepsilon m_1$ and $|V'| = 3\varepsilon m_2$. First, we verify that

$$\left| d_{D,q}(U,W) \frac{d_{H,q}(U',W')}{d_{H,q}(U,W)} - d_{D,q}(U',W') \right| \le \varepsilon \tag{7}$$

implies that

$$|d_{H',q}(U,W) - d_{H',q}(U',W')| \le 3\varepsilon. \tag{8}$$

Indeed, straightforward calculation using the (ε, q)-regularity of H and (7) give

$$\begin{aligned}
&|d_{H',q}(U,W) - d_{H',q}(U',W')| \\
&\quad = |(d_{H,q}(U,W) - d_{D,q}(U,W)) - (d_{H,q}(U',W') - d_{D,q}(U',W'))| \\
&\quad \le \varepsilon + |d_{D,q}(U,W) - d_{D,q}(U',W')| \\
&\quad \le \varepsilon + \left| d_{D,q}(U,W) - d_{D,q}(U,W)\frac{d_{H,q}(U',W')}{d_{H,q}(U,W)} \right| \\
&\qquad\qquad + \left| d_{D,q}(U,W)\frac{d_{H,q}(U',W')}{d_{H,q}(U,W)} - d_{D,q}(U',W') \right| \\
&\quad \le \varepsilon + \frac{d_{D,q}(U,W)}{d_{H,q}(U,W)} |d_{H,q}(U,W) - d_{H,q}(U',W')| + \varepsilon \\
&\quad \le \varepsilon + \frac{d_{D,q}(U,W)}{d_{H,q}(U,W)}\varepsilon + \varepsilon \\
&\quad \le 3\varepsilon.
\end{aligned}$$

Next, we will prove that (7) is unlikely to fail, because of the random choice of D. We set

$$\lambda = \min\left\{ \frac{9\varepsilon^3}{C}, \frac{3}{2} \right\}. \tag{9}$$

Then the two-sided estimate in Lemma 3.5 gives that

$$\left| |D \cap E_H(U',W')| - \frac{e_H(U',W')|D|}{e_H(U,W)} \right| < \lambda \frac{e_H(U',W')|D|}{e_H(U,W)}$$

fails with probability

$$\le 2\exp\left(-\frac{\lambda^2}{3} \frac{e_H(U',W')|D|}{e_H(U,W)} \right). \tag{10}$$

Since

$$\begin{aligned}
&\left| d_{D,q}(U',W') - d_{D,q}(U,W)\frac{d_{H,q}(U',W')}{d_{H,q}(U,W)} \right| \\
&\qquad\qquad = \frac{1}{9\varepsilon^2 q m_1 m_2} \left| |D \cap E_H(U',W')| - \frac{e_H(U',W')|D|}{e_H(U,W)} \right|,
\end{aligned}$$

and because of (ii) and (9), we have

$$\lambda \frac{e_H(U',W')}{9q\varepsilon^2 m_1 m_2} \frac{|D|}{e_H(U,W)} \le \lambda \frac{e_H(U',W')}{9q\varepsilon^2 m_1 m_2} \le \lambda \frac{e_H(U,W)}{9q\varepsilon^2 m_1 m_2} \le \varepsilon,$$

we infer that (7) and consequently (8) fails with small probability given in (10).

We now sum over all possible choices for U' and W' and use the conditions of this case (i.e. $|D| > \varepsilon^3 q m_1 m_2$, $e_H(U', W') > \varepsilon^3 q m_1 m_2$) and (ii). We have that

$$\mathbb{P}\left(H' \text{ is not } (3\varepsilon, q)\text{-regular}\right) \quad \leq \quad 2^{m_1+m_2} \cdot 2\exp\left(-\frac{\lambda^2\varepsilon^6}{3C} q m_1 m_2\right) < 1$$

for m_1, m_2 sufficiently large, since $q = q(m_0) \gg 1/m_0$. This implies that, for m_0 large enough, there is a set D such that H' is $(3\varepsilon, q)$-regular, as required.

□

3.3. Proof of the main result

The proof of Theorem 1.2 is based on Lemma 2.3, which we prove later in Section 4. The main idea is to "find" a regular subgraph J satisfying (I)–(IV) of the Counting Lemma, in the arbitrary subgraph F with

$$|E(F)| \geq \left(1 - \frac{1}{l-1} + \delta\right) q \binom{n}{2}.$$

Proof of Theorem 1.2. Let $l \geq 2$ and $1/(l-1) > \delta > 0$ be fixed and suppose $q = q(n) \gg ((\log n)^4/n)^{1/(l-1)}$. First we define some constants that will be used in the proof.

We start by setting

$$\alpha = \frac{\delta}{8}, \tag{11}$$

$$\sigma = 10^{-6}. \tag{12}$$

(As a matter of fact, our proof is not sensitive to the value of the constant σ; in fact, as long as $0 < \sigma < 1$, every choice works.) We want to use the Counting Lemma, Lemma 2.3, in order to determine the value of ε. Set $\alpha^{\mathrm{CL}} = \alpha$ and $\sigma^{\mathrm{CL}} = \sigma$, then Lemma 2.3 yields $\varepsilon^{\mathrm{CL}}$. We set

$$\varepsilon = \min\left\{\frac{\varepsilon^{\mathrm{CL}}}{3}, \frac{\delta}{80}\right\} \tag{13}$$

and

$$C = \frac{4+\delta}{4}. \tag{14}$$

We then apply the sparse regularity lemma (Theorem 2.2) with $\varepsilon^{\mathrm{SRL}} = \varepsilon$, $C^{\mathrm{SRL}} = C$ and $t_0^{\mathrm{SRL}} = \max\{\sqrt{8l^2/\delta}, 40/\delta\}$. Theorem 2.2 then gives ξ^{SRL} and we define

$$\xi = \xi^{\mathrm{SRL}}.$$

Moreover, Theorem 2.2 yields

$$T_0^{\mathrm{SRL}} \geq t = t^{\mathrm{SRL}} \geq t_0^{\mathrm{SRL}} = \max\left\{\sqrt{\frac{8l^2}{\delta}}, \frac{40}{\delta}\right\}. \tag{15}$$

For the rest of the proof all the constants defined above (α, σ, ε, C, ξ, and t) are fixed.

Fact 3.1, Lemma 3.2, and Lemma 2.3 imply that a graph G in $\mathcal{G}(n,q)$ satisfies the following properties (P1)–(P3) with probability $1-o(1)$:

(P1) $|E(G)| \geq (1+o(1))\, q\binom{n}{2}$,
(P2) G is (ξ, C)-bounded, and
(P3) G satisfies the property considered in Lemma 2.3.

We will show that if a graph G satisfies (P1)–(P3), then any $F \subseteq G$ with $|E(F)| \geq (1-1/(l-1)+\delta)q\binom{n}{2}$ contains at least $cq^{\binom{l}{2}}n^l$ (for some constant $c = c(\delta, l)$) copies of K_l, and Theorem 1.2 will follow.

To achieve this, we first regularise F by applying Theorem 2.2 with $\varepsilon^{\mathrm{SRL}} = \varepsilon$, $C^{\mathrm{SRL}} = C$ and $t_0^{\mathrm{SRL}} = \max\{\sqrt{8l^2/\delta}, 40/\delta\}$. Consequently F admits an (ε, q)-regular (ε, t)-equitable partition $(V_i)_0^t$. We set $m = n/t = |V_i|$ for $i \neq 0$.

Let F_{cluster} be the cluster graph of F with respect to $(V_i)_0^t$ defined as follows

$$\begin{aligned} V\left(F_{\mathrm{cluster}}\right) &= \{1, \ldots, t\}, \\ E\left(F_{\mathrm{cluster}}\right) &= \Big\{\{i,j\}\colon\ (V_i, V_j) \text{ is } (\varepsilon,q)\text{-regular } \wedge\ e_F(V_i,V_j) \geq \alpha q m^2\Big\}. \end{aligned}$$

Our next aim is to apply the classical Turán theorem to guarantee the existence of a $K_l \subseteq F_{\mathrm{cluster}}$. For this we define a subgraph F' of F. Set

$$E(F') = \bigcup\{E_F(V_i, V_j)\colon \{i,j\} \in E(F_{\mathrm{cluster}})\}$$

We now want to find a lower bound for $|E(F')|$. There are four possible reasons for an edge $e \in E(F)$ not to be in $E(F')$:

(R1) e has at least one vertex in V_0,
(R2) e is contained in some vertex class V_i for $1 \leq i \leq t$,
(R3) e is in $E(V_i, V_j)$ for an (ε, q)-irregular pair (V_i, V_j), or
(R4) e is in $E(V_i, V_j)$ for sparse a pair (*i.e.*, $e(V_i, V_j) < \alpha q m^2$).

We bound the number of discarded edges of type (R1)–(R3) by applying that G is (ξ, C)-bounded (Property (P2)):

$$\begin{aligned} \#\text{ of edges of type (R1)} &\leq Cq\varepsilon n^2, \\ \#\text{ of edges of type (R2)} &\leq Cq\left(\frac{n}{t}\right)^2 \cdot t, \\ \#\text{ of edges of type (R3)} &\leq Cq\left(\frac{n}{t}\right)^2 \cdot \varepsilon\binom{t}{2}. \end{aligned}$$

Furthermore, we bound the number of discarded edges of type (R4), by

$$\#\text{ of edges of type (R4)} \leq \alpha q\left(\frac{n}{t}\right)^2 \cdot \binom{t}{2}.$$

This, combined with $n \geq 2$, (11), (13), (14), (15), and $\delta < 1$ implies that

$$\begin{aligned}|E(F)\setminus E(F')| &\leq \left(C\left(\varepsilon+\frac{1}{t}+\frac{\varepsilon}{2}\right)+\frac{\alpha}{2}\right)qn^2\\ &\leq \left(C\left(2\varepsilon+\frac{1}{t}\right)+\frac{\alpha}{2}\right)\cdot 4q\binom{n}{2}\\ &\leq \left((4+\delta)\left(\frac{\delta}{40}+\frac{\delta}{40}\right)+\frac{\delta}{4}\right)q\binom{n}{2}\leq \frac{\delta}{2}q\binom{n}{2},\end{aligned}$$

and thus

$$|E(F')|\geq \left(1-\frac{1}{l-1}+\frac{\delta}{2}\right)q\binom{n}{2}.$$

We use the last inequality and once again (P2) to achieve the desired lower bound for $|E(F_{\text{cluster}})|$. Indeed,

$$|E(F_{\text{cluster}})|\geq \frac{e(F')}{Cq(n/t)^2}=\left(1-\frac{1}{l-1}+\frac{\delta}{2}\right)\left(1-\frac{1}{n}\right)\left(1+\frac{\delta}{4}\right)^{-1}\frac{t^2}{2},$$

and then, for n large enough ($n > 16/\delta^2$), by using $t^2 \geq 8l^2/\delta$, we deduce that

$$\begin{aligned}|E(F_{\text{cluster}})| &> \left(1-\frac{1}{l-1}+\frac{\delta}{2}\right)\left(1-\frac{\delta}{4}\right)\frac{t^2}{2}\\ &\geq \left(1-\frac{1}{l-1}+\frac{\delta}{8}\right)\frac{t^2}{2}\\ &\geq \left(1-\frac{1}{l-1}\right)\frac{t^2}{2}+\frac{l^2}{2}.\end{aligned}\tag{16}$$

The last inequality implies, by Turán's theorem [18], that there is a subgraph K_l in F_{cluster}. Let $\{i_1,\dots,i_l\}$ be the vertex set of this K_l in F_{cluster}. Then we set $J_0 = F[V_{i_1},\dots,V_{i_l}] \subseteq F$. Now, every pair $(V_{i_j}, V_{i_{j'}})$ for $1 \leq j < j' \leq l$ satisfies the conditions of Lemma 3.4 with $\varepsilon^{\text{Lem3.4}} = \varepsilon$ and $\alpha^{\text{Lem3.4}} = \alpha$. Thus there is a subgraph $J \subseteq J_0 \subseteq F$ that is $(3\varepsilon, q)$-regular and $e_J(V_{i_j}, V_{i'_j}) = \alpha q m^2$. Since $\varepsilon \leq \varepsilon^{\text{CL}}/3$ and J satisfies conditions (I)–(IV) of the Counting Lemma, Lemma 2.3, with the constants chosen above ($\alpha^{\text{CL}} = \alpha$, $\sigma^{\text{CL}} = \sigma$, and $\varepsilon^{\text{CL}} \geq 3\varepsilon$), there are at least

$$(1-\sigma)p^{\binom{l}{2}}m^l=\frac{(1-\sigma)\alpha^{\binom{l}{2}}}{t^l}q^{\binom{l}{2}}n^l\geq \frac{(1-\sigma)\alpha^{\binom{l}{2}}}{\left(T_0^{\text{SRL}}\right)^l}q^{\binom{l}{2}}n^l$$

different copies of K_l in $J \subseteq F$. Observe that α, σ and T_0 depend on δ and l but not on n. Consequently, there are $c(\delta,l)q^{\binom{l}{2}}n^l \gg 1$ (where $c(\delta,l) = (1-\sigma)\alpha^{\binom{l}{2}}/\left(T_0^{\text{SRL}}\right)^l$) copies of K_l in F, as required by Theorem 1.2. □

4. The counting lemma

Our aim in this section is to prove Lemma 2.3. In order to do this, we will need two lemmas. We introduce these in the first two subsections. Then, in Section 4.3, we will illustrate the proof of the Counting lemma on the particular case $l = 4$. Finally, we give the proof of Lemma 2.3 in Section 4.4.

4.1. The pick-up lemma

Before we state the 'Pick-Up Lemma', Lemma 4.3, let us state a simple one-sided estimate for the hypergeometric distribution, which will be useful in the proof of Lemma 4.3.

Lemma 4.1 (A hypergeometric tail lemma). *Let b, d, and u be positive integers and suppose we select a d-set D uniformly at random from a set U of cardinality u. Suppose also that we are given a fixed b-set $B \subseteq U$. Then we have for $\lambda > 0$*

$$\mathbb{P}\left(|D \cap B| \geq \lambda \frac{bd}{u}\right) \leq \left(\frac{\mathrm{e}}{\lambda}\right)^{\lambda bd/u}. \tag{17}$$

Proof. For the proof we refer the reader to [11]. □

We now state and prove the Pick-Up Lemma. Let $k \geq 2$ be a fixed integer and let m be sufficiently large. Let $V_1, \ldots, V_k$ be pairwise disjoint sets all of size m and let $\mathcal{B}$ be a subset of $V_1 \times \cdots \times V_k$. For $1 > p = p(m) \gg 1/m$ set $T = pm^2$ and consider the probability space

$$\Omega = \binom{V_1 \times V_k}{T} \times \cdots \times \binom{V_{k-1} \times V_k}{T},$$

where $\binom{V_i \times V_k}{T}$ denotes the family of all subsets of $V_i \times V_k$ of size T, and all the $R = (R_1, \ldots, R_{k-1}) \in \Omega$ are equiprobable, *i.e.*, have probability

$$\binom{m^2}{T}^{-(k-1)}.$$

For every $R = (R_1, \ldots, R_{k-1}) \in \Omega$ the *degree with respect to* R_i $(1 \leq i < k)$ of a vertex v_k in V_k is

$$d_{R_i}(v_k) = |\{v_i \in V_i \colon (v_i, v_k) \in R_i\}|. \tag{18}$$

Definition 4.2 ($\Pi(\zeta, \mu, K)$). *For ζ, μ, K with $1 > \zeta$, $\mu > 0$ and $K > 0$, we say that property $\Pi(\zeta, \mu, K)$ holds for $R = (R_1, \ldots, R_{k-1}) \in \Omega$ if*

$$\widetilde{V}_k = \widetilde{V}_k(K) = \{v_k \in V_k \colon \ d_{R_i}(v_k) \leq Kpm, \ \forall 1 \leq i \leq k-1\}$$

and

$$\mathcal{B}(R) = \{b = (v_1, \ldots, v_k) \in \mathcal{B} \colon \ v_k \in \widetilde{V}_k \ \wedge \ (v_j, v_k) \in R_j, \ \forall \ 1 \leq j \leq k-1\}$$

satisfy the inequalities

$$|\widetilde{V}_k| \geq (1-\mu)m, \tag{19}$$

$$|\mathcal{B}(R)| \leq \zeta p^{k-1} m^k. \tag{20}$$

We think of $\mathcal{B}(R)$ as the members of $\mathcal{B}$ that have been *picked-up* by the random element $R \in \Omega$. We will be interested in the probability that the property $\Pi(\zeta, \mu, K)$ fails for a fixed $\mathcal{B}$ in the uniform probability space Ω.

Lemma 4.3 (Pick-Up Lemma). *For every β, ζ and μ with $1 > \beta, \zeta, \mu > 0$ there exist $1 > \eta = \eta(\beta, \zeta, \mu) > 0$, $K = K(\beta, \mu) > 0$ and m_0 such that if $m \geq m_0$ and*

$$|\mathcal{B}| \leq \eta m^k, \tag{21}$$

then

$$\mathbb{P}(\Pi(\zeta, \mu, K) \text{ fails for } R \in \Omega) \leq \beta^{(k-1)T}. \tag{22}$$

For the proof we need a few definitions. Suppose β and μ are given. We define

$$\theta = \frac{1}{2}\beta^{k-1}, \tag{23}$$

$$K = \max\left\{\frac{3(k-1)\log 1/\theta}{\mu}, \mathrm{e}^2\right\}. \tag{24}$$

Since $p \gg 1/m$ the definition of $K \geq 3(k-1)\log(1/\theta)/\mu$ implies that

$$(k-1)\binom{m}{\mu m/(k-1)} \exp\left(-\frac{\mu T K \log K}{2(k-1)}\right) \leq \theta^T \tag{25}$$

holds for m sufficiently large.

Using the definition of d_{R_i} in (18) we construct for each $i = 1, \ldots, k-1$ a subset of V_k by putting

$$V_k^{(i)} = \{v_k \in V_k^{(i-1)}\colon\ d_{R_i}(v_k) \leq Kpm\},$$

where $V_k^{(0)} = V_k$. Observe that $V_k = V_k^{(0)} \supseteq V_k^{(1)} \supseteq \cdots \supseteq V_k^{(k-1)} = \widetilde{V}_k$. In the view of Lemma 4.3 we define the following "bad" events in Ω.

Definition 4.4 (A_i, B). *For each $i = 0, \ldots, k-1$ and K, $\mu > 0$, $\zeta > 0$, let $A_i = A_i(\mu, K)$, $B = B(\zeta, K) \subseteq \Omega$ be the events*

$$\begin{aligned} A_i\colon &\quad |V_k^{(i)}| &<&\quad \big(1 - i\mu/(k-1)\big)m, \\ B\colon &\quad |\mathcal{B}(R)| &>&\quad \zeta p^{k-1} m^k. \end{aligned}$$

Observe that the definition of $V_k^{(0)} = V_k$ implies

$$\mathbb{P}(A_0) = 0. \tag{26}$$

We restate Lemma 4.3 by using the notation introduced in Definition 4.4.

Lemma 4.3′ (Pick-up Lemma, event version). *For every β, ζ and μ with $1 > \beta, \zeta, \mu > 0$ there exist $1 > \eta = \eta(\beta, \zeta, \mu) > 0$, $K = K(\beta, \mu) > 0$ and m_0 such that if $m \geq m_0$ and*

$$|\mathcal{B}| \leq \eta m^k, \tag{27}$$

then

$$\mathbb{P}(A_{k-1}(\mu, K) \vee B(\zeta, K)) \leq \beta^{(k-1)T}. \tag{28}$$

We need some more preparation before we prove Lemma 4.3$'$. Suppose β, ζ, μ are given by Lemma 4.3$'$ and θ, K are fixed by (23) and (24). For each $i = 1, \ldots, k-1$ we consider the set $\mathcal{B}_i \subseteq \mathcal{B}$ consisting of those k-tuples $b \in \mathcal{B}$ which were partially "picked up" by edges of $R_1, \ldots, R_i$. For technical reasons we consider only those k-tuples containing vertices $v_k \in V_k^{(i-1)}$, *i.e.*, with $d_{R_j}(v_k) \leq Kpm$ for $j = 1, \ldots, i-1$. More formally, we let

$$\mathcal{B}_i = \{b = (v_1, \ldots, v_k) \in \mathcal{B}\colon\ v_k \in V_k^{(i-1)} \wedge\ (v_j, v_k) \in R_j,\ \forall\ 1 \leq j \leq i\}.$$

We also set $\mathcal{B}_0 = \mathcal{B}$.

The definitions of $\widetilde{V}_k = V_k^{(k-1)} \subseteq V_k^{(k-2)}$ and $\mathcal{B}_{k-1}$ imply

$$\mathcal{B}(R) \subseteq \mathcal{B}_{k-1}. \tag{29}$$

(Equality may fail in (29) because we may have $V_k^{(k-2)} \setminus V_k^{(k-1)} \neq \emptyset$.) For each $i = k, \ldots, 1$ define ζ_{i-1} by

$$\begin{aligned} \zeta_{k-1} &= \zeta, \\ \zeta_{i-1} &= \frac{k-1-(i-1)\mu}{4(k-1)K^{i-1}} \zeta_i^2 \theta^{4K^{i-1}/\zeta_i}. \end{aligned} \tag{30}$$

Furthermore, consider for each $i = 0, \ldots, k-1$ the event $B_i = B_i(\zeta_i, K) \subseteq \Omega$ defined by

$$B_i\colon \quad |\mathcal{B}_i| > \zeta_i p^i m^k. \tag{31}$$

In order to prove Lemma 4.3$'$ we need two more claims, which we will prove later.

Claim 4.5. *For all* $1 \leq i \leq k-1$, *we have*

$$\mathbb{P}(A_i) = \mathbb{P}\left(|V_k^{(i)}| < \left(1 - \frac{i\mu}{k-1}\right) m\right) \leq \theta^T.$$

Claim 4.6. *For all* $1 \leq i \leq k-1$, *we have*

$$\mathbb{P}(B_i \mid \neg A_{i-1} \wedge \neg B_{i-1}) \leq \theta^T.$$

Assuming Claims 4.5 and 4.6, we may easily prove Lemma 4.3$'$.

Proof of Lemma 4.3$'$. Set $\eta = \zeta_0$ where ζ_0 is given by (30). The definition of $\mathcal{B}_0 = \mathcal{B}$ and (27) implies $|\mathcal{B}_0| \leq \zeta_0 m^k$ and consequently by the definition of the event B_0 in (31)

$$\mathbb{P}(B_0) = 0. \tag{32}$$

Because of (29) and $\zeta_{k-1} = \zeta$ in (30) we have

$$\mathbb{P}(B) \leq \mathbb{P}(B_{k-1}). \tag{33}$$

Using the formal identity

$$\mathbb{P}(B_i) = \mathbb{P}(B_i \wedge (\neg A_{i-1} \wedge \neg B_{i-1})) + \mathbb{P}(B_i \wedge (A_{i-1} \vee B_{i-1})),$$

we observe that

$$\mathbb{P}(B_i) \leq \mathbb{P}(B_i \mid \neg A_{i-1} \wedge \neg B_{i-1}) + \mathbb{P}(A_{i-1}) + \mathbb{P}(B_{i-1}) \tag{34}$$

for each $i = 1, \dots, k-1$. It follows by applying (33) and (34) that

$$\begin{aligned}\mathbb{P}(A_{k-1} \vee B) &\leq \mathbb{P}(A_{k-1}) + \mathbb{P}(B_{k-1}) \\ &\leq \mathbb{P}(A_{k-1}) + \sum_{i=1}^{k-1} \Big(\mathbb{P}(B_i \mid \neg A_{i-1} \wedge \neg B_{i-1}) + \mathbb{P}(A_{i-1}) \Big) + \mathbb{P}(B_0).\end{aligned}$$

Claims 4.5 and 4.6, and (26), (32) and (23) finally imply

$$\mathbb{P}(A_{k-1} \vee B) \leq 2(k-1)\theta^T \leq 2(k-1)\left(\frac{\beta^{k-1}}{2}\right)^T \leq \beta^{(k-1)T}$$

for m sufficiently large, as required. □

We now prove Claim 4.5 and then Claim 4.6.

Proof of Claim 4.5. Fix a set $V^* \subseteq V_k$ of size $\mu m/(k-1)$. For a fixed j $(1 \leq j \leq i)$ assume that $d_{R_j}(v_k) > Kpm$ for every v_k in V^*. This clearly implies the event

$$E_j(V^*): \quad |R_j \cap (V_j \times V^*)| > Kpm\frac{\mu m}{k-1} = K\frac{\mu T}{k-1}. \tag{35}$$

The T pairs of R_j are chosen uniformly in $V_j \times V_k$, so the hypergeometric tail lemma, Lemma 4.1, applies, and using the fact that $\mathrm{e} \leq K^{1/2}$ by (24) we get

$$\mathbb{P}\left(E_j(V^*)\right) \leq \left(\frac{\mathrm{e}}{K}\right)^{K\mu T/(k-1)} \leq \exp\left(-\frac{\mu T K \log K}{2(k-1)}\right). \tag{36}$$

Set $E_j = \bigvee E_j(V^*)$, where the union is taken over all $V^* \subseteq V_k$ of size $\mu m/(k-1)$. Then

$$\mathbb{P}(E_j) \leq \binom{m}{\mu m/(k-1)} \exp\left(-\frac{\mu T K \log K}{2(k-1)}\right) \tag{37}$$

holds for each $j = 1, \dots, i$, and this implies

$$\mathbb{P}\left(\bigvee_{j=1}^{i} E_j\right) \leq i\binom{m}{\mu m/(k-1)} \exp\left(-\frac{\mu T K \log K}{2(k-1)}\right).$$

Finally, the fact that $A_i \subseteq \bigvee_{j=1}^{i} E_j$ and the choice of K with (25) gives that

$$\mathbb{P}(A_i) \leq i\binom{m}{\mu m/(k-1)} \exp\left(-\frac{\mu T K \log K}{2(k-1)}\right) \leq \theta^T,$$

as required. □

Proof of Claim 4.6. Recall β, ζ and μ are given by Lemma 4.3′ and θ, K and ζ_i are fixed by (23), (24) and (30). In order to prove Claim 4.6 we fix i $(1 \leq i \leq k-1)$ and we assume $\neg A_{i-1}$ and $\neg B_{i-1}$ occur. This means by Definition 4.4 and (31) that

$$|V_k^{(i-1)}| \geq \left(1 - \frac{(i-1)\mu}{k-1}\right) m = \left(\frac{k-1-(i-1)\mu}{k-1}\right) m, \tag{38}$$

$$|\mathcal{B}_{i-1}| \leq \zeta_{i-1} p^{i-1} m^k. \tag{39}$$

We have to show that

$$|\mathcal{B}_i| \le \zeta_i p^i m^k \tag{40}$$

holds for R in the uniform probability space Ω with probability $\ge 1-\theta^T$.

First we define the auxiliary constant

$$L_i = \left(\frac{1}{\theta}\right)^{4K^{i-1}/\zeta_i}. \tag{41}$$

The definition of θ in (23) and the facts that $0 < \zeta_i < 1$ for each $i = 1, \dots, k-1$ and $K > 1$ imply that

$$L_i \ge \left(\frac{2}{\beta^{k-1}}\right)^4 > \mathrm{e}^2 \tag{42}$$

holds.

We define the degree of a pair in $V_i \times V_k^{(i-1)}$ with respect to $\mathcal{B}_{i-1}$ by

$$d_{\mathcal{B}_{i-1}}(w_i, w_k) = \Big|\{b = (v_1, \dots, v_k) \in \mathcal{B}_{i-1}\colon\ v_i = w_i \text{ and } v_k = w_k\}\Big|.$$

We can bound the value of the average degree by (38) and (39):

$$\begin{aligned}\operatorname{avg}\left\{d_{\mathcal{B}_{i-1}}(v_i, v_k)\colon\ (v_i, v_k) \in V_i \times V_k^{(i-1)}\right\} &= \frac{|\mathcal{B}_{i-1}|}{m|V_k^{(i-1)}|} \\ &\le \frac{k-1}{k-1-(i-1)\mu}\zeta_{i-1}p^{i-1}m^{k-2}.\end{aligned} \tag{43}$$

We also can bound $\Delta_{\mathcal{B}_{i-1}}(V_i, V_k^{(i-1)}) = \max\{d_{\mathcal{B}_{i-1}}(v_i, v_k)\colon\ (v_i, v_k) \in V_i \times V_k^{(i-1)}\}$ by the following observation. Let (v_i, v_k) be an arbitrary element in $V_i \times V_k^{(i-1)}$. Then, by the definition of $V_k^{(i-1)}$, we have

$$d_{\mathcal{B}_{i-1}}(v_i, v_k) \le d_{R_1}(v_k) \cdot \ldots \cdot d_{R_{i-1}}(v_k) \cdot m^{k-2-(i-1)} \le (Kpm)^{i-1} m^{k-i-1}. \tag{44}$$

Inequality (44) implies

$$\Delta_{\mathcal{B}_{i-1}}\left(V_i, V_k^{(i-1)}\right) \le K^{i-1} p^{i-1} m^{k-2}. \tag{45}$$

Let F be the set of pairs of "high degree". More precisely, set

$$F = \left\{(v_i, v_k) \in V_i \times V_k^{(i-1)}\colon\ d_{\mathcal{B}_{i-1}} > \frac{\zeta_i}{2} p^{i-1} m^{k-2}\right\}.$$

A simple averaging argument applying (43) yields

$$|F| \le \frac{2(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i}|V_i||V_k^{(i-1)}| \le \frac{2(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i} m^2. \tag{46}$$

On the other hand, if we set $\bar{F} = V_i \times V_k^{(i-1)} \setminus F$ then the definition of F and (45) imply

$$\begin{aligned}
|\mathcal{B}_i| &= \sum_{(v_i,v_k)\in R_i\cap\bar{F}} d_{\mathcal{B}_{i-1}}(v_i,v_k) + \sum_{(v_i,v_k)\in R_i\cap F} d_{\mathcal{B}_{i-1}}(v_i,v_k) \\
&\le \frac{\zeta_i}{2} p^{i-1}m^{k-2}|R_i\cap\bar{F}| \;+\; K^{i-1}p^{i-1}m^{k-2}|R_i\cap F| \\
&\le \frac{\zeta_i}{2} p^{i-1}m^{k-2}T \;+\; K^{i-1}p^{i-1}m^{k-2}|R_i\cap F| \\
&= \left(\frac{\zeta_i}{2} + \frac{K^{i-1}}{T}|R_i\cap F|\right) p^i m^k. \qquad (47)
\end{aligned}$$

Next we prove that

$$\mathbb{P}\left(|R_i\cap F| > \frac{\zeta_i T}{2K^{i-1}}\right) \le \theta^T, \qquad (48)$$

which, together with (47), yields our claim, namely, that

$$\mathbb{P}\left(|\mathcal{B}_i| > \zeta_i p^i m^k\right) \le \theta^T. \qquad (49)$$

We now prove inequality (48). Without loss of generality we assume equality holds in (46). Then the hypergeometric tail lemma, Lemma 4.1, implies that

$$\begin{aligned}
\mathbb{P}\left(|R_i\cap F| > L_i\frac{|F|T}{m^2}\right) &= \mathbb{P}\left(|R_i\cap F| > L_i\frac{2(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i}T\right) \\
&\le \left(\frac{\mathrm{e}}{L_i}\right)^{L_i\frac{2(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i}T} \\
&\le \exp\left(-\frac{L_i(\log L_i)(k-1)\zeta_{i-1}T}{(k-1-(i-1)\mu)\zeta_i}\right), \qquad (50)
\end{aligned}$$

where in the last inequality we used that $L_i \ge \mathrm{e}^2$ (see (42)). The definitions of ζ_{i-1} and L_i in (30) and (41) yield

$$\frac{L_i(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i} = \frac{L_i\zeta_i}{4K^{i-1}}\theta^{4K^{i-1}/\zeta_i} = \frac{\zeta_i}{4K^{i-1}}.$$

We use the last inequality to derive

$$\begin{aligned}
\frac{L_i(\log L_i)(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i} &= \log\frac{1}{\theta}, \\
L_i\frac{2(k-1)\zeta_{i-1}}{(k-1-(i-1)\mu)\zeta_i} &= \frac{\zeta_i}{2K^{i-1}},
\end{aligned}$$

which, combined with inequality (50), gives (48). □

4.2. The k-tuple lemma for subgraphs of random graphs

Let $G \in \mathcal{G}(n,q)$ be the binomial random graph with edge probability $q = q(n)$, and suppose $H = (U,W;F)$ is a bipartite, not necessarily induced subgraph of G with $|U| = m_1$ and $|W| = m_2$. Furthermore, denote the density of H by $p = e(H)/m_1m_2$.

We now consider subsets of W of fixed cardinality $k \ge 1$, and classify them according to the size of their joint neighbourhood in H. For this purpose we define

$$\mathcal{B}^{(k)}(U,W;\gamma) = \left\{ b = \{v_1,\ldots,v_k\} \in W: \ \left| d_U^H(b) - p^k m_1 \right| \geq \gamma p^k m_1 \right\},$$

where $d_U^H(b)$ denotes the size of the joint neighbourhood of b in H, that is,

$$d_U^H(b) = \left| \bigcap_{i=1}^{k} \Gamma_H(v_i) \right|.$$

The following lemma states that in a typical $G \in \mathcal{G}(n,q)$ the set $\mathcal{B}^{(k)}(U,W;\gamma)$ is "small" for any sufficiently large (ε,q)-regular subgraph $H = (U,W;F)$ of a dense enough random graph G. Recall that if G is a graph and U, $W \subset V(G)$ are two disjoint sets of vertices, then $G[U,W]$ denotes the bipartite graph naturally induced by (U,W).

Lemma 4.7 (The k-tuple lemma). *For any constants $\alpha > 0$, $\gamma > 0$, $\eta > 0$, and $k \geq 1$ and function $m_0 = m_0(n)$ such that $q^k m_0 \gg (\log n)^4$, there exists a constant $\varepsilon > 0$ for which the random graph $G \in \mathcal{G}(n,q)$ satisfies the following property with probability $1 - o(1)$: If for a bipartite subgraph $H = (U,W;F)$ of G the conditions*

(*i*) $e(H) \geq \alpha e(G[U,W])$,

(*ii*) *H is (ε,q)-regular,*

(*iii*) $|U| = m_1 \geq m_0$ *and* $|W| = m_2 \geq m_0$

apply, then

$$|\mathcal{B}^{(k)}(U,W;\gamma)| \leq \eta \binom{m_2}{k} \tag{51}$$

also applies.

Proof. The proof of Lemma 4.7 is given in [11]. □

4.3. Outline of the proof of the counting lemma for $l = 4$

The proof of the Lemma 2.3 contains some technical definitions. In order to make the reading more comprehensible, we first informally illustrate the basic ideas of the proof for the case $l = 4$, before we give the proof for a general $l \geq 2$ in Section 4.4.

Consider the following situation: Let V_1, V_2, V_3, and V_4 be pairwise disjoint sets of vertices of size m. Let J be a 4-partite graph with vertex set $V(J) = V_1 \cup V_2 \cup V_3 \cup V_4$. We think of J as a not necessarily induced subgraph of a random graph in $\mathcal{G}(n,q)$ with $T = pm^2$ edges between each V_i and V_j $(1 \leq i < j \leq 4)$, where $p = \alpha q$. We will describe a situation in which we will be able to assert that J contains the "right" number of K_4's. Here and everywhere below by the "right" number we mean "as expected in a random graph of density p"; notice that, for the number of K_4's, this means $\sim p^6 m^4$. Observe that, however, J is a not necessarily induced subgraph of a graph in $\mathcal{G}(n,q)$, and this makes our task hard. As it turns out, it will be more convenient to imagine that J is generated in $l - 1 = 3$ stages. First we choose the edges from V_4 to $V_1 \cup V_2 \cup V_3$. Then we choose the edges from V_3 to $V_1 \cup V_2$, and in the third stage we disclose the edges between V_2 and V_1.

The key idea of the proof is to consider "bad" tuples, which we create in every stage.

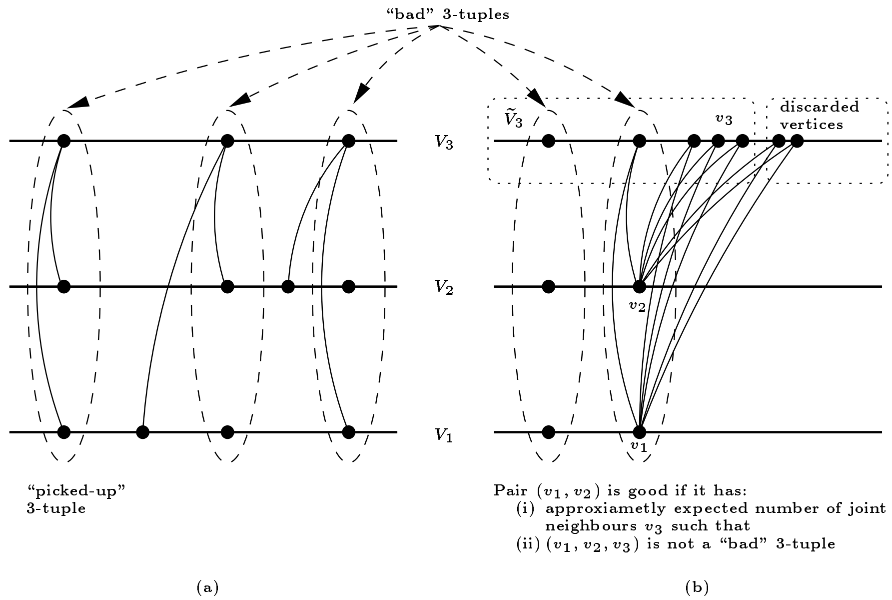

Figure 1

After we chose the edges from V_4 to the other vertex classes, we define "bad" 3-tuples in $V_1 \times V_2 \times V_3$: a 3-tuple is "bad" if its joint neighbourhood in V_4 is much smaller than expected. Then, with the right choice of constants, Proposition 4.11 for $k = 3$ and $J = J[V_4, V_1 \cup V_2 \cup V_3]$ will ensure that there are not too many "bad" 3-tuples. (Proposition 4.11 is a corollary of the the k-tuple lemma, Lemma 4.7.)

We next generate the edges between V_3 and $V_1 \cup V_2$. We want to define "bad" pairs in $V_1 \times V_2$. Here it becomes slightly more complicated to distinguish "bad" from "good". This is because there are two things that might go wrong for a pair in $V_1 \times V_2$. First of all, again the joint neighbourhood (now in V_3) of a pair in $V_1 \times V_2$ might be too small. On the other hand, it could have the right number of joint neighbours in V_3, but many of these neighbours "complete" the pair to a "bad" 3-tuple. Here the Pick-Up Lemma comes into play for $k = 3$ (see Proposition 4.10): this lemma will ensure that, given the set of "bad" 3-tuples (which was already defined in the first stage) is small, we will not "pick-up" too many of these (see Figure 1(a)), while choosing the edges between V_3 and $V_1 \cup V_2$. (We say that a triple (v_1, v_2, v_3) has been *picked-up* if (v_1, v_3) and (v_2, v_3) are in the edge set generated between V_3 and $V_1 \cup V_2$.)

Here the situation complicates somewhat. The Pick-Up Lemma forces us to discard a small portion (less or equal μ^{PU} fraction) of vertices in V_3. Thus, in order to avoid the first type of "badness" (too small joint neighbourhood) as a 2-tuple in $V_1 \times V_2$ it is not enough to have the right number of joint neighbours in V_3; we need the right number of joint neighbours in $\widetilde{V}_3$, which is V_3 without the $\mu^{\mathrm{PU}}m$ vertices (at most) we lose by applying the Pick-Up Lemma (see Figure 1(b)). This will be ensured by the the k-tuple lemma (to be more precise, Proposition 4.11), now for $k = 2$ and $J = J[\widetilde{V}_3, V_1 \cup V_2]$.

Later, in the general case, we will refer to the set of "bad" i-tuples in $V_1 \times \cdots \times V_i$

as $\mathcal{B}_i$ (see Definition 4.8 below). We define $\mathcal{B}_i$ as the union of the sets $\mathcal{B}_i^{(a)}$ and $\mathcal{B}_i^{(b)}$, defined as follows. We put in $\mathcal{B}_i^{(a)}$ the i-tuples that are "bad" because they have a joint neighbourhood in $\widetilde{V}_{i+1}$ that is too small; the set $\mathcal{B}_i^{(b)}$ is defined as the set of i-tuples in $V_1 \times \cdots \times V_i$ that "bad" because they extend to too many "bad" $(i+1)$-tuples (*i.e.*, $(i+1)$-tuples in $\mathcal{B}_{i+1}$).

As described above, we define $\mathcal{B}_i$ $(i = l-1, \dots, 1)$ by reverse induction, starting with $\mathcal{B}_{l-1}$, and going down to $\mathcal{B}_1$. With the right choice of constants, there will not be too many "bad" vertices in V_1.

Having ensured that most of the m vertices in V_1 are not "bad" (*i.e.*, do not belong to $\mathcal{B}_1$) we are now able to count the number of K_4's. We will use the following deterministic argument, which will later be formalized in Lemma 4.13. Consider a vertex v_1 in V_1 that is not "bad". This vertex has approximately the expected number of neighbours in $\widetilde{V}_2$ (*i.e.*, $\sim pm$), and not too many of these neighbours constitute, together with v_1, a "bad" 2-tuple. In other words, this means that v_1 extends to $\sim pm$ copies of K_2 in $(V_1 \times V_2) \setminus \mathcal{B}_2$. This implies that each such K_2 has the right number of joint neighbours in $\widetilde{V}_3$ (*i.e.*, $\sim p^2 m$), and consequently extends to the right number of K_3's in $(V_1 \times V_2 \times V_3) \setminus \mathcal{B}_3$. Repeating the last argument, each of these K_3's extends into $\sim p^3 m$ different copies of K_4. Since we have ensured that most of the m vertices in V_1 are not "bad", we have $\sim m \cdot pm \cdot p^2 m \cdot p^3 m = p^{\binom{4}{2}} m^4$ copies of K_4.

4.4. Proof of the counting lemma

In this section we will prove Lemma 2.3. In the section 'Concepts and Constants', we introduce the key definitions and describe the logic of all important constants which will appear later in the proof. Afterwards we prove two technical propositions in the section 'Tools'. These propositions correspond to the lemmas in Sections 4.1 and 4.2, and their use will give a short proof of the Counting Lemma, to be presented in the section 'Main proof'.

Concepts and constants. Let $t \geq l \geq 2$ be fixed integers and let n be sufficiently large. Let α and ε be positive constants. Let $G \in \mathcal{G}(n,q)$ be the binomial random graph with edge probability $q = q(n)$, and suppose J is an l-partite subgraph of G with vertex classes $V_1, \dots, V_l$. For all $1 \leq i < j \leq l$ we denote by J_{ij} the bipartite graph induced by V_i and V_j. Consider the following assertions for J.

(I) $|V_i| = m = n/t$ for all $1 \leq i \leq l$,
(II) $q^{l-1} n \gg (\log n)^4$,
(III) J_{ij} $(1 \leq i < j \leq l)$ has $T = pm^2$ edges, where $1 > \alpha q = p \gg 1/n$, and
(IV) J_{ij} $(1 \leq i < j \leq l)$ is (ε, q)-regular.

Let $\sigma > 0$ be given. We define the constants

$$\gamma = \mu = \nu = \frac{1}{3}\left(1 - (1-\sigma)^{1/l}\right), \tag{52}$$

and, for $1 \leq i \leq l-2$, we put

$$\beta_{i+1} = \left(\frac{1}{2}\left(\frac{\alpha}{\mathrm{e}}\right)^{\binom{l}{2}-\binom{i}{2}}\right)^{1/i}. \tag{53}$$

In order to prove Lemma 2.3 we need some definitions. These definitions always depend on a fixed subgraph J of our random graph $G \in \mathcal{G}(n,q)$ satisfying (I)–(IV). However, we will drop references to J because we want to simplify the notation (*e.g.*, we write V_i instead of V_i^J). Also, for each $i = 1, \dots, l$ we denote $V_1 \times \cdots \times V_i$ by $\mathcal{W}_i$.

In the proof we consider for a fixed J sets of "bad" i-tuples $\mathcal{B}_i \subseteq \mathcal{W}_i$ $(1 \le i \le l-1)$. We define these sets recursively from $\mathcal{B}_{l-1}$ to $\mathcal{B}_1$. As mentioned above in the discussion of the $l = 4$ case, there are two reasons that make a given i-tuple in $\mathcal{W}_i$ "bad". First of all, its joint neighbourhood in V_{i+1} might be too small (see the definition of $\mathcal{B}_i^{(a)}$ in Definition 4.8) and, secondly, it could extend into too many "bad" $(i+1)$-tuples in $\mathcal{B}_{i+1}$ (see the definition of $\mathcal{B}_i^{(b)}$ in Definition 4.8). Note that the "bad" $(i+1)$-tuples have already been defined, as we are using reverse induction in these definitions.

Next we apply the Pick-Up Lemma for $k = i+1$ $(1 \le i \le l-2)$ with $\mu_{i+1}^{\mathrm{PU}} = \mu$ and $\beta_{i+1}^{\mathrm{PU}} = \beta_{i+1}$ (and yet unspecified $\zeta_{i+1}^{\mathrm{PU}}$). As a result we obtain $K_{i+1}^{\mathrm{PU}} = K_{i+1}^{\mathrm{PU}}(\beta_{i+1}^{\mathrm{PU}}, \mu_{i+1}^{\mathrm{PU}})$ and the set

$$\widetilde{V}_{i+1} = \widetilde{V}_{i+1}^{\mathrm{PU}}(K_{i+1}^{\mathrm{PU}}) \subseteq V_{i+1}$$

of undiscarded vertices with

$$|\widetilde{V}_{i+1}| \ge (1-\mu)m.$$

We need a few more definitions before we define $\mathcal{B}_i$, $\mathcal{B}_i^{(a)}$ and $\mathcal{B}_i^{(b)}$ (recursively for $i = l-1, \dots, 1$). Let $\widetilde{\Gamma}_{i+1}(b)$ be the joint neighbourhood of $b = (v_1, \dots, v_i) \in \mathcal{W}_i$ in $\widetilde{V}_{i+1}$ with respect to J, more precisely

$$\widetilde{\Gamma}_{i+1}(b) = \{w \in \widetilde{V}_{i+1} \colon\ (v_j, w) \in E(J_{j,i+1}),\ \forall\ 1 \le j \le i\}.$$

For a fixed set $\mathcal{B} \subseteq \mathcal{W}_{i+1}$ and $b = (v_1, \dots, v_i) \in \mathcal{W}_i$ we denote the *degree* $d_{\mathcal{B}}(b)$ of b in $\mathcal{B}$ with respect to J by

$$d_{\mathcal{B}}(b) = \left|\left\{w \in \widetilde{\Gamma}_{i+1}(b) \colon\ (v_1, \dots, v_i, w) \in \mathcal{B}\right\}\right|.$$

Next we define (still for a fixed J) the sets of "bad" i-tuples $\mathcal{B}_i = \mathcal{B}_i(\gamma, \mu, \nu) \subseteq \mathcal{W}_i$ mentioned earlier. Although we do not apply the Pick-Up Lemma for $k = l$, for the sake of convenience we consider the neighbourhood of elements in $\mathcal{W}_{l-1}$ in $\widetilde{V}_l$, instead of in V_l.

Definition 4.8 ($\mathcal{B}_{l-1}$, $\mathcal{B}_i^{(a)}$, $\mathcal{B}_i^{(b)}$, $\mathcal{B}_i$). *Let γ, μ, ν be given by* (52)*. We define recursively the following sets of "bad" tuples for $i = l-1, \dots, 1$:*

$$\begin{aligned}
\mathcal{B}_{l-1} &= \mathcal{B}_{l-1}(\gamma,\mu) &&= \left\{b \in \mathcal{W}_{l-1} \colon\ \left|\widetilde{\Gamma}_l(b)\right| < (1-\gamma-\mu)p^{l-1}m\right\},\\
\mathcal{B}_i^{(a)} &= \mathcal{B}_i^{(a)}(\gamma,\mu) &&= \left\{b \in \mathcal{W}_i \colon\ \left|\widetilde{\Gamma}_{i+1}(b)\right| < (1-\gamma-\mu)p^{i}m\right\},\\
\mathcal{B}_i^{(b)} &= \mathcal{B}_i^{(b)}(\nu) &&= \left\{b \in \mathcal{W}_i \colon\ d_{\mathcal{B}_{i+1}}(b) \ge \nu p^i m\right\},\\
\mathcal{B}_i &= \mathcal{B}_i(\gamma,\mu,\nu) &&= \mathcal{B}_i^{(a)}(\gamma,\mu) \cup \mathcal{B}_i^{(b)}(\nu).
\end{aligned}$$

We also consider "bad" events in $\mathcal{G}(n,q)$ defined on the basis of the size of the sets $\mathcal{B}_{l-1}(\gamma,\mu)$, $\mathcal{B}_i^{(a)}(\gamma,\mu)$, $\mathcal{B}_i^{(b)}(\nu)$, and $\mathcal{B}_i(\gamma,\mu,\nu)$ defined above. In the following definition we mean by J an arbitrary subgraph of $G \in \mathcal{G}(n,q)$ satisfying conditions (I)–(IV).

Definition 4.9. *Let γ, μ, ν be given by (52) and let $\eta_i > 0$ $(i = l-1, \ldots, 1)$ be fixed. We define the events*

$$\begin{aligned}
X_{l-1}(\gamma, \mu, \eta_{l-1})&\colon \exists\ J \subseteq G \ s.t.\ |\mathcal{B}_{l-1}| > (\eta_{l-1}/2)m^{l-1},\\
X_i^{(a)}(\gamma, \mu, \eta_i)&\colon \exists\ J \subseteq G \ s.t.\ \left|\mathcal{B}_i^{(a)}\right| > (\eta_i/2)m^i,\\
X_i^{(b)}(\gamma, \mu, \nu, \eta_i, \eta_{i+1})&\colon \exists\ J \subseteq G \ s.t.\ |\mathcal{B}_{i+1}| \le \eta_{i+1}m^{i+1} \ \wedge\ |\mathcal{B}_i^{(b)}| > (\eta_i/2)m^i,\\
X_i(\gamma, \mu, \nu, \eta_i, \eta_{i+1}) &= X_i^{(a)}(\gamma, \mu, \eta_i) \ \vee\ X_i^{(b)}(\gamma, \mu, \nu, \eta_i, \eta_{i+1}).
\end{aligned}$$

For simplicity, we let

$$\begin{aligned}
X_{l-1}^{(a)} &= X_{l-1} = X_{l-1}(\gamma, \mu, \eta_{l-1}),\\
X_i^{(a)} &= X_i^{(a)}(\gamma, \mu, \eta_i) \quad \textit{for } i = 1, \ldots, l-1,\\
X_i^{(b)} &= X_i^{(b)}(\gamma, \mu, \nu, \eta_i, \eta_{i+1}) \quad \textit{for } i = 1, \ldots, l-2,
\end{aligned}$$

and

$$X_i = X_i(\gamma, \mu, \nu, \eta_i, \eta_{i+1}) \quad \textit{for } i = 1, \ldots, l-1.$$

Owing to the special role of X_1 later in the proof, we let

$$X_{\text{bad}} = X_{\text{bad}}(\gamma, \mu, \nu, \eta_1, \eta_2) = X_1(\gamma, \mu, \nu, \eta_1, \eta_2).$$

We will now describe the remaining constants used in the proof. Notice that α and σ were given and we have already fixed γ, μ, and ν in (52) and β_i for $2 \le i \le l-1$ in (53). The (yet unspecified) parameters η_i and ε will be determined by Propositions 4.10 and 4.11. First we set $\eta_1 = \nu$. Then Proposition 4.10 (PU_{i+1}) inductively describes $\eta_{i+1} = \eta_{i+1}(\beta_{i+1}, \gamma, \mu, \nu, \eta_i)$ for $i = 1, \ldots, l-2$ such that $\mathbb{P}(X_i^{(b)}) = o(1)$. Finally, for $i = 1, \ldots, l-1$, Proposition 4.11 (TL_i) implies the choice for $\varepsilon_i = \varepsilon_i(\alpha, \gamma, \mu, \eta_i)$ such that $\mathbb{P}(X_i^{(a)}) = o(1)$. We set

$$\varepsilon = \min\{\varepsilon_i \colon i = 1, \ldots, l-1\}.$$

A diagram illustrating the definition scheme for the constants above is given in Figure 2.

$$\begin{tikzcd}
\alpha, \sigma, \gamma, \mu, \nu, \beta_2, \ldots, \beta_{l-1} \arrow[d] \\
\eta_1 = \nu \arrow[r, "\mathrm{PU}_2"] \arrow[d, "\mathrm{TL}_1"] & \eta_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \eta_i \arrow[r, "\mathrm{PU}_{i+1}"] \arrow[d, "\mathrm{TL}_i"] & \eta_{i+1} \arrow[r] \arrow[d] & \cdots \arrow[r, "\mathrm{PU}_{l-1}"] & \eta_{l-1} \arrow[d, "\mathrm{TL}_{l-1}"] \\
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_i & \varepsilon_{i+1} & \cdots & \varepsilon_{l-1}
\end{tikzcd}$$

$$\underbrace{\varepsilon_1 \quad \varepsilon_2 \quad \cdots \quad \varepsilon_i \quad \varepsilon_{i+1} \quad \cdots \quad \varepsilon_{l-1}}_{\varepsilon = \min \varepsilon_i}$$

Figure 2 Flowchart of the constants

Thus, ε is defined for any given α and σ, as claimed in Lemma 2.3. From now on, these constants are fixed for the rest of the proof of Lemma 2.3.

Tools. We need some auxiliary results before we prove Lemma 2.3. For this purpose we state variants of the Pick-Up Lemma, Lemma 4.3, and of the k-tuple lemma, Lemma 4.7, in the form that we apply these later. These variants will be referred to as (PU_{i+1}) and (TL_i).

The next proposition follows from Lemma 4.3 for $k = i+1$ $(1 \leq i \leq l-2)$.

Proposition 4.10 (PU_{i+1})**.** *Fix $1 \leq i \leq l-2$. Let α, $\sigma > 0$ be arbitrary, let γ, μ, ν and β_{i+1} be given by (52) and (53), and let η_i be defined as stated in Section 4.4 (see Figure 2). Then there exists $\eta_{i+1} = \eta_{i+1}(\beta_{i+1}, \gamma, \mu, \nu, \eta_i) > 0$ such that for every $t \geq l$ a random graph G in $\mathcal{G}(n,q)$ satisfies the following property with probability $1 - o(1)$: If J is a subgraph of G satisfying (I)–(IV) and $\mathcal{B}_{i+1}(\gamma, \mu, \nu) \subseteq \mathcal{W}_{i+1}$ is such that*

$$|\mathcal{B}_{i+1}(\gamma, \mu, \nu)| \leq \eta_{i+1} m^{i+1}, \tag{54}$$

then the number of i-tuples b in $\mathcal{W}_i$ with

$$d_{\mathcal{B}_{i+1}}(b) \geq \nu p^i m$$

is less than

$$\frac{\eta_i}{2} m^i,$$

which means

$$\left|\mathcal{B}_i^{(b)}(\nu)\right| \leq \frac{\eta_i}{2} m^i. \tag{55}$$

Furthermore,

$$|\widetilde{V}_{i+1}| \geq (1-\mu)m$$

holds.

We restate Proposition 4.10, by using the events $X_i^{(b)}$ from Definition 4.9. Observe that inequalities (54) and (55) correspond to $X_i^{(b)}$, so that $\mathbb{P}(X_i^{(b)}) = o(1)$ is equivalent to the first part of Proposition 4.10′.

Proposition 4.10′ (PU_{i+1})**.** *Fix $1 \leq i \leq l-2$. Let α, $\sigma > 0$ be arbitrary, let γ, μ, ν and β_{i+1} be given by (52) and (53), and let η_i be defined as stated in Section 4.4 (see Figure 2). Then there exists $\eta_{i+1} = \eta_{i+1}(\beta_{i+1}, \gamma, \mu, \nu, \eta_i) > 0$ such that for every $t \geq l$*

$$\mathbb{P}\left(X_i^{(b)}(\gamma, \mu, \nu, \eta_i, \eta_{i+1})\right) = o(1)$$

and

$$\mathbb{P}\left(|\widetilde{V}_{i+1}| < (1-\mu)m\right) = o(1).$$

Proof. We apply Lemma 4.3 for $k = i+1$ and with the following choice of β^{PU}, ζ^{PU},

μ^{PU}:

$$\beta^{\mathrm{PU}} = \beta_{i+1}, \tag{56}$$
$$\zeta^{\mathrm{PU}} = \frac{\eta_i \nu}{2}, \tag{57}$$
$$\mu^{\mathrm{PU}} = \mu. \tag{58}$$

Lemma 4.3 then gives η^{PU}, from which we define the constant η_{i+1} we are looking for by putting

$$\eta_{i+1} = \eta^{\mathrm{PU}}.$$

We assume inequality (54) holds. In other words, the number of the "bad" $(i+1)$-tuples in $\mathcal{W}_{i+1}$ is

$$|\mathcal{B}_{i+1}| \le \eta_{i+1} m^{i+1} = \eta^{\mathrm{PU}} m^{i+1}. \tag{59}$$

On the other hand, if we assume that (55) does not hold (*i.e.*, the event $X_i^{(b)}$ occurs), then the number of $(i+1)$-tuples in $\mathcal{B}_{i+1}$ that have been "picked-up" has to exceed

$$\frac{\eta_i}{2} m^i \cdot \nu p^i m = \zeta^{\mathrm{PU}} p^i m^{i+1}. \tag{60}$$

The Pick-Up Lemma bounds the number of these configurations in

$$\binom{V_1 \times V_{i+1}}{T} \times \cdots \times \binom{V_i \times V_{i+1}}{T}$$

by

$$\left(\beta^{\mathrm{PU}}\right)^{iT} \cdot \binom{m^2}{T}^i = \left(\beta_{i+1}\right)^{iT} \binom{m^2}{T}^i. \tag{61}$$

We now estimate the number of all possible graphs J satisfying (I)–(IV) for which (59) holds but the number of members in $\mathcal{B}_{i+1}$ that have been "picked-up" exceeds (60). There are fewer than $\binom{n}{m}^l$ different ways to fix the l vertex classes of J. Furthermore, observe that $\mathcal{B}_{i+1}$ is determined by all the edges in $J_{jj'}$ ($i < j' \le l$, $1 \le j < j' \le l$, which gives $\binom{l}{2} - \binom{i+1}{2}$ different pairs jj'). Thus we have at most $\binom{m^2}{T}^{\binom{l}{2}-\binom{i+1}{2}}$ possibilities to determine $\mathcal{B}_{i+1}$. This, combined with (61), (III), and (53), yields that

$$\begin{aligned} \mathbb{P}\left(X_i^{(b)}\right) &\le \binom{n}{m}^l \binom{m^2}{T}^{\binom{l}{2}-\binom{i+1}{2}} \cdot \left(\beta_{i+1}\right)^{iT} \binom{m^2}{T}^i \cdot q^{\left(\binom{l}{2}-\binom{i}{2}\right)T} \\ &\le 2^{nl} \left(\frac{\mathrm{e} m^2 q}{T}\right)^{\left(\binom{l}{2}-\binom{i}{2}\right)T} \left(\beta_{i+1}\right)^{iT} \le 2^{nl} \left(\left(\frac{\mathrm{e}}{\alpha}\right)^{\binom{l}{2}-\binom{i}{2}} \left(\beta_{i+1}\right)^i\right)^T \le 2^{nl-T}. \end{aligned}$$

Since l is fixed and $T \gg m = n/t$, we have

$$\mathbb{P}\left(X_i^{(b)}\right) = o(1).$$

Note that the set $\widetilde{V}_{i+1}$ was determined by the application of the Pick-Up Lemma. Therefore, the second assertion in Proposition 4.10′ also follows from the proof above. $\square$

The following is an easy consequence of Lemma 4.7 for $k = i$ $(1 \le i \le l-1)$.

Proposition 4.11 (TL_i). *Fix $1 \le i \le l-1$. Let α, $\sigma > 0$ be arbitrary, let γ, μ be given by* (52), *and let η_i be defined as stated in Section 4.4 (see Figure 2). Then there exists $\varepsilon_i = \varepsilon_i(\alpha, \gamma, \mu, \eta_i) > 0$ such that for every $t \ge l$ a random graph G in $\mathcal{G}(n,q)$ satisfies the following property with probability $1 - o(1)$: If $\varepsilon \le \varepsilon_i$ and J is a subgraph of G satisfying (I)–(IV), then the number of i-tuples b in $\mathcal{W}_i$ with*

$$\left|\widetilde{\Gamma}_{i+1}(b)\right| < (1-\gamma-\mu)p^i m$$

is less than

$$\frac{\eta_i}{2} m^i,$$

which means that

$$\left|\mathcal{B}_i^{(a)}(\gamma,\mu)\right| \le \frac{\eta_i}{2} m^i. \tag{62}$$

We can reformulate Proposition 4.11 in a shorter way by using the event $X_i^{(a)}$ (see Definition 4.9).

Proposition 4.11′ (TL_i). *Fix $1 \le i \le l-1$. Let α, $\sigma > 0$ be arbitrary, let γ, μ be given by* (52) *and let η_i be defined as stated in Section 4.4 (see Figure 2). Then there exists $\varepsilon_i = \varepsilon_i(\alpha, \gamma, \mu, \eta_i) > 0$ such that for every $t \ge l$ and $\varepsilon \le \varepsilon_i$*

$$\mathbb{P}\left(X_i^{(a)}(\gamma,\mu,\eta_i)\right) = o(1).$$

Proof. We apply the k-tuple lemma, Lemma 4.7, with $k = i$, $\alpha^{\mathrm{TL}} = \alpha/3$, $\gamma^{\mathrm{TL}} = \gamma$ and

$$\eta^{\mathrm{TL}} = \eta_i/(2i^i). \tag{63}$$

The k-tuple lemma gives an $\varepsilon^{\mathrm{TL}}$ and we set $\varepsilon_i = \min\{(\varepsilon^{\mathrm{TL}})^3, \alpha/2, 1/27\}$. Let $\varepsilon \le \varepsilon_i$ and J be a subgraph of $G \in \mathcal{G}(n,q)$ satisfying (I)–(IV). Set $U = \widetilde{V}_{i+1}$ and $W = \bigcup_{j=1}^{i} V_j$. By (IV), the graph $J_{jj'}$ $(1 \le j < j' \le i)$ is (ε, q)-regular. A straightforward argument (using $\varepsilon \le 1/27$ and Lemma 3.2 for $C = 3/2$) shows that with probability $1 - o(1)$ the subgraph $J[U,W]$ is at least $(\sqrt[3]{\varepsilon}, q)$-regular and therefore $(\varepsilon^{\mathrm{TL}}, q)$-regular, which yields condition (ii) of Lemma 4.7. Moreover, with probability $1 - o(1)$ we have, say,

$$|E(G[U,W])| \le \frac{3}{2} q(1-\mu)km^2,$$

and using the regularity of J we see that

$$|E(J[U,W])| \ge (\alpha - \varepsilon)q(1-\mu)km^2,$$

which by our choice of ε gives condition (i) of Lemma 4.7. Finally, with assertion (II) for J all assumptions of the k-tuple lemma are satisfied for $J[U,W]$.

Therefore, the k-tuple lemma implies that, with probability $1 - o(1)$, we have

$$\left|\left\{b \in \mathcal{W}_i \colon \left|\widetilde{\Gamma}_{i+1}(b)\right| \le (1-\gamma)p^i(1-\mu)m\right\}\right| \le \eta^{\mathrm{TL}} \binom{im}{i}.$$

The choice of η^{TL} in (63) gives

$$\left|\left\{b \in \mathcal{W}_i\colon \ \left|\widetilde{\Gamma}_{i+1}(b)\right| \leq (1-\gamma-\mu+\gamma\mu)p^i m\right\}\right| \leq \frac{\eta_i}{2}m^i,$$

and hence (62) holds with probability $1-o(1)$, by the simple observation that

$$\left|\widetilde{\Gamma}_{i+1}(b)\right| \leq (1-\gamma-\mu)p^i m \quad \text{implies} \quad \left|\widetilde{\Gamma}_{i+1}(b)\right| \leq (1-\gamma-\mu+\gamma\mu)p^i m.$$

$\square$

Main proof. Our proof of the Counting Lemma, Lemma 2.3, follows immediately from Lemmas 4.12 and 4.13 below. Lemma 4.12 is a probabilistic statement and asserts that the probability of the event $X_{\mathrm{bad}} \subseteq \mathcal{G}(n,q)$ is $o(1)$. On the other hand, Lemma 4.13 is deterministic and claims that if a graph G is not in X_{bad} and J is a not necessarily induced subgraph of G satisfying (I)–(IV), then J contains the right number of copies of K_l. We apply the technical propositions from the last section in the proof of the probabilistic Lemma 4.12 below.

Lemma 4.12. *For arbitrary α and $\sigma > 0$, let γ, μ, ν be given by (52), and let ε and η_i $(i = 2, \ldots, l-1)$ be defined as stated in Section 4.4. Let G be a random graph in $\mathcal{G}(n,q)$. Then*

$$\mathbb{P}(G \in X_{\mathrm{bad}}(\gamma,\mu,\nu)) = o(1).$$

Proof. Formal logic implies

$$\begin{array}{lllllllll} X_{\mathrm{bad}} & \subseteq & X_1^{(a)} & \vee & (X_1^{(b)} \wedge \neg X_2) & \vee & X_2^{(a)} & \vee & (X_2^{(b)} \wedge \neg X_3) \\ & & & & & \vee & \vdots & \vee & \vdots \\ & & & & & \vee & X_{l-2}^{(a)} & \vee & (X_{l-2}^{(b)} \wedge \neg X_{l-1}) \quad \vee \quad X_{l-1}, \end{array}$$

and thus, by Propositions 4.10 and 4.11 (notice $X_{l-1} = X_{l-1}^{(a)}$ by Definition 4.9), we have

$$\mathbb{P}\left(X_{\mathrm{bad}}\right) \leq \sum_{i=1}^{l-2}\left(\mathbb{P}(X_i^{(a)}) + \mathbb{P}(X_i^{(b)})\right) + \mathbb{P}(X_{l-1}) = o(1).$$

$\square$

Lemma 4.13. *For arbitrary α and $\sigma > 0$, let γ, μ, ν be given by (52), and let ε and η_i $(i = 2, \ldots, l-1)$ be defined as stated in Section 4.4. Then every subgraph J of a graph $G \notin X_{\mathrm{bad}}(\gamma,\mu,\nu)$ satisfying conditions (I)–(IV) contains at least*

$$(1-\sigma)p^{\binom{l}{2}}m^l$$

copies of K_l.

Proof. We shall prove by induction on i that the following statement holds for all $1 \leq i \leq l$:

($\mathcal{S}_i$) Let J be a subgraph of $G \notin X_{\mathrm{bad}}$ such that (I)–(IV) apply. Then there are at least $(1-\gamma-\mu-\nu)^i p^{\binom{i}{2}} m^i$ different i-tuples in $\mathcal{W}_i \setminus \mathcal{B}_i$ that induce a K_i in $J[V_1, \dots, V_i]$.

Suppose $i = 1$. Note that $\neg X_{\mathrm{bad}}$ implies that $|V_1 \cap \mathcal{B}_1| \le \eta_1 m = \nu m$. Therefore $V_1 \setminus \mathcal{B}_1$ contains at least $(1-\nu)m \ge (1-\gamma-\mu-\nu)p^0 m^1$ copies of K_1.

We now proceed to the induction step. Assume $i \ge 2$ and ($\mathcal{S}_{i-1}$) holds. Therefore, $\mathcal{W}_{i-1} \setminus \mathcal{B}_{i-1}$ contains at least $(1-\gamma-\mu-\nu)^{i-1} p^{\binom{i-1}{2}} m^{i-1}$ different $(i-1)$-tuples $b = (v_1, \dots, v_{i-1})$, each constituting the vertex set of a K_{i-1} in $J[V_1, \dots, V_{i-1}]$. For every $b \in \mathcal{W}_{i-1} \setminus \mathcal{B}_{i-1}$, we have

(i) $|\widetilde{\Gamma}_i(b)| \ge (1-\gamma-\mu)p^{i-1}m$, and

(ii) $d_{\mathcal{B}_i}(b) < \nu p^{i-1} m$.

Therefore, every such b extends to at least $(1-\gamma-\mu-\nu)p^{i-1}m$ different $b' \in \mathcal{W}_i \setminus \mathcal{B}_i$ that correspond to a $K_i \subseteq J[V_1, \dots, V_i]$. This implies ($\mathcal{S}_i$), and hence our induction is complete.

Assertion ($\mathcal{S}_l$) and the choice of γ, μ, and ν in (52) give at least

$$(1-\gamma-\mu-\nu)^l p^{\binom{l}{2}} m^l = (1-\sigma)p^{\binom{l}{2}} m^l$$

copies of K_l in J. $\square$

Clearly, Lemmas 4.12 and 4.13 together imply the Counting Lemma, Lemma 2.3.

5. The d-degenerate case

In this section we describe how the proof of Theorem 1.2 extends to the proof of Theorem 1.2′. The detailed proof of Theorem 1.2′ is given in [14]. First we outline the proof of Theorem 1.2′, assuming a counterpart for the Counting Lemma, Lemma 2.3, which we state below.

Let d be an integer and H a d-degenerate graph on h vertices. Let $t \ge h \ge 2$ be fixed integers and let n be sufficiently large. Let α and ε be constants greater than 0. Suppose J is an h-partite subgraph of G with vertex classes $V_1, \dots, V_h$ satisfying the following conditions:

(I′) $|V_i| = m = n/t$ for all i,

(II′) $q^d n \gg (\log n)^4$,

(III′) for all $1 \le i < j \le h$,

$$|E(J_{ij})| = \begin{cases} T = pm^2 & \text{if } \{w_i, w_j\} \in E(H) \\ 0 & \text{if } \{w_i, w_j\} \notin E(H), \end{cases}$$

where $1 > \alpha q = p \gg 1/n$, and

(IV′) J_{ij} $(1 \le i < j \le h)$ is (ε, q)-regular.

We now state the appropriate counting lemma for the d-degenerate case.

Lemma 2.3′ (Counting lemma, d-degenerate case). *For every α, $\sigma > 0$, integer d and d-degenerate graph H on h vertices, there exists $\varepsilon > 0$ such that for every $t \ge h$ a* *random graph G in $\mathcal{G}(n,q)$ satisfies the following property with probability $1-o(1)$: Every subgraph $J \subseteq G$ satisfying conditions (I′)–(IV′) contains at least*

$$(1-\sigma)p^{|E(H)|}m^h$$

copies of H.

Sketch of the proof of Theorem 1.2′. Let d be a fixed positive integer and suppose H is a d-degenerate graph of order h. Let the vertices of H be ordered $w_1,\dots,w_h$ such that each w_i has at most d neighbours in $\{w_1,\dots,w_{i-1}\}$.

At first, we follow the proof of Theorem 1.2 and observe that, by (16), the Erdős–Stone–Simonovits theorem (see (1)) implies that F_{cluster} contains at least one copy of H if we choose t_0^{SRL} big enough. This yields, in the same way as in the original proof, that F contains an h-partite $(\varepsilon^{\text{Lem2.3}'},q)$-regular graph J with $|E(J_{ij})| = \alpha^{\text{Lem2.3}'}qm^2$ if $\{w_i,w_j\} \in E(H)$ and $E(J_{ij}) = \emptyset$ if $\{w_i,w_j\} \notin E(H)$. For $1 \le i \le h$, we identify the vertex class V_i in J with the vertex $w_i \in V(H)$.

We then apply Lemma 2.3′ with appropriate $\alpha^{\text{Lem2.3}'}$ and $0 < \sigma < 1$ to deduce Theorem 1.2′. □

Finally, we outline of the proof of Lemma 2.3′.

Sketch of the proof of Lemma 2.3′. We prove Lemma 2.3′ in the same way as Lemma 2.3. Observe that conditions (I) and (IV) are unchanged in Lemma 2.3′. Conditions (III) and (III′) state that J is a "blown-up" copy of the subgraph we are considering, namely, K_l and H, respectively. The main difference is between (II) and (II′).

The crucial part of the proof of the original counting lemma is the definition of "bad" tuples in Definition 4.8. Recall that the proof of Lemma 2.3 used the Pick-Up Lemma (Lemma 4.3). There we had to discard a small portion of the vertices of V_i (of high degree to some V_j, $j < i$) to obtain $\widetilde{V}_i \subseteq V_i$. For $1 \le i \le |V(K_l)|$, we considered two types of "bad" $(i-1)$-tuples in $\mathcal{W}_{i-1} = V_1 \times \cdots \times V_{i-1}$. The first type, the ones put in $\mathcal{B}_{i-1}^{(a)}$, was determined by the size of their joint neighbourhood in $\widetilde{V}_i$. On the other hand, an $(i-1)$-tuple in $\mathcal{W}_{i-1}$ was bad 'of the second type', and was put in $\mathcal{B}_{i-1}^{(b)}$, if it was contained in too many "bad" i-tuples in $\mathcal{B}_i$.

We use the property that H is d-degenerate to change the definition of $\mathcal{B}_i^{(a)}$. In the proof of Lemma 2.3 we wanted to extend inductively each K_{i-1} in $\mathcal{W}_{i-1}$ that is not "bad" to the right number of copies of K_i in $\mathcal{W}_i \setminus \mathcal{B}_i$. For this purpose we had to consider the joint neighbourhood of all vertices in the $(i-1)$-tuple. The graph H is d-degenerate, and we fixed an ordering $w_1,\dots,w_h$ of $V(H)$ so that each w_i has at most d neighbours in $\{w_1,\dots,w_{i-1}\}$. This implies that it is sufficient to consider the joint neighbourhood of at most d elements of the $(i-1)$-tuple to determine its "badness", or its membership in $\mathcal{B}_{i-1}^{(a)}$. For $i = 1,\dots,h$, we define the index sets I_i consisting of the the indices of the neighbours of w_i in $\{w_1,\dots,w_{i-1}\}$. Also, for a fixed $(i-1)$-tuple $(v_1,\dots,v_{i-1}) \in \mathcal{W}_{i-1}$, we consider the joint neighbourhood of $\bigcap \Gamma(v_j) \cap \widetilde{V}_i =: \bigcap \widetilde{\Gamma}(v_j)$, where the intersection is taken over $j \in I_i$. More precisely, we define $\mathcal{B}_i^{(a)}$ as follows:

$$\begin{aligned} I_i &= \{j \in [i-1]\colon\ \{w_j, w_i\} \in E(H)\}, \\ \mathcal{B}_{i-1}^{(a)}(\gamma,\mu) &= \left\{(v_1,\dots,v_{i-1}) \in \mathcal{W}_{i-1}\colon\ \left|\bigcap_{j\in I_i} \widetilde{\Gamma}_i(v_j)\right| < (1-\gamma-\mu)p^{|I_i|}m\right\}. \end{aligned}$$

Obviously,

$$|I_i| \le d \quad \text{for}\ \ 1 \le i \le h \tag{64}$$

holds. The definition of $\mathcal{B}_i^{(b)}$ remains almost unchanged; again, for some $\mathcal{B} \subseteq \mathcal{W}_{i+1}$ and $b = (v_1,\dots,v_i) \in \mathcal{W}_i$, we set $d_{\mathcal{B}}(b) = |\{w \in \widetilde{\Gamma}_{i+1}\colon\ (v_1,\dots,v_i,w) \in \mathcal{B}\}|$ and we only adjust the exponent of p:

$$\mathcal{B}_i^{(b)} = \mathcal{B}_i^{(b)}(\nu) = \left\{b \in \mathcal{W}_i\colon\ d_{\mathcal{B}_{i+1}}(b) \ge \nu p^{|I_i|}m\right\}.$$

Then we define the corresponding events exactly as in Definition 4.9.

The proof of Lemma 2.3 consists of two propositions (Propositions 4.10 and 4.11) and two lemmas (Lemmas 4.12 and 4.13). We now discuss the proofs of the corresponding results with the new definition for the families $\mathcal{B}_i^{(a)}$ and $\mathcal{B}_i^{(b)}$ under (I′)–(IV′) instead of (I)–(IV), and with K_l replaced by an arbitrary d-degenerate graph H. We define the following constants, slightly different compared to the ones in the original proof (see (52) and (53)):

$$\gamma = \mu = \nu = \frac{1}{3}\left(1 - (1-\sigma)^{1/h}\right), \tag{65}$$

and, for $1 \le i \le l-2$ and $|I_{i+1}| > 0$,

$$\beta_{i+1} = \left(\frac{1}{2}\left(\frac{\alpha}{\mathrm{e}}\right)^{\sum_{j=i}^{h}|I_j|}\right)^{1/|I_{i+1}|}. \tag{66}$$

The other constants are defined in the same way as described in Section 4.4 (see Figure 2, with l replaced by h).

We now discuss the proofs of the results that correspond to Propositions 4.10 and 4.11 and Lemmas 4.12 and 4.13.

Proposition 4.10. The proof is an application of the Pick-Up Lemma, Lemma 4.3, for $k = i+1$. The Pick-Up Lemma does not require condition (II). It is already valid for $q(n) \gg 1/n$, which is still guaranteed by (II′). It is easy to see that $X_i^{(b)}$ is impossible if we set $\eta_{i+1} = \eta_i\nu/2$ and if $|I_{i+1}| = 0$. If $|I_{i+1}| > 0$, then essentially the same calculation with the new β_{i+1} defined in (66) gives the proposition. We apply the Pick-Up Lemma for the space $\prod_{j\in I_{i+1}} \binom{V_j\times V_{i+1}}{T}$ and the projection of $\mathcal{B}_{i+1}$ onto $\prod_{j\in I_{i+1}} V_j \times V_{i+1}$.

Proposition 4.11. The proof is a straightforward application of the k-tuple lemma, Lemma 4.7. In the original proof we apply the k-tuple lemma for $k = i$ $(1 \le i \le l-1)$ and we needed condition (II) (namely, $q^{l-1}n \gg (\log n)^4$) for $i = l-1$. Here, the new definition of $\mathcal{B}_{i-1}^{(a)}$ from above comes into play. Inequality (64) ensures that we consider at most the joint neighbourhood of d vertices. This means that we apply the k-tuple lemma for $k \le d$ and thus condition (II′) (namely, $q^d n \gg (\log n)^4$) is sufficient.

Lemma 4.12. For the proof we only apply Propositions 4.10 and 4.11. In order to adjust the proof, we simply replace l by h.

Lemma 4.13. This lemma is a deterministic statement. It is not affected by the change from (II) to (II′), but the induction there is formulated in such a way that it relies on the structure (symmetries) of K_l. We fix this and reformulate $(\mathcal{S}_i)$ to

($\mathcal{S}'_i$) Let J be a subgraph of $G \notin X_{\text{bad}}$ such that (I′)–(IV′) apply. Then there are at least $(1-\gamma-\mu-\nu)^i p^{\sum_{j=1}^{i}|I_j|} m^i$ different i-tuples in $\mathcal{W}_i \setminus \mathcal{B}_i$ which induce a $H[\{w_1, \ldots, w_i\}]$ in $J[V_1, \ldots, V_i]$.

Thus, the induction works exactly the same way and $(\mathcal{S}'_h)$ implies the result, by our choice of the constants in (65) (there we again replace l with h and $\binom{l}{2}$ with $|E(H)|$). $\square$

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