The Size-Ramsey Number of Trees

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Abstract. If G and H are graphs, let us write $G \to (H)_2$ if G contains a monochromatic copy of H in any 2-colouring of the edges of G. The **size-Ramsey number** $r_{\mathrm{e}}(H)$ of a graph H is the smallest possible number of edges a graph G may have if $G \to (H)_2$. Suppose T is a tree of order $|T| \geq 2$, and let t_0, t_1 be the cardinalities of the vertex classes of T as a bipartite graph, and let $\Delta(T)$ be the maximal degree of T. Moreover, let Δ_0, Δ_1 be the maxima of the degrees of the vertices in the respective vertex classes, and let $\beta(T) = t_0\Delta_0 + t_1\Delta_1$. Beck [7] proved that $\beta(T)/4 \leq r_{\mathrm{e}}(T) = O\{\beta(T)(\log|T|)^{12}\}$, improving on a previous result of his [6] stating that $r_{\mathrm{e}}(T) \leq \Delta(T)|T|(\log|T|)^{12}$. In [6], Beck conjectures that $r_{\mathrm{e}}(T) = O\{\Delta(T)|T|\}$, and in [7] he puts forward the stronger conjecture that $r_{\mathrm{e}}(T) = O\{\beta(T)\}$. Here, we prove the first of these conjectures, and come quite close to proving the second by showing that $r_{\mathrm{e}}(T) = O\{\beta(T)\log\Delta(T)\}$.

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1. Introduction

In this note we are concerned with a numerical problem in Ramsey theory: we shall study the size-Ramsey number of trees. Before we proceed, let us introduce some notation and definitions. For a graph G, we write $|G|$ for the **order** $|V(G)|$ of G, and $e(G)$ for its **size** $e(G)$. Let a real $0 \leq \gamma \leq 1$ and an integer $r \geq 2$ be fixed, and suppose G and H are graphs. We write $G \to_\gamma H$ if any subgraph $J \subset G$ of G with size $e(J) \geq \gamma e(G)$ contains an isomorphic copy of H as a subgraph, and we write $G \to (H)_r$ if G contains a monochromatic copy of H in any edge-colouring of G with r colours.

Settling a one-hundred-dollar problem of Erdős, ten years ago Beck [6] proved the following striking result. Let P^t be the path of order t. If $0 < \gamma \leq 1$ and $p = C_\gamma/n$, where $C_\gamma > 0$ is a constant that depends only on γ, then the random graph $G_p = G_{n,p} \in \mathcal{G}(n,p)$ is almost surely such that $G \to_\gamma P^t$ for $t = \lfloor C_\gamma^{-1} n \rfloor$. As an immediate corollary, one has that for any fixed $r \geq 2$ the **r-size-Ramsey number**

$$r_e(H, r) = \min\{e(G)\colon G \to (H)_r\} \tag{1}$$

of the path $H = P^t$ is $O(c_r t)$ for some constant c_r that depends only on r. Also in [6], Beck shows by non-constructive means that the size-Ramsey number $r_e(T) = r_e(T, 2)$ of a tree $T = T^t$ of order t and maximal degree $\Delta = \Delta(T)$ is not greater than $\Delta t(\log t)^{12}$. (The proof of this result is complex and it is heavily based on the probabilistic method: besides random graphs, the Erdős–Lovász sieve is used.) Thus, for trees of bounded maximal degree, the size-Ramsey number is nearly linear in $|T|$. Indeed, Beck conjectures in [6] that $r_e(T) = O\{\Delta(T)|T|\}$.

More recently, Friedman and Pippenger [11] improved on Beck's result by showing that, for trees T of bounded maximal degree, it does indeed hold that the size-Ramsey number $r_e(T)$ is linear in $|T|$. The proof in [11] is based on a new, beautiful tree-universality result for expanding graphs. With this result in hand, basically following Alon and Chung [3] and Beck [6], Friedman and Pippenger [11] prove the following result. Let $0 < \gamma \leq 1$, and $1 \leq \Delta < t$ be given. Then, for suitable primes p and q, the Ramanujan graph $X = X^{p,q}$ constructed by Lubotzky, Phillips, and Sarnak [16] is such that (i) $e(X) \leq c\Delta^4\gamma^{-3}t$, where c is an absolute constant, and (ii) $X \to_\gamma T$ for any tree T of order $|T| \leq t$ and maximal degree $\Delta(T) \leq \Delta$. Note that this is a 'density' type result rather than a Ramsey-theoretical one, by which we mean that it concerns the property $G \to_\gamma T$ rather than $G \to (T)_r$. Clearly, the Friedman–Pippenger result above implies that $r_{\mathrm{e}}(T) = O\{\Delta(T)^4|T|\}$ for any tree T. This bound has very recently been improved by Ke [14], who proved a density type result implying that $r_{\mathrm{e}}(T) = O\{\Delta(T)^2|T|\}$. Our first main result in this note, Theorem 9 (see also Corollary 10), improves this to $r_{\mathrm{e}}(T) = O\{\Delta(T)|T|\}$, thus verifying the conjecture of Beck [6].

Now, note that if T is the $(t-1)$-star $K^{1,t-1}$ $(t \geq 2)$, then clearly $r_{\mathrm{e}}(T) = 2t-3$, whereas $\Delta(T)|T| = t(t-1)$. Thus the bound $r_{\mathrm{e}}(T) = O\{\Delta(T)|T|\}$ may be far from sharp for some trees T. Beck [7] has identified what seems to be the 'correct' parameter of a tree T that determines the order of $r_{\mathrm{e}}(T)$. If T has bipartition $V(T) = V_0(T) \cup V_1(T)$ and $t_\sigma = |V_\sigma(T)|$, $\Delta_\sigma = \Delta_\sigma(T) = \max\{d(v)\colon v \in V_\sigma(T)\}$ $(\sigma \in \{0,1\})$, let $\beta(T) = t_0\Delta_0 + t_1\Delta_1$. Note that for instance $\beta(K^{1,t-1}) = 2(t-1)$ and moreover $\beta(T) \leq \Delta(T)|T|$ for any tree T. Improving his previous result, Beck [7] proved that

$$\beta(T)/4 \leq r_{\mathrm{e}}(T) \leq C\beta(T)(\log|T|)^{12} \tag{2}$$

for any tree T and some absolute constant C, and thus determined $r_{\mathrm{e}}(T)$ up to a $(\log|T|)^{12}$ factor. Beck conjectures in [7] that the lower bound in (2) gives the correct order of $r_{\mathrm{e}}(T)$, *i.e.* that $r_{\mathrm{e}}(T) = O\{\beta(T)\}$. We are unable to prove this conjecture, but here we considerably improve the upper bound in (2) by showing that $r_{\mathrm{e}}(T) \leq C\beta(T)\log\Delta(T)$ for some absolute constant C. Furthermore, the result of Beck that gives the upper bound in (2) is intrinsically Ramsey-theoretical, whereas ours is a density type result.

Our method is based on the Friedman–Pippenger tree-universality result. We in fact obtain a variant of that result using the same argument, and then we show how our bounds follow from this variant and a simple result concerning random bipartite graphs. Our methods are non-constructive owing to the use of random graphs, but we remark that, for most trees T, there are explicit constructions of graphs G that give $r_{\mathrm{e}}(T) = O\{\Delta(T)|T|\}$. These constructions are based on the Ramanujan graphs $X^{p,q}$ of Lubotzky, Phillips, and Sarnak [16], and certain Cayley graphs of Abelian groups due to Alon [2]. For trees T with $\Delta(T)$ about $|T|^{1-1/d}$ for some integer $d \geq 2$, we may prove that $r_{\mathrm{e}}(T) = O\{\Delta(T)|T|\}$ constructively by considering projective geometries. (See Section 5).

This note is organised as follows. Our variant of Friedman and Pippenger's tree-universality result is stated and proved in Section 2, and in the following section we give the results concerning random bipartite graphs that we need. For the inequalities used in Section 3 as well as for definitions not given here in detail, we refer the reader to [8]. In Section 4 we prove our main results. Comments concerning explicitly constructed graphs that may replace the random graphs used in our proofs are given in Section 5.

2. A tree-universality result

We shall assume throughout that all our bipartite graphs have been given a fixed bipartition. More specifically, if G is a bipartite graph, its associated bipartition will be $V(G) = V_0(G) \cup V_1(G)$. In particular, if T is a tree of order $t = |T| \geq 2$, we assume fixed one of the canonical bipartitions $V(T) = V_0(T) \cup V_1(T)$ of T. Also, if G is a bipartite graph with associated bipartition $V(G) = V_0(G) \cup V_1(G)$, we let $\Delta_\sigma(G) = \max\{d_G(v)\colon v \in V_\sigma(G)\}$ and put $n_\sigma(G) = |V_\sigma(G)|$ ($\sigma \in \{0,1\}$). We sometimes write $G = G^{n_0,n_1}$ if $n_\sigma(G) = n_\sigma$ ($\sigma \in \{0,1\}$). If T is a tree, and $\Delta_\sigma(T) \leq \Delta_\sigma$, $n_\sigma(T) \leq t_\sigma$ ($\sigma \in \{0,1\}$), we say that T is a $(t_0, \Delta_0; t_1, \Delta_1)$-**tree**.

Now let J be a bipartite graph with associated bipartition $V(J) = V_0(J) \cup V_1(J)$. If for every $X \subset V_\sigma(J)$ with $|X| \leq b_\sigma$ ($\sigma \in \{0,1\}$) we have $|\Gamma_J(X)| \geq f_\sigma|X|$, we say that J is a $(b_0, f_0; b_1, f_1)$**-expanding** bipartite graph. The main result in this section is the following.

Theorem 1. *Suppose* $1 \leq \Delta_0 \leq t_1$ *and* $1 \leq \Delta_1 \leq t_0$ *are fixed integers. Then every non-empty* $(2t_1/\Delta_0, 2\Delta_0; 2t_0/\Delta_1, 2\Delta_1)$*-expanding bipartite graph contains as a subgraph every* $(t_0, \Delta_0; t_1, \Delta_1)$*-tree.*

The rest of this section is devoted to the proof of the above result. Thus, suppose t_σ, Δ_σ ($\sigma \in \{0,1\}$) are as in Theorem 1, let T be a fixed $(t_0, \Delta_0; t_1, \Delta_1)$-tree, and let J be a $(2t_1/\Delta_0, 2\Delta_0; 2t_0/\Delta_1, 2\Delta_1)$-expanding bipartite graph. We shall show that J contains a copy of T as a subgraph.

Let $S \subset T$ be a subtree of T. A function $f\colon V(S) \to V(J)$ is an **embedding** of S in J if f is injective, it preserves the adjacency relation, and moreover it preserves the vertex classes, *i.e.* $f(V_\sigma(T) \cap V(S)) \subset V_\sigma(J)$ for $\sigma \in \{0,1\}$. In what follows the indices will be reduced modulo 2, so that, for instance, $V_0(J) = V_2(J) = \cdots$. Let us now suppose that an embedding $f\colon S \to J$ is given, and suppose $X \subset V_\sigma(J)$ ($\sigma \in \{0,1\}$). Then we let $A_f(X) = |\Gamma_J(X) \setminus f(V(S))|$. Also, for $x \in J$ we let $D_f(x) = d_S(f^{-1}(x))$ if $x \in f(V(S))$ and $D_f(x) = 0$ otherwise, and put $B_f(X) = \sum_{x \in X}(\Delta_\sigma - D_f(x))$. Finally, we set $C_f(X) = A_f(X) - B_f(X)$.

Let us say that X is **f-solvent** if $C_f(X) \geq 0$, and **f-bankrupt** otherwise. If $C_f(X) = 0$, we shall say that X is **f-critical**. A central definition is the following. We shall say that f is **good** if every $X \subset V(J)$ with $X \subset V_\sigma(J)$ ($\sigma \in \{0,1\}$) is f-solvent whenever $|X| \leq 2t_{\sigma+1}/\Delta_\sigma$. We have now arrived at the main claim in the proof of Theorem 1.

Claim. (i) *If $S \subset T$ is a subtree of T with $|S| = 1$, then there is a good embedding $f\colon S \to J$ of S in J.* (ii) *Let $S \subset T$ be a subtree of T, and suppose $f\colon S \to J$ is a good embedding of S in J. If $S \subset S' \subset T$, where S' is a tree with $|S'| = |S|+1$, then there is a good embedding $g\colon S' \to J$ of S' in J that extends f.*

As Theorem 1 immediately follows from (i) and (ii) above, we proceed to prove this claim. Let us first consider (i). Suppose $V(S) = \{x\} \subset V_\sigma(T)$ ($\sigma \in \{0,1\}$). Then let $f\colon V(S) \to V(J)$ be any function such that $f(x) \in V_\sigma(J)$. We claim that f is a good embedding. To check this, let $X \subset V_\rho(J)$ ($\rho \in \{0,1\}$) be such that $|X| \leq 2t_{\rho+1}/\Delta_\rho$. Let us check that $C_f(X) \geq 0$. If $X = \emptyset$, then clearly $C_f(X) = A_f(X) = B_f(X) = 0$, and so we assume that $X \neq \emptyset$. Note that then $A_f(X) = |\Gamma_J(X) \setminus \{f(x)\}| \geq 2\Delta_\rho|X| - 1 \geq \Delta_\rho|X| \geq B_f(X)$. Thus $C_f(X) \geq 0$, and f is indeed good. We now turn to (ii).

Suppose $f\colon S \to J$ and $S \subset S' \subset T$ are as in the statement of (ii). Then clearly there is a leaf v of S' such that $V(S') \setminus V(S) = \{v\}$. Let $w \in S$ be the unique neighbour of v in S'. Suppose $v \in V_{\sigma'}(T)$ ($\sigma' \in \{0,1\}$). Now let us consider all embeddings $g\colon S' \to J$ of S' in J that extend f, and let $\mathcal{G}$ be the set of such extensions. Our claim is that $\mathcal{G}$ contains a good extension. Although strictly speaking this is not necessary, let us first check that $\mathcal{G} \neq \emptyset$ as a warm-up. Since $\{f(w)\}$ is f-solvent, we have that $|\Gamma_J(f(w)) \setminus f(V(S))| = A_f(\{f(w)\}) \geq B_f(\{f(w)\}) = \Delta_{\sigma'+1} - d_S(w) \geq 1$, which implies that indeed f has at least one extension $g \in \mathcal{G}$. We now check that $\mathcal{G}$ contains a *good* extension.

Suppose for a contradiction that every $g \in \mathcal{G}$ admits a g-bankrupt set $X_g \subset V(J)$ such that $X_g \subset V_\rho(J)$, where $\rho = \rho(g) \in \{0,1\}$ and $|X_g| \leq 2t_{\rho+1}/\Delta_\rho$. Since f is good, the sets X_g ($g \in \mathcal{G}$) are all f-solvent.

We now do a little simple calculation. For brevity, if P is a statement we set $[P] = 0$ if P is false and $[P] = 1$ if P is true. Fix $X \subset V_\rho(J)$ ($\rho \in \{0,1\}$) and let $g \in \mathcal{G}$. Note that then $A_g(X) = |\Gamma_J(X) \setminus g(V(S'))| = |\Gamma_J(X) \setminus f(V(S))| -$

$[g(v) \in \Gamma_J(X)]$. Moreover $B_g(X) = \sum_{x\in X}(\Delta_\rho - D_g(x)) = B_f(X) - [f(w) \in X] - [g(v) \in X]$. Thus we have that

$$C_g(X) = C_f(X) - [g(v) \in \Gamma_J(X)] + [f(w) \in X] + [g(v) \in X].$$

From this, and the fact that X_g is g-bankrupt but f-solvent, we conclude that

$$X_g \text{ is } f\text{-critical, } g(v) \in \Gamma_J(X_g) \setminus X_g, \text{ and } f(w) \notin X_g \tag{3}$$

for every $g \in \mathcal{G}$. Note in particular that $\rho(g) = \sigma' + 1$ for all $g \in \mathcal{G}$; that is, we have $X_g \subset V_{\sigma'+1}(J)$ for all $g \in \mathcal{G}$.

Lemma 2. *Suppose $X \subset V_\rho(J)$, where $\rho \in \{0,1\}$, is f-critical and satisfies $|X| \leq 2t_{\rho+1}/\Delta_\rho$. Then $|X| \leq t_{\rho+1}/\Delta_\rho$.*

Proof. We have $A_f(X) = |\Gamma_J(X) \setminus f(V(S))| \geq 2\Delta_\rho|X| - t_{\rho+1}$, and $B_f(X) = \sum_{x\in X}(\Delta_\rho - D_f(x)) \leq \Delta_\rho|X|$. Since $A_f(X) - B_f(X) = C_f(X) = 0$, the lemma follows. □

We now check that C_f is a submodular function when restricted to each of the power sets $\mathcal{P}(V_\rho(J))$ of $V_\rho(J)$ ($\rho \in \{0,1\}$).

Lemma 3. *Suppose X, $Y \subset V_\rho(J)$, where $\rho \in \{0,1\}$. Then*

$$C_f(X \cap Y) + C_f(X \cup Y) \leq C_f(X) + C_f(Y). \tag{4}$$

Proof. Note first that B_f is a modular function on $\mathcal{P}(V_\rho(J))$ ($\rho \in \{0,1\}$). Now (4) follows from the observation that $\Gamma_J(X\cap Y) \subset \Gamma_J(X)\cap\Gamma_J(Y)$ and $\Gamma_J(X\cup Y) = \Gamma_J(X) \cup \Gamma_J(Y)$. □

An easy consequence of the above two lemmas is the following.

Corollary 4. *Suppose X, $Y \subset V_\rho(J)$, where $\rho \in \{0,1\}$, are f-critical and moreover $|X|$, $|Y| \leq t_{\rho+1}/\Delta_\rho$. Then $X \cup Y$ is f-critical and $|X \cup Y| \leq t_{\rho+1}/\Delta_\rho$.*

Proof. Since $|X \cup Y| \leq 2t_{\rho+1}/\Delta_\rho$ and f is good, the set $X \cup Y$ is f-solvent. Similarly $X \cap Y$ is f-solvent. Now (4) and the fact that X and Y are f-critical imply that $X\cap Y$ and $X\cup Y$ are f-critical as well. Lemma 2 now gives that $|X \cup Y| \leq t_{\rho+1}/\Delta_\rho$, as required. □

We are now ready to finish the proof of (ii) of our claim. Let $X^* = \bigcup_{g\in\mathcal{G}} X_g \subset V_{\sigma'+1}(J)$. By Corollary 4, we have that X^* is f-critical and $|X^*| \le t_{\sigma'}/\Delta_{\sigma'+1}$. Recall that $f(w) \notin X_g$ for any $g \in \mathcal{G}$ (cf. (3)), and so $f(w) \notin X^*$. Let $X' = X^* \cup \{f(w)\} \subset V_{\sigma'+1}(J)$. We now claim that

$$\Gamma_J(X') \setminus f(V(S)) = \Gamma_J(X^*) \setminus f(V(S)). \tag{5}$$

It suffices to check that if $y \in \Gamma_J(f(w)) \setminus f(V(S))$, then $y \in \Gamma_J(X^*)$. So suppose $y \in \Gamma_J(f(w)) \setminus f(V(S))$ is fixed. Then note that there is an extension $g \in \mathcal{G}$ of f for which $g(v) = y$. But then by (3) we conclude that $y = g(v) \in \Gamma_J(X_g) \subset \Gamma_J(X^*)$. Thus (5) does indeed hold, and we have $A_f(X') = A_f(X^*)$. Now note that $B_f(X') = B_f(X^*) + (\Delta_{\sigma'+1} - d_S(w)) > B_f(X^*)$. Thus $C_f(X') < C_f(X^*) = 0$. However, $|X'| = |X^*| + 1 \le t_{\sigma'}/\Delta_{\sigma'+1} + 1 \le 2t_{\sigma'}/\Delta_{\sigma'+1}$, and hence, as f is good, we have that $C_f(X') \ge 0$. This contradiction completes the proof of Theorem 1.

3. Bipartite graphs with uniformly distributed edges

Let integers n_0, $n_1 \ge 1$ and r_0, $r_1 \ge 1$ be such that $n_0 r_0 = n_1 r_1$. Set $r = \max\{r_0, r_1\}$, and let $p = (r_0/n_1)\log r = (r_1/n_0)\log r$. We shall assume throughout that $r \ge 3$ and $p \le 1$. Consider the space $\mathcal{G}(n_0, n_1; p)$ of random bipartite graphs $G = G_{n_0,n_1,p}$ with vertex classes $V_0(G)$ and $V_1(G)$ satisfying $|V_\sigma(G)| = n_\sigma$ ($\sigma \in \{0,1\}$), and where each edge is independently present with probability p. Our aim in this section is to show the following technical lemma, which we shall do with the aid of the random graphs $G_{n_0,n_1,p}$. If G is a graph and U, $W \subset V(G)$, we let $e_G(U,W)$ denote the number of edges that have one endvertex in U and the other in W.

Lemma 5. *There is an absolute constant $r^* \ge 1$ for which the following holds. Let n_0, n_1, r_0, $r_1 \ge 1$ be integers with $n_0 r_0 = n_1 r_1$, and set $p = (r_0/n_1)\log r = (r_1/n_0)\log r$ where $r = \max\{r_0, r_1\}$. Suppose $0 < \alpha \le 1$ satisfies αr_0, $\alpha r_1 \ge 1$. Then if $r \ge r^*$ there is a bipartite graph $G = G^{n_0,n_1}$ such that (i) $1/2 \le e(G)/n_0 r_0 \log r \le 2$, and (ii) if $\sigma \in \{0,1\}$, for any $U \subset V_\sigma(G)$, $W \subset V_{\sigma+1}(G)$ with $1 \le u = |U| \le u_\sigma = \lfloor n_{\sigma+1}/\mathrm{e} r_\sigma \rfloor = \lfloor n_\sigma/\mathrm{e} r_{\sigma+1} \rfloor$ and $w = |W| = \lfloor \alpha r_\sigma u \rfloor$, we have*

$$e(U,W) = e_G(U,W) < puw + 12\mathrm{e}(r_\sigma uw)^{1/2}\log r. \tag{6}$$

Proof. Note that if $p \geq 1$, we may take for G a complete bipartite graph with vertex classes of order n_0 and n_1. Thus we assume that $p < 1$. We shall show that then, provided r is large enough, the probability that $G = G_p = G_{n_0,n_1,p} \in \mathcal{G}(n_0, n_1; p)$ satisfies (i) and (ii) is positive. For $\sigma \in \{0,1\}$, let Q_σ be the property given in assertion (ii). We shall in fact show that, as $r \to \infty$, we have (iii) $\mathbb{P}(Q_\sigma) = 1 - o(1)$ $(\sigma \in \{0,1\})$ and (iv) $\mathbb{P}(1/2 \leq e(G)/r_0 n_0 \log r \leq 2) = 1 - o(1)$.

Let us check (iii) first, and notice that by symmetry we may assume $\sigma = 0$. Let $1 \leq u \leq u_0$ and $w = w(u) = \lfloor \alpha r_0 u \rfloor$. Set $A = 12\mathrm{e}$, $\mu = puw$, and $b = A(r_0uw)^{1/2}\log r$. Let $U \subset V_0(G)$ and $W \subset V_1(G)$ be such that $|U| = u$ and $|W| = w$. Let $P_u = \mathbb{P}(e(U,W) \geq \mu + b)$, and set $E_0 = \sum_{1\leq u\leq u_0} \binom{n_0}{u}\binom{n_1}{w} P_u$. Our aim is to show that $E_0 = o(1)$ as $r \to \infty$. Let

$$\eta = \frac{b}{\mu} = \frac{A(r_0uw)^{1/2}\log r}{(r_0/n_1)uw\log r} = An_1(r_0uw)^{-1/2}. \tag{7}$$

We estimate $E_0 = \sum_u \binom{n_0}{u}\binom{n_1}{w} P_u$ by breaking the sum into two parts. Let $E_0^{(1)} = \sum^* \binom{n_0}{u}\binom{n_1}{w} P_u$, where $\sum^*$ denotes sum over all $1 \leq u \leq u_0$ with $\eta \leq \mathrm{e}^2$, and let $E_0^{(2)} = E_0 - E_0^{(1)}$. Note that below we may assume that r is large enough for our inequalities to hold.

(1) *We have* $E_0^{(1)} = o(1)$ *as* $r \to \infty$.

Here we assume throughout that $\eta \leq \mathrm{e}^2$. We start by claiming that $P_u = \mathbb{P}(e(U,W) \geq \mu + b) \leq \exp\big\{-\big(A^2/3\mathrm{e}^4\big)n_1\log r\big\}$. To check this claim, let us first consider the case in which $\eta \leq 1$. Note that from (7) we have that $\eta^2\mu = A^2 n_1 \log r$. Then Hoeffding's inequality [13] (see also [17]) gives that $P_u \leq \exp\big\{-\big(A^2/3\big)n_1\log r\big\}$, and the claim follows in this case. Now consider the case $1 \leq \eta \leq \mathrm{e}^2$. Here $P_u \leq \mathbb{P}(e(U,W) \geq 2\mu) \leq \exp\{-\mu/3\}$, again by Hoeffding's inequality. Note that $b/\mu = \eta \leq \mathrm{e}^2$ gives that $\mu \geq b/\mathrm{e}^2 = (A/\mathrm{e}^2)(r_0uw)^{1/2}\log r$. Also, we have $An_1(r_0uw)^{-1/2} = \eta \leq \mathrm{e}^2$, and therefore $\mu \geq (A^2/\mathrm{e}^4)n_1\log r$. Thus our claimed upper bound for P_u follows.

We now estimate $E_0^{(1)}$. If $n_0 \leq n_1$ this is very quick: note that

$$E_0^{(1)} = \sum\nolimits^* \binom{n_0}{u}\binom{n_1}{w} P_u \leq 2^{n_0+n_1}\exp\left\{-\frac{A^2}{3\mathrm{e}^4}n_1\log r\right\} \leq \big(4r^{-6}\big)^{n_1} = o(1)$$

as $r \to \infty$. Thus let us assume that $n_0 > n_1$. Then

$$E_0^{(1)} \leq 2^{n_1}\sum\nolimits^* \binom{n_0}{u} P_u \leq 3 \times 2^{n_1}\exp\left\{-\frac{A^2}{3\mathrm{e}^4}n_1\log r\right\}\left(\frac{\mathrm{e}n_0}{u_0}\right)^{u_0}$$

$$\leq 3 \times 2^{n_1} \left(\mathrm{e}^2 r_1\right)^{n_1/\mathrm{e}r_0} \exp\left\{-\frac{A^2}{3\mathrm{e}^4} n_1 \log r\right\}$$
$$\leq 3 \times 2^{n_1} \exp\left\{\frac{2n_1}{\mathrm{e}r_0} + \frac{n_1}{\mathrm{e}r_0}\log r - \frac{A^2}{3\mathrm{e}^4} n_1 \log r\right\},$$

which tends to 0 as $r \to \infty$.

(2) *We have* $E_0^{(2)} = o(1)$ *as* $r \to \infty$.

We now assume that $\eta \geq \mathrm{e}^2$. Let v be such that $b = \mathrm{e}v\mu/\log v$. Then $\mathrm{e}v/\log v = b/\mu = \eta \geq \mathrm{e}^2$, and hence we may suppose $v \geq \mathrm{e}$. Also, we have $\mathrm{e}v\mu/\log v = b \geq 1 \geq \mathrm{e}/v$, and so $v^2\mu \geq \log v$. Thus $P_u \leq \mathbb{P}(e(U,W) \geq b) \leq \exp\{-v\mu\}$ (see Theorem 7(*ii*) in Chapter I of [8]). Now, we have $v\mu = (b/\mathrm{e})\log v \geq 12(r_0uw)^{1/2}(\log r)(\log v) \geq 12(r_0uw)^{1/2}(\log r)$. Thus $v \geq 12(r_0uw)^{-1/2}n_1$, and hence

$$P_u \leq \exp\{-v\mu\} \leq \left(\frac{(r_0uw)^{1/2}}{12n_1}\right)^{12(r_0uw)^{1/2}\log r}. \tag{8}$$

Note that if $w \leq 1$ then $u \leq 1$, and hence (6) holds trivially; that is, we have $P_u = 0$ in this case. Thus we assume $\alpha r_0 u \geq \lfloor \alpha r_0 u \rfloor = w \geq 2$. Hence $w \geq \alpha r_0 u/2$, and we have from (8) that

$$P_u \leq \left(\frac{\alpha^{1/2} r_0 u}{12 n_1}\right)^{6\alpha r_0 u \log r}.$$

Thus $E_0^{(2)} = \sum^\dagger \binom{n_0}{u}\binom{n_1}{w} P_u$, where $\sum^\dagger$ indicates sum over all $2 \leq u \leq u_0$ with $\eta \geq \mathrm{e}^2$, is at most

$$\sum\nolimits^\dagger \left(\frac{\mathrm{e}n_0}{u}\right)^u \left(\frac{\mathrm{e}n_1}{\alpha r_0 u}\right)^{\alpha r_0 u} P_u \leq \sum_{u=2}^{u_0} \left(\frac{\mathrm{e}n_0}{u}\right)^u \left\{\frac{\mathrm{e}n_1}{\alpha r_0 u}\left(\frac{ur_0}{n_1}\right)^{6\log r}\right\}^{\alpha r_0 u}$$
$$\leq \sum_{u=2}^{u_0} \left(\frac{\mathrm{e}n_0}{u}\right)^u \left(\frac{1}{\alpha}\right)^{\alpha r_0 u} \left(\frac{ur_0}{n_1}\right)^{4\alpha r_0 u \log r}$$
$$\leq \sum_{u=2}^{u_0} \left(\mathrm{e}r_1 \left(\frac{1}{\alpha}\right)^{\alpha r_0} \left(\frac{ur_0}{n_1}\right)^{3\alpha r_0 \log r}\right)^u,$$

which is at most $\sum_{2\leq u\leq u_0} \mathrm{e}^u\{\alpha r^2\}^{-\alpha r_0 u} \leq \sum_{u\geq 2}(\mathrm{e}/r)^u = o(1)$ as $r \to \infty$.

Thus we have shown that $E_0 = E_0^{(1)} + E_0^{(2)} = o(1)$ as $r \to \infty$, and hence that (*iii*) above does indeed hold. To see (*iv*), it suffices to note that $e(G)$ has binomial distribution $\mathrm{Bi}(n_0n_1, p)$ with parameters n_0n_1 and p and that $\mathbb{E}(e(G)) = pn_0n_1 = n_0r_0\log r \to \infty$ as $r \to \infty$. □

When $n_0 = n_1$ in Lemma 5, we may require a more restrictive condition on $G = G^{n,n}$, as shows Lemma 6 below. One may prove this lemma by suitably altering the proof of Lemma 5 and hence we omit its proof.

Lemma 6. *There is an absolute constant $r^* \geq 1$ for which the following holds. Let $r^* \leq r \leq n$, set $p = r/n$, and suppose $0 < \alpha \leq 1$ satisfies $\alpha r \geq 1$. Then there is a bipartite graph $G = G^{n,n}$ such that (i) $1/2 \leq e(G)/nr \leq 2$, and (ii) if $\sigma \in \{0,1\}$, for any $U \subset V_\sigma(G)$, $W \subset V_{\sigma+1}(G)$ with $1 \leq u = |U| \leq n/2r$ and $w = |W| = \lfloor \alpha r u \rfloor$, we have $e_G(U,W) < puw + 12\mathrm{e}(ruw)^{1/2}$.* □

4. The main results

Let H be a bipartite graph with bipartition $V(H) = V_0(H) \cup V_1(H)$, and suppose $V_0(H)$, $V_1(H) \neq \emptyset$. Let $\bar{d}_\sigma(H) = |V_\sigma(H)|^{-1} \sum\{d_H(v)\colon v \in V_\sigma(H)\}$ be the average degree of the vertices in $V_\sigma(H)$ ($\sigma \in \{0,1\}$). The following simple but useful lemma was observed by Beck [7]. We include Beck's proof of this lemma for completeness.

Lemma 7. *Let H be as above. Then there is a non-empty induced subgraph $J \subset H$ of H such that (*) for any $U \subset V_\sigma(J) = V_\sigma(H) \cap V(J)$ ($\sigma \in \{0,1\}$), we have $e_J(U, V(J)) \geq (1/2)\bar{d}_\sigma(H)|U|$.*

Proof. We define a sequence $H = H_0 \supset H_1 \supset \cdots$ of induced subgraphs of H as follows. Let $H_0 = H$, and suppose $H_0 \supset \cdots \supset H_{i-1}$ ($i \geq 1$) have been defined. If $|H_{i-1}| = 0$, or else $|H_{i-1}| > 0$ and condition (*) in our lemma holds, we terminate the sequence. Suppose $|H_{i-1}| > 0$ but (*) fails. Then let $U \subset V(H_{i-1})$ be such that $U \subset V_\sigma(H_{i-1})$ for some $\sigma \in \{0,1\}$, and moreover $e_{H_{i-1}}(U, V(H_{i-1})) < (1/2)\bar{d}_\sigma(H)|U|$. We now let $H_i = H_{i-1} - U$. This defines a sequence $H = H_0 \supset \cdots \supset H_t$ of induced subgraphs of H. If $J = H_t$ is as required, we are done. Thus suppose $|H_t| = 0$. But then

$$e(H) = \sum_{1 \leq i \leq t} \big(e(H_{i-1}) - e(H_i)\big) < \frac{1}{2}\bar{d}_0(H)|V_\sigma(H)| + \frac{1}{2}\bar{d}_1(H)|V_\sigma(H)| = e(H),$$

which is a contradiction, and hence we necessarily have $|H_t| > 0$. □

We may now prove our first main result.

Theorem 8. *Let $0 < \gamma \leq 1$ be given. Then there is a constant $c_\gamma > 0$ depending only on γ for which the following holds. For all integers $1 \leq \Delta_0 \leq t_1$ and $1 \leq \Delta_1 \leq t_0$, there is a bipartite graph G such that (i) $e(G) \leq c_\gamma(\Delta_0 t_0 + \Delta_1 t_1)\log \Delta$, where $\Delta = \max\{\Delta_0, \Delta_1\}$, and (ii) $G \to_\gamma T$ for any $(t_0, \Delta_0, t_1, \Delta_1)$-tree T.*

Proof. Let γ, t_σ, Δ_σ ($\sigma \in \{0,1\}$) be as above. In proving our result, we may assume that $\Delta \geq \mathrm{e}^2 r^*$, where r^* is as given in Lemma 5. Moreover, we may further assume that $\Delta_0 t_0 \geq \Delta_1 t_1$. We now start the proof proper.

Let $0 < \alpha = \alpha(\gamma) \leq \min\{\mathrm{e}\gamma/4, 4/\mathrm{e}^2\}$ be the largest real number such that $(1/12\mathrm{e})^2(\gamma/4 - \alpha/\mathrm{e})^2 \geq \alpha$. For $\sigma \in \{0,1\}$, let r_σ be the smallest integer power of 2 such that $\alpha r_\sigma \geq 2\Delta_\sigma$. Clearly $r_\sigma \leq (4/\alpha)\Delta_\sigma$ ($\sigma \in \{0,1\}$). Now let n_0 be the smallest power of 2 such that $n_0 \geq 2\mathrm{e}t_0 r_1/\Delta_1$. Then we have $n_0 \leq (16\mathrm{e}/\alpha)t_0$. Finally, let $n_1 = r_0 n_0 / r_1$. Then note that $n_1/\mathrm{e}r_0 \geq n_0/\mathrm{e}r_1 \geq 2t_0/\Delta_1 \geq 2t_1/\Delta_0$. Now let $G = G^{n_0,n_1}$ be the bipartite graph whose existence is guaranteed in Lemma 5. Then

$$e(G) \leq 2n_0 r_0 \log r \leq 2\left(\frac{16\mathrm{e}t_0}{\alpha}\right)\left(\frac{4\Delta_0}{\alpha}\right)\log\left(\frac{4\Delta}{\alpha}\right) \leq c'_\gamma(t_0\Delta_0 + t_1\Delta_1)\log\Delta,$$

where $c'_\gamma = 128\mathrm{e}\alpha^{-2}\log(4/\alpha)$, since $\log(4\Delta/\alpha) \leq (\log(4/\alpha))\log\Delta$ as $\alpha \leq 4\mathrm{e}^{-2}$ and $\Delta \geq \mathrm{e}^2$. Thus (i) in our result holds for G if $c_\gamma \geq c'_\gamma$. We now check (ii). Thus let $H \subset G$ be a fixed subgraph of G with $e(H) \geq \gamma e(G)$. Let $J \subset H$ be the subgraph of H given by Lemma 7. We claim that then the graph J is $(2t_1/\Delta_0, 2\Delta_0; 2t_0/\Delta_1, 2\Delta_1)$-expanding.

Suppose for a contradiction that $U \subset V_\sigma(J)$ is such that $u = |U| \leq 2t_{\sigma+1}/\Delta_\sigma$ and $|\Gamma_J(U)| < 2\Delta_\sigma|U|$, where $\sigma \in \{0,1\}$. Then, let $W \subset V_{\sigma+1}(J)$ be such that $w = |W| = \lfloor \alpha r_\sigma u \rfloor$ and $\Gamma_J(U) \subset W$. Now observe that

$$\begin{aligned}\frac{\gamma}{4} r_\sigma(\log r)u &\leq \frac{\gamma}{2}\bar{d}_\sigma(G)u \leq \frac{1}{2}\bar{d}_\sigma(H)u \leq e_J(U, V(J)) \leq e_G(U,W)\\ &< puw + 12\mathrm{e}(r_\sigma uw)^{1/2}\log r \leq \frac{\alpha}{\mathrm{e}} r_\sigma(\log r)u + 12\mathrm{e}(r_\sigma uw)^{1/2}\log r.\end{aligned}$$

Thus $(\gamma/4 - \alpha/\mathrm{e})r_\sigma(\log r)u < 12\mathrm{e}(r_\sigma uw)^{1/2}\log r$, and hence we have $\alpha r_\sigma u \leq (1/12\mathrm{e})^2(\gamma/4 - \alpha/\mathrm{e})^2 r_\sigma u < w = \lfloor \alpha r_\sigma u \rfloor$, which is a contradiction. Therefore we conclude that indeed J is $(2t_1/\Delta_0, 2\Delta_0; 2t_0/\Delta_1, 2\Delta_1)$-expanding. Now (ii) above follows from Theorem 1. □

Our second main result is as follows.

Theorem 9. *Let $0 < \gamma \leq 1$ be given. Then there is a constant $c_\gamma > 0$ depending only on γ for which the following holds. For any $1 \leq \Delta < t$, there is a bipartite graph G such that (i) $e(G) \leq c_\gamma \Delta t$, and (ii) $G \to_\gamma T$ for any tree T of order $|T| \leq t$ and maximal degree $\Delta(T) \leq \Delta$.*

Proof. Let γ, Δ, t be as in the statement of our result. We may assume that $\Delta \geq r^*$, where r^* is as in Lemma 6. Let $0 < \alpha = \alpha(\gamma) \leq \gamma/2$ be the largest real such that $(1/12\mathrm{e})^2(\gamma/4 - \alpha/2)^2 \geq \alpha$. Now let $r = \lceil 2\Delta/\alpha \rceil \leq 4\Delta/\alpha$, and $n = \lceil 4tr/\Delta \rceil \leq 32t/\alpha$. Note that then $r^* \leq r \leq n$, $0 < \alpha \leq 1$, and let $G = G^{n,n}$ be the corresponding bipartite graph given by Lemma 6. Then $e(G) \leq 2rn \leq 2(4\Delta/\alpha)(32t/\alpha) \leq 256\alpha^{-2}\Delta t$. We now check that $G \to_\gamma T$ for any tree T with $|T| \leq t$ and $\Delta(T) \leq \Delta$. Let $H \subset G$ be a subgraph of G with $e(H) \geq \gamma e(G)$. Let $J \subset H$ be the subgraph of H given by Lemma 7. It now suffices to check that J is a $(2t/\Delta, 2\Delta; 2t/\Delta, 2\Delta)$-expanding bipartite graph.

Suppose for a contradiction that $U \subset V_\sigma(J)$ is such that $|\Gamma_J(U)| < 2\Delta|U|$ although $u = |U| \leq 2t/\Delta$, where $\sigma \in \{0,1\}$. Then, let $W \subset V_{\sigma+1}(J)$ be such that $w = |W| = \lfloor \alpha r u \rfloor$ and $\Gamma_J(U) \subset W$. Now observe that

$$\frac{\gamma}{4} ru \leq \frac{\gamma}{2}\bar{d}_\sigma(G)u \leq \frac{1}{2}\bar{d}_\sigma(H)u \leq e_J(U, V(J)) \leq e_G(U, W)$$
$$< puw + 12\mathrm{e}(ruw)^{1/2} \leq \frac{\alpha}{2} ru + 12\mathrm{e}(ruw)^{1/2}.$$

Thus $\alpha r u \leq (1/12\mathrm{e})^2(\gamma/4 - \alpha/2)^2 ru < w = \lfloor \alpha r u \rfloor$, which is a contradiction.

□

An immediate corollary to Theorems 8 and 9 is the upper bound for the size-Ramsey number of trees given in Corollary 10 below. The lower bound in this corollary is due to Beck [7], and its very short proof is included for convenience.

Corollary 10. *For any $r \geq 2$, there is a constant c_r depending only on r such that*

$$\beta(T)/4 \leq \hat{r}_{\mathrm{e}}(T, r) \leq c_r \min\{\beta(T) \log \Delta(T), |T|\Delta(T)\}$$

for all trees T.

Proof. Let an integer $r \geq 2$ and a tree T be fixed. Let T have bipartition $V(T) = V_0(T) \cup V_1(T)$, and let $t_\sigma = |V_\sigma(T)|$, $\Delta_\sigma = \Delta_\sigma(T)$ $(\sigma \in \{0,1\})$. We may assume that $t_0\Delta_0 \geq t_1\Delta_1$. Let us prove that $\hat{r}_{\mathrm{e}}(T, 2) \geq \beta(T)/4$. Suppose $G \to (T)_2$.

Let $U = \{v \in G\colon d_G(v) \geq \Delta_0\}$ and set $W = V(G) \setminus U$. Colour the edges of G between U and W red, and the rest of the edges of G blue. Suppose $T' \subset G$ is a monochromatic copy of T in G, and let $\varphi\colon T \to T'$ be an isomorphism. Let $v_0 \in V_0(T)$ have degree $d_T(v) = \Delta_0$. Clearly $\varphi(v_0) \in U$, and hence regardless of the colour of T', we have $\varphi(V_0(T)) \subset U$. Thus $e(G) \geq |U|\Delta_0/2 \geq |V_0(T)|\Delta_0/2 = t_0\Delta_0/2 \geq \beta(T)/4$. The upper bound for $r_e(T, r)$ follows immediately from Theorems 8 and 9. □

5. Concluding remarks

The existence of the graph G in Theorem 9 is proved by non-constructive means. For some values of γ, Δ, and t we may however take for G a suitable Ramanujan graph $X = X^{p,q}$. More precisely, there is an absolute constant $\varepsilon > 0$ with the following property. For any given $0 < \gamma \leq 1$, there exist constants $c_\gamma > 0$, Δ_γ, $t_\gamma \geq 1$ depending only on γ such that, for any integers $1 \leq \Delta < t$ with $\Delta \geq \Delta_\gamma$, $t \geq t_\gamma$, and $\Delta \leq t^\varepsilon$, there are primes p and q such that a bipartite Ramanujan graph $X = X^{p,q}$ constructed by Lubotzky, Phillips, and Sarnak [16] is such that (*i*) $e(X) \leq c_\gamma \Delta t$, and (*ii*) $X \to_\gamma T$ for any tree T of order $|T| \leq t$ and maximal degree $\Delta(T) \leq \Delta$. The only extra work involved in proving the statement above has to do with the existence of the primes p and q. We refer the reader to Section 4 of [12], where a similar number-theoretic problem is treated. The key result there is a beautiful theorem of Bombieri [9] (see also Davenport [10], §28) on the distribution of primes in arithmetic progressions. With this result in hand, our task there is quite straightforward, and in fact the same method applies here, proving the above assertion. Let us also remark that the techniques presented in this note also show that the incidence graphs of certain projective geometries are explicit examples that prove Theorem 9 for Δ and t with Δ about $t^{1-1/d}$ for some integer $d \geq 2$. (See Theorem 2.3 in Alon [1].)

Finally, we should like to mention that Professor Noga Alon [2] has kindly pointed out to us that one may improve the observations above as follows. One of the methods that is given for the construction of almost k-wise independent random variables in Alon, Goldreich, Håstad, and Peralta [4], namely, Construction 3, can be suitably modified to give an elementary construction of an r-regular n-vertex graph whose second largest eigenvalue in absolute value μ_1 is such that $|\mu_1| \leq ur^{1/2}$, where $n = 2^{uk}$ and $r = 2^{2k}$, and u and k are any fixed integers with $u \geq 3$ and $k \geq 1$. Here one considers a suitable Cayley graph of a certain Abelian group, and uses the result given in Problem 11.8 of Lovász [15] (see Section 3.2 of Alon and Roichman [5]). These graphs, the Ramanujan graphs of Lubotzky, Phillips, and Sarnak and, as pointed out by Professor Alon, their powers may be used to prove appropriate variants of Lemma 6 constructively, thus allowing one to give a constructive proof of Theorem 9 for a wide range of values of γ, Δ, and t.

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