Contemporary Mathematics
Volume **749**, 2020
https://doi.org/10.1090/conm/749/15074

The principal-symbol index map for an algebra of pseudodifferential operators

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This paper is dedicated to the memory of Heinz-Otto Cordes (1925-2018)

Abstract. Let B be a compact Riemannian manifold, let Ω denote the cylinder $\mathbb{R} \times B$, Δ_Ω its Laplace operator and $\Lambda = (1 - \Delta_\Omega)^{-1/2}$. Let $\mathfrak{A}$ denote the C*-algebra of bounded operators on $L^2(\mathbb{R} \times B)$ generated by all the classical pseudodifferential operators on $\mathbb{R} \times B$ of the form $L\Lambda^N$, N a nonnegative integer and L an N-th order differential operator whose (local) coefficients approach 2π-periodic functions at $+\infty$ and $-\infty$. Let $\mathfrak{E}$ denote the kernel of the continuous extension of the principal symbol to $\mathfrak{A}$.

The problem of computing the K-theory index map $\delta_1(K_1(\mathfrak{A}/\mathfrak{E})) \to K_0(\mathfrak{E}) \simeq \mathbb{Z}^2$ on an element of $K_1(\mathfrak{A}/\mathfrak{E})$ is reduced to the problem of computing the Fredholm indices of two elliptic operators on the compact manifold $S^1 \times B$.

For $B = S^1$, Hess went further and proved in her thesis that $K_0(\mathfrak{A}) \cong \mathbb{Z}^5$ and $K_1(\mathfrak{A}) \cong \mathbb{Z}^4$.

Introduction

More than fifty years after the Atiyah-Singer index theorem, people are still looking for generalizations of their theorem for classes of non-compact manifolds, for manifolds with boundary, corners or edges, or for manifolds with singularities in a more general sense. There are, in fact, a great variety of different classes of problems, and of different approaches or tecnhiques that could be used.

One of the approaches is to consider C*algebras generated by zero-order pseudodiferential operators which contain an *order-reducing device*. That means that there are linear isomorphisms between Sobolev spaces so that certain classes of differential operators multiplied on the right by them fall into the C*algebra. In the case of a compact manifold, the kernel of the continuous extension to the C*-algebra of the principal symbol map coincides with the ideal of compact operators, and the Fredholm index can be regarded as the K-theory index map associated to the exact sequence (1) below. For a noncompact manifold, one might argue that the relevant index is no longer the Fredholm index, but rather the K-theory index map taking values in K_0 of the kernel of the principal symbol.

2010 *Mathematics Subject Classification.* Primary 46L80, 19K56, 47G30.
Key words and phrases. K-Theory, C*-algebras, Index Theory, Pseudodifferential Operators.

Monthubert and Nistor followed that track in [**24, 25**], using algebras of pseudodifferential operators on groupoids which relate to (regular) algebras of pseudodifferential operators on a noncompact manifold. In their case, the relevant compactification of the original manifold is itself a manifold with corners. In this paper we consider an algebra of pseudodifferential operators on a cylinder $\mathbb{R} \times B$, where B is compact, which includes operators with periodic symbols. In our case the relevant compactification of the manifold is not even locally path connected.

The K-groups of our algebra, for the special cases of B equals a point and of B equals S^1, have been computed in [**23**] and [**13**], respectively.

The algebra considered here is an example of a *comparison algebra*, in the sense of Cordes, who pioneered [**4–6, 9, 10**] the use of Banach-algebra techniques in the study of the spectral theory of partial differential operators and singular integral operators on noncompact manifolds. Those algebras were defined in [**8**] as C*-algebras of bounded operators on $L^2(M)$ for a noncompact manifold M equipped with a smooth positive measure and a positive second-order elliptic differential operator H. Typically, M is equipped with a riemannian metric and its induced measure, and H is equal to the identity minus the Laplacian. One defines $\Lambda = H^{-1/2}$, where H also denotes the Friedrichs extension of the original operator defined on $C_c^\infty(M)$, which of course coincides with the unique self-adjoint extension in case the operator is essentially self-adjoint. Given a class of smooth functions $\mathcal{A}^\sharp$ and a class of first-order differential operators $\mathcal{D}^\sharp$ on M satisfying certain conditions, a comparison algebra is defined as the C*-subalgebra of the algebra of bounded operators on $L^2(M)$ generated by all operators of multiplication by $a \in \mathcal{A}^\sharp$ and by all $D\Lambda$, $D \in \mathcal{A}^\sharp$. In his monograph [**8**], Cordes develops general techniques to describe the ideal structure of a comparison algebra and how to find differential operators *within reach* of the algebra, meaning differential operators which, composed with an isomorphism from L^2 to a suitable Sobolev space, belong to the algebra.

In case the manifold is compact, there is only one possible choice of classes: $\mathcal{A}^\sharp = C^\infty(M)$ and $\mathcal{D}^\sharp$ equals to the set of all first order differential operators with smooth coefficients. In this case, the resulting comparison algebra coincides [**21**] with the C*algebra generated by all zero-order classical pseudodifferential operators on M, which we will denote by $\Psi(M)$. The principal symbol of operators, a priori a *-homomorphism defined on the subalgebra of classical pseudodifferential operators and taking values in smooth functions on the co-sphere bundle S^*M, extends to a surjective C*-algebra homomorphism $\sigma_M : \Psi(M) \to C(S^*M)$ with kernel equal to the ideal of compact operators $\mathfrak{K}_M$. In other words, we have the short exact sequence of C*-algebras

$$0 \longrightarrow \mathfrak{K}_M \hookrightarrow \Psi(M) \xrightarrow{\sigma_M} C(S^*(M)) \longrightarrow 0. \tag{1}$$

In Cordes' approach, the map σ_M arises as the Gelfand map for the commutative C*algebra $\Psi(M)/\mathfrak{K}_M$. Classically, (1) follows from the equality between the norm, modulo compact operators, of a singular integral operator on a compact manifold and the supremum norm of its symbol, proven by Gohberg [**15**] and Seeley [**29**]. Proofs of that result in the language of pseudodifferential operators appeared in [**16, 18**].

1. Preliminaries

In this section, we define the C*-algebra $\mathfrak{A}$ and review results about its structure that will be needed in the next and final section.

Let B denote a compact Riemannian manifold of dimension n without boundary. Consider the cylinder $\Omega = \mathbb{R} \times B$ and denote by $\Delta_\Omega = \Delta_{\mathbb{R}} + \Delta_B$ its Laplacian (we use the classical analysis convention sign for the Laplacian, $-\Delta_\Omega \geq 0$) Define $\mathfrak{A}$ as the C*-subalgebra of $\mathfrak{B}(L^2(\Omega))$ (we denote by $\mathfrak{B}(\mathcal{H})$ the C*-algebra of all bounded operators on a Hilbert space $\mathcal{H}$) generated by:

(i) all $A_1 = a(M_x)$, operators of multiplication by $a \in C^\infty(B)$,
$$[a(M_x)f](t,x) = a(x)f(t,x), \quad (t,x) \in \mathbb{R} \times B;$$
(ii) all $A_2 = b(M_t)$, operators of multiplication by $b \in C([-\infty,+\infty])$ (this denotes the set of all continuous functions on $\mathbb{R}$ which have limits at $-\infty$ and $+\infty$),
$$[b(M_t)f](t,x) = b(t)f(t,x), \quad (t,x) \in \mathbb{R} \times B;$$
(iii) all operators $A_3 = e^{ikM_t}$, $k \in \mathbb{Z}$, defined by
$$[e^{ikM_t}f](t,x) = e^{ikt}f(t,x), \quad (t,x) \in \mathbb{R} \times B;$$
(iv) $A_4 = \Lambda := (1-\Delta_\Omega)^{-1/2}$;
(v) $A_5 = \frac{1}{i}\frac{\partial}{\partial t}\Lambda$;
(vi) $D_x\Lambda$, where $D_x = -i\sum_{k=1}^n c^k(x)\partial_{x_k}$ is a smooth vector field on B.

Since Ω is a complete manifold, its Laplacian, defined on $C_c^\infty(\Omega)$, has a unique extension (also denoted by Δ_Ω) as an unbounded self-adjoint operator on $L^2(\Omega)$ ([**14**], [**8**, Theorem IV.1.8]) and one can then define Λ by functional calculus . The operators in (v) and (vi), a priori defined on the dense subset $\Lambda^{-1}(C_c^\infty(\Omega))$, have continuous extensions to $L^2(\Omega)$ [**8**, Section V.1].

Moreover, if L denotes an M-th order differential operator on Ω whose (smooth) coefficients approach 2π-periodic functions as t tends to $-\infty$ and as t tends to $+\infty$, the operator $L\Lambda^M$ can be extended to a bounded operator on $L^2(\Omega)$, which belongs to $\mathfrak{A}$. These $L\Lambda^M$ are zero-order *classical* pseudodifferential operators (meaning that their local symbols have asymptotic expansions in homogeneous components).

Applying Cordes' comparison algebra techniques and using results from [**7**, **11**], the statements about the structure of $\mathfrak{A}$ contained in the rest of this Section were proven in [**20**] (see also [**19**]).

Let $\mathfrak{E}$ denote the commutator ideal of $\mathfrak{A}$, i.e, the smallest closed ideal containing all commutators $AB - BA$, A and B in $\mathfrak{A}$. Let $\overline{\Omega}$ denote the compactfication $[-\infty,+\infty] \times B$ of Ω, and let S^*M denote the cosphere bundle of any manifold M. We may then write
$$S^*\overline{\Omega} = \{(t,x,\tau,\xi);\ (t,\tau) \in T^*[-\infty,+\infty],\ (x,\xi) \in T^*B,\ |\tau|^2 + |\xi|^2 = 1\}.$$

Theorem 1. *The spectrum of the commutative C^*-algebra $\mathfrak{A}/\mathfrak{E}$ is homeomorphic to the following subset of $(S^*\overline{\Omega}) \times S^1$:*
$$\mathbf{M} = \{(t,x,\tau,\xi,e^{i\theta});\ \theta \in \mathbb{R} \text{ and } \theta = t \text{ if } |t| < \infty\}.$$
Denoting by
$$\sigma : \mathfrak{A} \to C(\mathbf{M}) \tag{2}$$

the composition of the Gelfand isomorphism $\mathfrak{A}/\mathfrak{E} \simeq C(\mathbf{M})$ with the projection $\mathfrak{A} \to \mathfrak{A}/\mathfrak{E}$, for each of the generators of $\mathfrak{A}$ listed above, $\sigma(A_k)(t,x,\tau,\xi,e^{i\theta})$, $k = 1, \cdots, 6$, is equal to, respectively,

$$a(x),\ b(t),\ e^{ij\theta},\ 0,\ \tau,\ \sum_{k=1}^{n} c^j(x)\xi_j. \tag{3}$$

Since M is a compactification of $S^*\Omega$, we may regard $C(\mathbf{M})$ as a subspace of $C(S^*\Omega)$. Theorem 1 can then be rephrased by saying that the usual principal symbol of classical pseudodifferential operators, defined on a dense subalgebra of $\mathfrak{A}$, has a continuous extension σ to $\mathfrak{A}$ whose image is equal to $C(\mathbf{M})$ and whose kernel is $\mathfrak{E}$.

Let $F_d : L^2(S^1) \to \ell^2(\mathbb{Z})$ denote the discrete Fourier transform

$$(F_d u)_j = \int_0^1 u(e^{2\pi i\varphi})e^{-2\pi ij\varphi}d\varphi\ ,\ u \in L^2(\mathbb{S}^1) \text{ and } j \in \mathbb{Z}$$

(S^1 is equipped with the Lebesgue measure pushforwarded by $[0,1) \ni \varphi \mapsto e^{2\pi i\varphi} \in S^1$; hence F_d is unitary). For each sequence $(b_j)_{j\in\mathbb{Z}}$ with limits as $j \to -\infty$ and as $j \to +\infty$, we define bounded operators $b(M_j)$ on $\ell^2(\mathbb{Z})$ and $b(D_\varphi)$ on $L^2(S^1)$ by

$$[b(M_j)u]((u_j)_{j\in\mathbb{Z}}) = (b_ju_j)_{j\in\mathbb{Z}} \quad \text{and} \quad b(D_\varphi) = F_d^{-1}b(M_j)F_d.$$

Given $a \in C(S^1)$, let $a(M_\varphi) \in \mathfrak{B}(L^2(S^1))$ denote the operator of multiplication by a. Finally let $\mathfrak{S}$ denote the C*-subalgebra of $\mathcal{L}(L^2(S^1))$ generated by the operators of the form $a(M_\varphi)$ and $b(D_\varphi)$, for $a \in C(S^1)$ and $b = (b_j)_{j\in\mathbb{Z}}$ a sequence as above.

In Cordes' language, $\mathfrak{S}$ is the only comparison algebra over S^1. The principal symbol, defined on the dense *-subalgebra of all classical zero-order pseudos, extends to a surjective C*-algebra homomorphism (see the Introduction)

$$\sigma_S : \mathfrak{S} \to C(S^1 \times \{-1,+1\}), \tag{4}$$

whose kernel is equal to the set $\mathfrak{K}_{S^1}$ of all compact operators on $L^2(S^1)$. On our set of generators, one has $[\sigma(a(M_\varphi))](e^{2\pi i\varphi},\pm1) = a(e^{2\pi i\varphi})$, for $a \in C(S^1)$, and $[\sigma(b(D_\varphi))](e^{2\pi i\varphi},\pm1) = b(\pm\infty)$, for $b = (b_j)_{j\in\mathbb{Z}}$ a sequence having limits as $j \to +\infty$ and as $j \to -\infty$. A detailed exposition of some of these facts can be found in [**1**].

For each $\varphi \in \mathbb{R}$, let U_φ be the operator on $L^2(S^1)$ given by $U_\varphi f(z) = z^{-\varphi}f(z)$, $z \in S^1$ (we take the principal branch of $z^{-\varphi} = e^{-\varphi\log z}$, $z \neq 0$). Let $Y_\varphi := F_dU_\varphi F_d^{-1}$. This defines a smooth family $\varphi \mapsto Y_\varphi$ of unitary operators on $\ell^2(\mathbb{Z})$ such that for all $k \in \mathbb{Z}$, $(Y_ku)_j = u_{j+k}$, and $Y_\varphi Y_\omega = Y_{\varphi+\omega}$. Given $u \in L^2(\mathbb{R})$, the sequence $\tilde{u}(\varphi) = (u(\varphi - j)_{j\in\mathbb{Z}})$ belongs to $\ell^2(\mathbb{Z})$ for almost every $\varphi \in [0,1)$ and the map

$$\begin{array}{rcl} W : L^2(\mathbb{R}) & \longrightarrow & L^2(S^1) \otimes \ell^2(\mathbb{Z}) \simeq L^2(S^1;\ell^2(\mathbb{Z})) \\ u & \longmapsto & Wu, \qquad (Wu)(e^{2\pi i\varphi}) = Y_\varphi\tilde{u}(\varphi) \end{array}$$

is a Hilbert space isomorphism.

We have used $\otimes$ to denote Hilbert-space tensor product. Given $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{K})$, we also denote by $A \otimes B$ the bounded operator on $\mathcal{H} \otimes \mathcal{K}$ defined by $(A \otimes B)(u \otimes v) = Au \otimes Bv$. And also, given C*-algebras $\mathfrak{A}_H \subset \mathfrak{B}(\mathcal{H})$ and $\mathfrak{A}_K \subset \mathfrak{B}(\mathcal{K})$, we denote by $\mathfrak{A}_H \otimes \mathfrak{A}_K \subset \mathfrak{B}(\mathcal{H} \otimes \mathcal{K})$ the closed linear span of all $A \otimes B$, $A \in \mathfrak{A}_H$ and $B \in \mathfrak{A}_K$.

We will denote by $\mathfrak{K}_X$ denote the sets of all compact operators on $L^2(X)$, for $X = \Omega$, S^1, $\mathbb{Z}$, B, $\mathbb{Z} \times B$, $S^1 \times B$. We will also write W and F meaning $W \otimes I_B$

and $F \otimes I_B$, where I_B denotes the identity map on $L^2(B)$. Similarly we will also denote by σ_S the map $\sigma_S \otimes I_{\mathfrak{K}_\mathbb{Z}} \otimes I_{\mathfrak{K}_B}$ defined on $\mathfrak{S} \otimes \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B$.

The *-isomorphism $\mathfrak{B}(L^2(\Omega)) \ni A \longmapsto WF^{-1}AFW^{-1} \in \mathfrak{B}(L^2(S^1) \otimes \ell^2(\mathbb{Z}) \otimes L^2(B))$ restricted to $\mathfrak{E}$ defines a C*-isomorphism

$$\mathfrak{E} \xrightarrow{\cong} \mathfrak{S} \otimes \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B. \tag{5}$$

The kernel of the surjective map $\sigma_S : \mathfrak{S} \otimes \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B \to C(S^1 \times \{-1,+1\}) \otimes \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B$ is equal to $\mathfrak{K}_{S^1} \otimes \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B \cong \mathfrak{K}_\Omega$ and hence σ_S induces a C*-algebra isomorphism

$$\mathfrak{E}/\mathfrak{K}_\Omega \xrightarrow{\cong} C(S^1 \times \{-1,+1\}) \otimes \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B \;\cong\; C(S^1 \times \{-1,+1\};\, \mathfrak{K}_\mathbb{Z} \otimes \mathfrak{K}_B). \tag{6}$$

The composition of this isomorphism with the canonical projection $\mathfrak{E} \to \mathfrak{E}/\mathfrak{K}_\Omega$ defines a C*-algebra homomorphism

$$\gamma : \mathfrak{E} \longrightarrow C(S^1 \times \{-1,+1\};\, \mathfrak{K}(L^2(\mathbb{Z} \times B))), \tag{7}$$

which can be extended, via the left regular representation, to a map from $\mathfrak{A}$ to the algebra of bounded operators on the Banach space $C(S^1 \times \{-1,+1\};\, \mathfrak{K}(L^2(\mathbb{Z} \times B))$. The image of this Banach-algebra homomorphism is contained in $C(S^1 \times \{-1,+1\};\, \mathfrak{B}(L^2(\mathbb{Z} \times B))$ (an $f \in C(S^1 \times \{-1,+1\}; \mathfrak{B}(L^2(\mathbb{Z} \times B))$ is here regarded as a multiplication operator on $C(S^1 \times \{-1,+1\}; \mathfrak{K}(L^2(\mathbb{Z} \times B))\,)$ and this defines a C*-algebra homomorphism, which we also denote by γ,

$$\gamma : \mathfrak{A} \longrightarrow C(S^1 \times \{-1,+1\};\, \mathfrak{B}(L^2(\mathbb{Z} \times B))). \tag{8}$$

We define yet another *-homomorphism

$$\Gamma : \mathfrak{A} \longrightarrow C(S^1 \times \{-1,+1\};\, \mathfrak{B}(L^2(S^1 \times B))) \tag{9}$$

$$\Gamma_A(e^{2\pi i\varphi}, \pm 1) = F_d^{-1}\Gamma_A(e^{2\pi i\varphi}, \pm 1)F_d, \quad e^{2\pi i\varphi} \in S^1.$$

The image of Γ is contained in $C(S^1 \times \{-1,+1\};\, \Psi(S^1 \times B))$, with $\Psi(S^1 \times B)$ as defined at the end of the Introduction. It follows almost by definition that the image of $\mathfrak{E}$ under Γ equals $C(S^1 \times \{-1,+1\};\, \mathfrak{K}_{S^1 \times B})$.

In order to give a more explicit description of Γ, we need two more definitions. For each $\varphi \in \mathbb{R}$, let us denote by $e^{i\varphi M_\theta}$ the multiplication operator (by a possibly discontinuous function)

$$[e^{i\varphi M_\theta}u](e^{i\theta}) = e^{i\varphi\theta}u(e^{i\theta}), \quad -\pi < \theta \le \pi, \quad u \in L^2(S^1).$$

And let us denote by D_θ the densely defined operator on $L^2(S^1)$

$$(D_\theta u)(e^{i\theta}) = -i\tilde{u}'(\theta), \quad \tilde{u}(\theta) = u(e^{i\theta}), \quad \theta \in \mathbb{R}, \quad u \in C^\infty(S^1).$$

For each of the generators of $\mathfrak{A}$ defined on page 227, the value of $\Gamma_{A_k}(e^{2\pi i\varphi}, \pm 1)$, $1 \le k \le 6$, equals

$$a(M_x), \quad b(\pm\infty)I, \quad e^{ijM_\theta}, \quad e^{-i\varphi M_\theta}[1 + (D_\theta - \varphi)^2 - \Delta_B]^{-1/2}e^{i\varphi M_\theta}, \tag{10}$$

$$e^{-i\varphi M_\theta}(D_\theta - \varphi)[1 + (D_\theta - \varphi)^2 - \Delta_B]^{-1/2}e^{i\varphi M_\theta},$$

$$e^{-i\varphi M_\theta}D_x[1 + (D_\theta - \varphi)^2 - \Delta_B]^{-1/2}e^{i\varphi M_\theta},$$

respectively (the first three classes of generators are mapped to constant functions).

For any positive integer k, the homomorphisms σ and Γ defined in (2) and (9) canonically induce homomorphisms, which we also denote by σ and Γ, on the algebra $M_k(\mathfrak{A})$ of k-by-k matrices with entries in $\mathfrak{A}$. Regarding $M_k(\mathfrak{A})$ as a C*-subalgebra of $\mathfrak{B}(L^2(\Omega)^k)$, it turns out that an $A \in M_k(\mathfrak{A})$ is a Fredholm operator if and only if $\sigma(A)$ and $\Gamma(A)$ are invertible [**20**, Theorem 3.2]. Moreover, the quotient of $M_k(\mathfrak{A})$ by the ideal of compact operators on $L^2(\Omega)^k$ is isomorphic to the image of the *total symbol*

$$\sigma \oplus \Gamma : M_k(\mathfrak{A}) \longrightarrow M_k(C(\mathbf{M})) \oplus M_k(C(S^1 \times \{-1,+1\};\ \Psi(S^1 \times B))). \tag{11}$$

2. The principal-symbol exact sequence

In this section we analyse the K-theory index map associated to the exact sequence of C*-algebras

$$0 \longrightarrow \mathfrak{E} \hookrightarrow \mathfrak{A} \xrightarrow{\sigma} C(\mathbf{M}) \longrightarrow 0, \tag{12}$$

induced by the continuous extension to $\mathfrak{A}$ of the usual principal symbol of pseudodifferential operators (see Theorem 1).

We first consider the exact sequence of C*-algebras associated to the map σ_S defined in (4),

$$0 \longrightarrow \mathfrak{K}_{S^1} \hookrightarrow \mathfrak{S} \xrightarrow{\sigma_S} C(S^1 \times \{-1,+1\}) \longrightarrow 0. \tag{13}$$

This sequence tells us that an $A \in \mathfrak{S}$ is a Fredholm operator if and only if $\sigma_S(A)$ never vanishes. A well-known index formula for zero-order pseudodifferential operators on the circle, which essentially goes back to Noether [**26**] (see also the example right after Theorem 19.2.4 in [**17**]), implies that the index of a Fredholm operator $A \in \mathfrak{S}$ is equal to the difference of the winding numbers of the restrictions of $\sigma_S(A)$ to the two copies of S^1 in $S^1 \times \{-1,+1\}$. A detailed proof of this fact without explicit mention of pseudodifferential or singular integral operators can be found in [**1**, Seção 10]. It then follows that there is a Fredholm operator of index 1 in $\mathfrak{S}$ and, therefore, the index map δ_1 in the standard exact sequence of K-groups associated to (13)

$$\begin{array}{ccccc}
\mathbb{Z} \cong K_0(\mathfrak{K}_{S^1}) & \longrightarrow & K_0(\mathfrak{S}) & \longrightarrow & K_0(C(S^1 \times \{-1,+1\})) \\
\delta_1 \uparrow & & & & \downarrow \delta_0 \\
K_1(C(S^1 \times \{-1,+1\})) & \longleftarrow & K_1(\mathfrak{S}) & \longleftarrow & K_1(\mathfrak{K}_{S^1}) = 0
\end{array}$$

is surjective. We then get $K_0(\mathfrak{S}) \cong \mathbb{Z}^2$ and $K_1(\mathfrak{S}) \cong \mathbb{Z}$.

Proposition 1. *The canonical projection $\pi : \mathfrak{E} \to \mathfrak{E}/\mathfrak{K}_\Omega$ induces a K_0-isomorphism.*

Proof. The previous considerations and the isomorphisms (5) and (6) imply that

$$K_0(\mathfrak{E}) \cong K_0(\mathfrak{E}/\mathfrak{K}_\Omega) \cong K_1(\mathfrak{A}/\mathfrak{E}) \cong \mathbb{Z}^2 \quad \text{and} \quad K_1(\mathfrak{E}) \cong \mathbb{Z}.$$

The standard exact sequence of K-groups associated to

$$0 \longrightarrow \mathfrak{K}_\Omega \hookrightarrow \mathfrak{E} \xrightarrow{\pi} \frac{\mathfrak{E}}{\mathfrak{K}_\Omega} \longrightarrow 0$$

then becomes

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\
\delta_1 \uparrow & & & & \downarrow \delta_0 \\
\mathbb{Z}^2 & \longleftarrow & \mathbb{Z} & \longleftarrow & 0
\end{array} \tag{14}$$

The index map δ_1 is non-zero (otherwise there would be an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0$). If it were not surjective, the image of the upper left homomorphism would be a finite non-trivial group, which could not be a subgroup of $\mathbb{Z}^2$. The exactness of the sequence then provides the desired result. $\square$

The following rephrasing of the fact that δ_1 in (14) is surjective is perhaps interesting in itself.

Corollary 1. *There exists $k > 0$ and a Fredholm operator of index 1 in $M_k(\mathfrak{E}) \subset \mathfrak{B}(L^2(\Omega)^k)$.*

We will need to use Banach algebra K-theory [**3**], in which K_0 consists of formal differences of classes of idempotents, while the elements of K_1 are classes of invertibles. The following lemma was used before in [**13**, **23**, **27**], see [**23**, Lemma 1] for a proof.

Lemma 1. *Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{J}$ a closed ideal of $\mathcal{A}$, and let $\delta_1 : K_1(\mathcal{A}/\mathcal{J}) \to K_0(\mathcal{J})$ denote the index map associated to the exact sequence $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{A}/\mathcal{J} \to 0$. If $u \in M_k(\mathcal{A}/\mathcal{J})$ is invertible, $\pi(a) = u$ and $\pi(b) = u^{-1}$, then*

$$\delta_1([u]_1) = \left[\begin{pmatrix} 2ab - (ab)^2 & a(2-ba)(1-ba) \\ (1-ba)b & (1-ba)^2 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

Next we consider the exact sequence obtained from (12) quotienting $\mathfrak{E}$ and $\mathfrak{A}$ by the compact ideal,

$$0 \longrightarrow \frac{\mathfrak{E}}{\mathfrak{K}_\Omega} \longrightarrow \frac{\mathfrak{A}}{\mathfrak{K}_\Omega} \xrightarrow{\pi} \frac{\mathfrak{A}}{\mathfrak{E}} \longrightarrow 0. \tag{15}$$

The following Proposition describes the connecting map $\delta_1 : K_1(\mathfrak{A}/\mathfrak{E}) \to K_0(\mathfrak{E}/\mathfrak{K}_\Omega)$ in terms of the index of Fredholm operators on $L^2(S^1 \times B)$. We recall that each element of $K_1(\mathfrak{A}/\mathfrak{E})$ is of the form $[[A]_\mathfrak{E}]_1$, where $A \in M_k(\mathfrak{A})$ is invertible modulo $M_k(\mathfrak{E})$ for some positive integer k and $[A]_\mathfrak{E}$ denotes its class in the quotient $M_k(\mathfrak{A})/M_k(\mathfrak{E})$.

Proposition 2. *Let $A \in M_k(\mathfrak{A})$ be such that $[A]_\mathfrak{E}$ is invertible in $M_k(\mathfrak{A}/\mathfrak{E})$ for some positive integer k. Then $\Gamma_A(z,-1)$ and $\Gamma_A(z,+1)$ are Fredholm operators on $L^2(S^1 \times B)^k$ for every $z \in S^1$ and*

$$\delta_1([[A]_\mathfrak{E}]_1) = (\operatorname{ind}\Gamma_A(1,-1)\,[E]_0, \operatorname{ind}\Gamma_A(1,+1)\,[E]_0) \tag{16}$$

where E is a rank one projection on $L^2(S^1 \times B)$ and ind *denotes the Fredholm index (It will become clear in the proof how the left side of* (16) *can be regarded as an element of $K_1(\mathfrak{E}/\mathfrak{K}_\Omega)$).*

Proof. Recall that the image of $\mathfrak{E}$ under Γ equals $C(S^1 \times \{-1,+1\}; \mathfrak{K}_{S^1\times B})$.

Let $B \in M_k(\mathfrak{A})$ be such that $I - AB$ and $I - BA$ belong to $M_k(\mathfrak{E})$. Then, $I - \Gamma_A\Gamma_B$ and $I - \Gamma_B\Gamma_A$ belong to $C(S^1 \times \{-1,+1\}; M_k(\mathfrak{K}_{S^1\times B}))$. This means that, at each $(z, \pm 1)$ in $S^1 \times \{-1,+1\}$, $\Gamma_A(z,\pm 1)$ is invertible modulo compacts, and hence it is a Fredholm operator.

Let $a = [A]_\mathfrak{K}$ and $b = [B]_\mathfrak{K}$. By Lemma 1,

$$\delta_1([[A]_\mathfrak{E}]_1) = \left[\begin{pmatrix} 2ab - (ab)^2 & a(2-ba)(1-ba) \\ (1-ba)b & (1-ba)^2 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0 \in K_0(\mathfrak{E}/\mathfrak{K}_\Omega) \tag{17}$$

(we are regarding $\mathfrak{E}/\mathfrak{K}_\Omega$ as an ideal of $\mathfrak{A}/\mathfrak{K}_\Omega$). Using the isomorphism (6) and the definitions of γ and Γ that follow (6), we see that the map

$$\mathfrak{E}/\mathfrak{K}_\Omega \ni [T] \quad \longmapsto \quad \Gamma_T \in C(S^1 \times \{-1,+1\};\, \mathfrak{K}_{S^1\times B})$$

is an isomorphism. Using this isomorphism as an identification, we may replace a and b by Γ_A and Γ_B in (17) and get $\delta_1([[A]_{\mathfrak{E}}]_1) =$

$$\left[\begin{pmatrix} 2\Gamma_A\Gamma_B - (\Gamma_A\Gamma_B)^2 & \Gamma_A(2-\Gamma_B\Gamma_A)(1-\Gamma_B\Gamma_A) \\ (1-\Gamma_B\Gamma_A)\Gamma_B & (1-\Gamma_B\Gamma_A)^2 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

$$\in K_0(C(S^1 \times \{-1,+1\};\, \mathfrak{K}_{S^1\times B})) \cong K_0(C(S^1;\, \mathfrak{K}_{S^1\times B})) \oplus K_0(C(S^1;\, \mathfrak{K}_{S^1\times B}))$$

Let us consider now the map $p : C(S^1;\, \mathfrak{K}_{S^1\times B}) \to \mathfrak{K}_{S^1\times B}$, $p(f) = f(1)$. The map $q : \mathfrak{K}_{S^1\times B} \to C(S^1;\, \mathfrak{K}_{S^1\times B})$ that regards a compact operator as a constant function is a right inverse for p. Moreover, the kernel of p is the suspension of $\mathfrak{K}$, and then its K_0-group is 0. All that imply that p induces a K_0-isomorphism. Using this isomorphism as an identification, we may write $\delta_1([[A]_{\mathfrak{E}}]_1)$ as the element of $K_0(\mathfrak{K}_{S^1\times B}) \oplus K_0(\mathfrak{K}_{S^1\times B})$

$$\left(\left[\begin{pmatrix} 2A^-B^- - (A^-B^-)^2 & A^-(2-B^-A^-)(1-B^-A^-) \\ (1-B^-A^-)B^- & (1-B^-A^-)^2 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0,\right.$$

$$\left.\left[\begin{pmatrix} 2A^+B^+ - (A^+B^+)^2 & A^+(2-B^+A^+)(1-B^+A^+) \\ (1-B^+A^+)B^+ & (1-B^+A^+)^2 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0\right),$$

where we have denoted $\Gamma_A(1,\pm 1)$ by $A^\pm$ and analogously for $\Gamma_B(1,\pm 1)$.

If we apply Lemma 1 to $\mathcal{A} = \mathcal{B}(L^2(S^1 \times B))$, $\mathcal{J} = \mathfrak{K}_{S^1\times B}$ and $u = [A^\pm]_{\mathfrak{K}}$, recalling that the K-theory index map is a generalization of the Fredholm index [**28**], we conclude that this element of $K_0(\mathfrak{K}_{S^1\times B}) \oplus K_0(\mathfrak{K}_{S^1\times B})$ that we have just written above is equal to the left side of (16). $\square$

Theorem 2 below follows immediately from Proposition 1 and Proposition 2. To simplify notation, in its statment we regard as identifications the canonical group isomorphisms

$$K_0(C(S^1 \times \{-1,+1\}; \mathfrak{K}_{S^1\times B})) \cong K_0(C(S^1; \mathfrak{K}_{S^1\times B})) \oplus K_0(C(S^1; \mathfrak{K}_{S^1\times B})) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

THEOREM 2. *Let $\delta_1' : K_1(\mathfrak{A}/\mathfrak{K}) \to K_0(\mathfrak{E})$ denote the index map associated to the exact sequence $0 \to \mathfrak{E} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{E} \to 0$, let $\Gamma_* : K_0(\mathfrak{E}) \to \mathbb{Z} \oplus \mathbb{Z}$ denote the isomorphism induced by the surjective map $\Gamma : \mathfrak{E} \to C(S^1 \times \{-1,+1\}; \mathfrak{K}_{S^1\times B})$, and denote $\delta_1 = \Gamma_* \circ \delta_1'$. Given $A \in M_k(\mathfrak{A})$ such that $[A]_{\mathfrak{E}}$ is invertible in $M_k(\mathfrak{A}/\mathfrak{E})$ for some positive integer k, then $A^- = \Gamma_A(z,-1)$ and $A^+ = \Gamma_A(z,+1)$ are Fredholm operators on $L^2(S^1 \times B)^k$ and*

$$\delta_1([[A]_{\mathfrak{E}}]_1) = (\mathsf{ind} A^-, \mathsf{ind} A^+) \tag{18}$$

where ind *denotes the Fredholm index.*

The final goal should be to describe the index map associated to (12) in purely topological terms. That can now be easily achieved if we invoke the Atiyah-Singer Theorem.

In the following we adopt topological K-theory definitions and notation of [**3**, Chapter I]. We denote, as usual, by T^*M, B^*M and S^*M, respectively, the tangent bundle, the bundle of unit closed balls in the tangent bundle, and the co-sphere bundle of any manifold M.

THEOREM 3. *Let δ_1 denote the index map associated to (12), and let $f \in M_k(C(\mathbf{M}))$ be invertible for some positive k. The restrictions f_- and f_+ of f to the points of $\mathbf{M}$ with, respectively, $t = -\infty$ and $t = +\infty$ are invertible elements of $M_k(C(S^*(S^1 \times B)))$. Let*

$$x_\pm = [(E^k, E^k, f_\pm)] \in K(B^*(S^1 \times B), S^*(S^1 \times B)) \cong K(T^*(S^1 \times B))$$

denote the classes of the triples $(E^k, E^k, f_\pm)$, where E^k denotes the trivial bundle $(S^1 \times B) \times \mathbb{C}^k$. Then

$$\delta^1([f]_1) = (\mathsf{ind}_t(x_+), \mathsf{ind}_t(x_-)) \in \mathbb{Z} \oplus \mathbb{Z},$$

where $ind_t : K(T^*(S^1 \times B)) \to \mathbb{Z}$ denotes the topological index of Atiyah and Singer [**2**].

PROOF. Let $A \in M_k(\mathfrak{A})$ be such that $\sigma(A) = f$. Define $A^\pm = \Gamma_A(1, \pm 1)$. We claim that A^- and A^+ belong to $M_k(\Psi(S^1 \times B))$ and $\sigma_{S^1 \times B}(A^\pm) = f_\pm$. By definition, $f_\pm = \sigma(A)|_{t=\pm\infty}$. The maps $\mathfrak{A} \ni A \mapsto \sigma(A)|_{t=\pm\infty}$, $\mathfrak{A} \ni A \mapsto \Gamma_A(1, \pm 1))|_{t=\pm\infty}$ and $\Psi(M) \ni T \mapsto \sigma_{S^1 \times B}(T)$ are C*-algebra homomorphisms. So, it is enough to verify our claim on a set of generators of $\mathfrak{A}$. For each of the generators $A_1, A_2, \cdots, A_6$ defined on page 227, the claim follows immediately from (10).

Our theorem would already follow from the Atiyah-Singer Theorem [**2**] if A^+ and A^- were classical pseudodiferential operators. For the general case, we need to show that the topological index of the class $[(E^k, E^k, \sigma(T))] \in K(T^*(S^1 \times B))$ is equal to the Fredholm index of any Fredholm operator $T \in \Psi(S^1 \times B)$ (the Atiyah-Singer Theorem gives that for those T that belong to the dense subalgebra of all classical pseudodifferential operators).

In the rest of this proof, we will replace $S^1 \times B$ by an arbitrary compact manifold M.

Let δ^{tbs} denote index mapping for the exact sequence of C*algebras

$$0 \longrightarrow C_0(T^*M) \longrightarrow C(B^*M) \longrightarrow C(S^*M) \longrightarrow 0$$

and let $\iota : K_0(C_0(T^*M)) \mapsto K(T^*M)$ be the canonical isomorphism between these two groups. We then have

$$\iota \circ \delta^{tbs}([g]_1) = [(E^k, E^k, g)], \quad \text{for any invertible } g \in M_k(C(S^*M)). \tag{19}$$

For a proof of (19) in a slightly more general setting, see [**22**, Proposition 15], for example.

Now let $T \in M_k(\Psi(M))$ be a Fredholm operator. Its Fredholm index $\mathsf{ind}(T)$ coincides with $\delta^M([\sigma_M(T)]_1)$, where δ^M denotes the index map of (1). Because every $f \in C^\infty(S^*M)$ is the symbol of a zero-order classical pseudodifferential operator, there exists such a pseudo T_0 with invertible symbol so close to T that $[\sigma(T)]_1 = [\sigma(T_0)]_1$. Then

$$\mathsf{ind}(T) = \delta^M([\sigma_M(T)]_1) = \delta^M([\sigma_M(T_0)]_1) = \mathsf{ind}_t([(E^k, E^k, \sigma_M(T_0))]) =$$

$$\mathsf{ind}_t(\iota \circ \delta^{tbs}([\sigma_M(T_0)]_1)) = \mathsf{ind}_t(\iota \circ \delta^{tbs}([\sigma_M(T)]_1)) = \mathsf{ind}_t([(E^k, E^k, \sigma_M(T_0))]).$$

The Atiyah-Singer Theorem was used in the third of the above equalities. Twice we used (19). □

EXAMPLE 1. Some statements made here about operators in $\mathfrak{A}$ are proven in [**20**], using results from [**8**].

Let $L : C^\infty(\Omega)^k \to C^\infty(\Omega)^k$ be a differential operator of the form

$$L = \sum_{j+|\alpha|\leq N} a_{j,\alpha}(t,x) \left(\frac{1}{i}\frac{\partial}{\partial t}\right)^j \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha ,$$

where each $a_{j,\alpha} \in M_k(C^\infty(\Omega))$ has support contained in $U \times \mathbb{R}$, U the domain of a chart in B. Moreover, let us assume that $a_{j,\alpha}$ is *semi-periodic* in the following sense. There exist 2π-periodic (in t) $a^\pm_{j,\alpha} \in M_k(C^\infty(\Omega))$ and, for a choice of real-valued $\chi^\pm \in C^\infty(\mathbb{R})$ such that $\chi^+(t) + \chi^-(t) = 1$ for all t and $\chi_\pm(t) = 0$ for $\mp t > 1$, there exists $a^0_{j,\alpha} \in M_k(C^\infty(\Omega))$ vanishing at infinity such that

$$a_{j,\alpha}(t,x) = \chi^+(t)a^+_{j,\alpha}(t,x) + \chi^-(t)a^-_{j,\alpha}(t,x) + a^0_{j,\alpha}(t,x), \quad \text{for all} \quad (t,x) \in \Omega.$$

The operator $A = L\Lambda^N$, a priori defined on the dense subspace $[\Lambda^{-N}(C^\infty_c(\Omega))]^k$ of $L^2(\Omega)^k$, extends to a bounded operator $A \in M_k(\mathfrak{A})$ ([**8**, Sections VII-3 and IX-3], [**20**, Theorem 3.7]). Each of the generators $A_1, \cdots, A_5$ defined on page 227 is a particular example of such an A, with $k = 1$ and N equal to 0 or 1. And because B has a finite atlas, each A_6 is a finite sum of such A's.

The symbol σ_A coincides with the continuous extension to the compactification $\mathbf{M}$ of $S^*\Omega$ of the principal symbol of L,

$$\sigma(A)(t,x,\tau,\xi) = \sum_{j+|\alpha|=N} a_{j,\alpha}(t,x)\tau^j\xi^\alpha, \quad (t,x,\tau,\xi) \in S^*\Omega$$

It follows that A is invertible modulo $\mathfrak{E}$ if and only if L is *uniformily elliptic* in the sense that

$$\inf\{\,|\sum_{j+|\alpha|=N} a_{j,\alpha}(t,x)\tau^j\xi^\alpha|,\ (t,x,\tau,\xi) \in S^*\Omega\} \ > \ 0.$$

We also have

$$\Gamma_A(1,\pm 1) = \sum_{j+|\alpha|=N} a^\pm_{j,\alpha}(t,\theta) \left(\frac{1}{i}\frac{\partial}{\partial \theta}\right)^j \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha \Lambda^N$$

(θ denotes the argument of a point in S^1). These are zero-order elliptic classical pseudodifferential operators on $S^1 \times B$ whose principal symbols are

$$\sigma(A)|_{t=\pm\infty}(e^{i\theta},x,\tau,\xi) = \sum_{j+|\alpha|=N} a^\pm_{j,\alpha}(\theta,x)\tau^j\xi^\alpha, \quad (e^{i\theta},x,\tau,\xi) \in S^*(S^1 \times B).$$

Having smooth symbols, their indices (and then $\delta_1([[A]_\mathfrak{E}]_1)$) can be more explicitly calculated as the integral of certain diferential forms on $S^*(S^1 \times B)$.

With the aid of a computer algebra system, Patrícia Hess [**13**] was able to follow this path and, using Fedosov's formula [**12**, formula (1)], find the index of a 2-by-2 pseudodifferential operator on $S^1 \times S^1$. That result was used in her proof that, for the case $B = S^1$, $K_0(\mathfrak{A}) \cong \mathbb{Z}^5$ and $K_1(\mathfrak{A}) \cong \mathbb{Z}^4$.

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