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Comparison Algebras with Periodic Symbols

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Severino Toscano do Rêgo Melo

Abstract

Let Ω denote the cylinder $\mathbb{R} \times \mathbb{B}$, where $\mathbb{B}$ is a compact Riemannian manifold, Δ_Ω its Laplacian and $\mathcal{H}$ the Hilbert space $L^2(\Omega)$. We define $\mathcal{C}_\mathcal{P}$ as the C*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by: (i) multiplications by the smooth functions on $\mathbb{B}$, by the 2π-periodic continuous functions on $\mathbb{R}$ and by the continuous functions on $[-\infty, +\infty]$; (ii) $\Lambda := (1 - \Delta_\Omega)^{-1/2}$; (iii) $\frac{\partial}{\partial t}\Lambda$, $t \in \mathbb{R}$; and (iv) $D\Lambda$, where D is any first order differential operator with smooth coefficients on $\mathbb{B}$.

The commutator ideal $\mathcal{E}_\mathcal{P}$ of $\mathcal{C}_\mathcal{P}$ is proven here to be *-isomorphic to $\mathcal{SL}\bar{\otimes}\mathcal{K}_\mathbb{Z}\bar{\otimes}\mathcal{K}_\mathbb{B}$, where $\mathcal{SL}$ denotes the algebra of singular integral operators on the circle and $\mathcal{K}_\mathbb{Z}$ and $\mathcal{K}_\mathbb{B}$ denote the sets of compact operators on $L^2(\mathbb{Z})$ and $L^2(\mathbb{B})$, respectively. This allows us to define on $\mathcal{C}_\mathcal{P}$ an operator-valued symbol, the γ-symbol, such that $\ker\gamma \cap \ker\sigma = \mathcal{K}(\mathcal{H})$. Here σ denotes the complex-valued symbol on $\mathcal{C}_\mathcal{P}$ that arises from the Gelfand map of the commutative C*-algebra $\mathcal{C}_\mathcal{P}/\mathcal{E}_\mathcal{P}$. We prove that $A \in \mathcal{C}_\mathcal{P}$ is Fredholm if and only if γ_A and σ_A are invertible.

We first consider the simpler case $\Omega = \mathbb{R}$. A unitary map W from $L^2(\mathbb{R})$ onto $L^2(\mathbb{S}^1)\bar{\otimes}L^2(\mathbb{Z})$ is defined such that the conjugate $W\mathcal{E}W^{-1}$ of the commutator ideal equals $\mathcal{SL}\bar{\otimes}\mathcal{K}_\mathbb{Z}$. In the case of $\Omega = \mathbb{R} \times \mathbb{B}$, we conjugate the commutator ideal with $W \otimes I_\mathbb{B}$, where $I_\mathbb{B}$ denotes the identity operator on $L^2(\mathbb{B})$, and obtain $\mathcal{SL}\bar{\otimes}\mathcal{K}_\mathbb{Z}\bar{\otimes}\mathcal{K}_\mathbb{B}$.

These results can be applied to differential operators on the line and on the cylinder. We then prove that an operator with coefficients in the algebra generated by the functions of type (i) above is Fredholm if and only if it is uniformly elliptic and a certain family of differential operators on $\mathbb{S}^1 \times \mathbb{B}$ is invertible. For the case of first order systems on the line, an index formula is given, involving the eigenvalues of two operators on the circle.

To the memory of my parents.

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Contents

Introduction

Let Ω denote a Riemannian manifold and $\triangle_\Omega$ its Laplacian. C*-algebras of bounded operators on $L^2(\Omega)$ generated by multiplication-operators and operators of the type $D(1-\triangle_\Omega)^{-1/2}$, where D denotes a first order differential expression, are called "Comparison Algebras"[2]. In this dissertation we are mainly concerned in finding necessary and sufficient conditions for operators in certain Comparison Algebras to be Fredholm.

The multiplication-generators of the Comparison Algebra on $\mathbb{R}$ studied in Chapter 1 are the *semi-periodic* functions, i. e. , functions in the algebra generated by the 2π-periodic continuous functions and the continuous functions on $[-\infty, +\infty]$. Any Mth-order linear ordinary differential operator L with semi-periodic coefficients is *within reach* of this algebra, i. e. , $L(1-\triangle_{\mathbb{R}})^{-M/2}$ belongs to it. It is a simple matter to consider systems instead of equations and, thus, we find that

$$L : H^M(\mathbb{R}, \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}, \mathbb{C}^N)$$

is Fredholm if and only if L is uniformly elliptic and a certain family of differential operators on $\mathbb{S}^1$ are invertible (Theorem 1.16). By H^M above, we denote the Mth Sobolev space. For the case of first-order systems, an explicit index-formula is given (Theorem 1.18). The first three sections of the Chapter 1 are essentially the content of [9] .

Cordes [3] studied a Comparison Algebra on a "polycylinder" $\Omega = \mathbb{R}^n \times \mathbb{B}$, where $\mathbb{B}$ is compact. In the case $n = 1$, the multiplication-generators are continuous on the compactification $[-\infty, +\infty] \times \mathbb{B}$ of Ω. In Chapter 2 , we add the 2π-periodic functions to this algebra and, using results of Chapter 1, extend his result for $n = 1$.

For the well-known example of Gohberg [14], the Fredholm property and the Fredholm index of an operator A are governed by its *symbol* σ_A, a complex-valued function over a compact space. The operator A is Fredholm if and only if its "σ-symbol" never vanishes. In Gohberg's case, as well for a variety of other examples (c. f. [5] [16] [24] [7]), the commutators of the algebra under consideration are compact. By the Gelfand-Naimark Theorem, one needs only to find a good description of the maximal ideal space of the quotient of the algebra by the compact ideal. The σ-symbol is then defined as the composition of the Gelfand map with the canonical projection, and a Fredholm criterion follows from Atkinson's Theorem. (See [11], for example.)

Cases of non-compact commutators have been studied by Cordes [4] [3] [8], Erkip [13] , Dudučava [12] and Power [20] . In our case, as in [4] , [3] and [8] , the algebra $\mathcal{A}$ has a two-link ideal chain

$$\mathcal{A} \supset \mathcal{E} \supset \mathcal{K} .$$

Here $\mathcal{E}$ denotes the *commutator ideal* (the smallest closed ideal of $\mathcal{A}$ containing all its commutators), and $\mathcal{K}$ denotes the ideal of compact operators on the Hilbert space $\mathcal{H}$ under consideration. The quotient $\mathcal{A}/\mathcal{E}$ is a commutative C*-algebra and a σ-symbol can again be defined as the composition of the Gelfand map with the canonical projection $\mathcal{A} \to \mathcal{A}/\mathcal{E}$. Invertibility of σ_A, $A \in \mathcal{A}$, is then a necessary condition for A to be Fredholm. The second quotient $\mathcal{E}/\mathcal{K}$, even though non-commutative, is isomorphic to an algebra of continuous functions on a compact space $\mathbf{E}$,taking values in the compact operators of another Hilbert space $\mathbf{h}$. In [3], [8] and in our examples here, this isomorphism can be extended to a homomorphism of the whole algebra into the continuous functions on $\mathbf{E}$, taking values in a Comparison Algebra of h. This homomorphism is called the "γ-symbol". The two symbols are not independent. In fact, we have

$$\ker \sigma \cap \ker \gamma = \mathcal{K} .$$

An operator $A \in \mathcal{A}$ is Fredholm if and only if σ_A and γ_A are invertible.

The results of Chapter 2 can be extended in a standard way to non-compact manifolds with cylindrical ends (c. f. [2] , VIII-3,4). Fredholm properties of elliptic operators on such manifolds have been studied, for example, by Lockhart-McOwen [17] and Melrose-Mendoza [19] . Differential operators on a cylinder with periodic coefficients have been considered by Taubes [26] .

Our Fredholm result for operators in the algebra considered in Chapter 1 (Theorem 1.14) could also be obtained as a consequence of a result previously announced by Rabinovič ([21] , Theorem 4). There he also gives a formula that makes it possible to calculate the index of those operators.

Let H denote the Hilbert transform on $L^2(\mathbb{R})$. The results presented in [15] give, in particular, a criterion for operators of the type

$$A = a - ibH$$

to be Fredholm, where a and b are continuous and periodic. (See also [23] for a generalization.) This criterion says that A is Fredholm if and only if $a+b$ and $a-b$ never vanish and the winding number with respect to the origin of $\frac{a+b}{a-b}$ equals zero. There is some similarity between this result and what we get for the operator

$$B = a + bS \ , \quad S = -i\frac{d}{dt}(1 - \frac{d^2}{dt^2})^{-1/2} \ ,$$

with a and b 2π-periodic and continuous. Theorem 1.14 and Remark 1.15 imply that $a+bS$ is Fredholm if and only if $a+b$ and $a-b$ never vanish and Γ_A is invertible. Here Γ_A is a function taking values in the singular integral operators on $\mathbb{S}^1$. The requirement that Γ_A be Fredholm and of index zero at every point is equivalent to $(a+b)(a-b) \neq 0$ and winding $\#(\frac{a+b}{a-b}) = 0$, what gives a necessary condition for B to be Fredholm.

Notation

The inner-products of our Hilbert spaces are linear in the second argument and denoted by $(\cdot,\cdot)$.

Some of the symbols in the following list are not defined in the main text. Others are, and have been inserted here for the reader's convenience. All "functions" below are complex-valued functions.

- $s(t) = t(1+t^2)^{-1/2}$
- $\mathbb{S}^1 :=$ the unit circle in $\mathbb{C}$
- $\mathbf{C}(X) := \{$continuous functions on $X\}$
- $\mathbf{C}^\infty(X) := \{$smooth functions on $X\}$
- $\mathbf{C}_o^\infty(X) := \{$smooth functions on X with compact support$\}$
- $\mathbf{CB}(X) := \{$continuous bounded complex-valued functions on $X\}$
- $\mathbf{CO}(X) := \{$functions in $\mathbf{CB}(X)$ vanishing at infinity$\}$
- $\mathbf{CS}(\mathbb{R}) := \{$continuous complex functions on $\mathbb{R}$ with limits at $+\infty$ and $-\infty\}$
- $\mathbf{CS}(\mathbb{Z}) := \{$sequences indexed by $\mathbb{Z}$ with limits at $+\infty$ and $-\infty\}$
- $\mathbf{P}_{2\pi} := \{2\pi$-periodic continuous functions on $\mathbb{R}\}$
- $\mathcal{A}^\natural$ denotes the algebra generated by $\mathbf{CS}(\mathbb{R})$ and $\mathbf{P}_{2\pi}$.
- $\tilde{a}$, for $a \in \mathcal{A}^\natural$: See equation 1.25 on page 16.
- F denotes the Fourier transform on $\mathbb{R}$: $Fu(\tau) = (2\pi)^{-1/2}\int e^{-i\tau t}u(t)\,dt$.

- $b(D)$: See page 6.
- T_j, for $j \in \mathbb{Z}$: See page 6
- F_d denotes the *discrete* Fourier transform. (See equation 1.5 on page 8.)
- $b(D_\theta)$, for $b \in \mathbf{CS}(\mathbb{Z})$: See page 8.
- Y_φ, for $\varphi \in \mathbb{R}$: See equation 1.6 on page 8.
- W : See equation 1.7 on page 8.
- $\mathcal{SL}$: See page 8
- $\mathbf{M}_{SL}$: See page 17
- $\mathbf{M}_A$: See page 16
- $\mathbf{W}_A$: See page 20
- $\mathbf{M}_P$: See page 42
- $\mathbf{W}_P$: See page 44
- $\otimes$, $\bar{\otimes}$: See Appendix.
- $\mathcal{L}(E) :=$ {bounded operators on the Banach space E}
- $\mathcal{K}(E) :=$ {compact operators on E}
- $\mathcal{L}_X := \mathcal{L}(L^2(X))$
- $\mathcal{K}_X := \mathcal{K}(L^2(X))$
- I_X denotes the identity operator on $L^2(X)$.
- $\mathbf{CB}(X, \mathcal{L}_Y)$, $\mathbf{CB}(X, \mathcal{K}_Y)$, $\mathbf{CO}(X, \mathcal{L}_Y)$, $\mathbf{CO}(X, \mathcal{K}_Y)$: See Appendix.
- A_k, B_k, for $k = 1, \ldots, 6$: See page 36.
- Λ: See page 35 , $\tilde{\Lambda}$: See page 36 .
- $\Lambda_{\mathbb{R}}$: See page 25 , $\Lambda_{\mathbb{S}^1}$: See page 28 , Λ_φ : See page 54 .

Chapter 1

A Comparison Algebra on the line with semi-periodic multiplications

1.1 Definition of the algebra $\mathcal{A}$ and a description of its commutator ideal

Let $\mathcal{A}$ denote the subalgebra of $\mathcal{L}_{\mathbb{R}} := \mathcal{L}(L^2(\mathbb{R}))$ obtained as the closure of the algebra generated by:

(i) multiplications by functions in $\mathbf{CS}(\mathbb{R})$,

(ii) multiplications by e^{ijt}, $j \in \mathbb{Z}$, and

(iii) operators of the form $b(D) := F^{-1}b(\tau)F$, $b \in \mathbf{CS}(\mathbb{R})$.

In order to give a description of $\mathcal{E}_{\mathcal{A}}$, the commutator ideal of $\mathcal{A}$, we consider the conjugate algebra $\hat{\mathcal{A}} := F^{-1}\mathcal{A}F$. The generators of $\hat{\mathcal{A}}$ that correspond to (i), (ii) and (iii) are respectively given by:

($\hat{i}$) operators of the form $a(D)$, $a \in \mathbf{CS}(\mathbb{R})$,

($\hat{ii}$) translation operators T_j, $(T_j u)(\tau) := u(\tau + j)$, and

($\hat{iii}$) multiplications by functions in $\mathbf{CS}(\mathbb{R})$.

It is obvious that the commutator $[b(D), T_j]$ equals zero and it is well known that

$$[a(D), b(\tau)] \in \mathcal{K}_{\mathbb{R}}\ , \quad \text{for} \quad a,\ b \in \mathbf{CS}(\mathbb{R}) \tag{1.1}$$

(c. f. [4] , Chapter III, for example). It is also clear that

$$[T_j, a(\tau)] = (a(\tau + j) - a(\tau))T_j, \quad \text{for} \quad a \in \mathbf{CS}(\mathbb{R}) \quad \text{and} \quad j \in \mathbb{Z}\ . \tag{1.2}$$

Note that this last commutator is not compact for $j \neq 0$ and nonconstant a.

Denoting by $\hat{\mathcal{E}_A}$ the commutator ideal of $\hat{\mathcal{A}}$, it is obvious that $\hat{\mathcal{E}_A} = F^{-1}\mathcal{E}_A F$.

Proposition 1.1 *The commutator ideal $\hat{\mathcal{E}_A}$ of the conjugate algebra $\hat{\mathcal{A}}$ coincides with the closure of*

$$\hat{\mathcal{E}}_{A,O} := \{ \sum_{j=-N}^{N} b_j(D)a_j(\tau)T_j + K; \quad N \in \mathbf{N},\ b_j \in \mathbf{CS}(\mathbf{R}),\ a_j \in \mathbf{CO}(\mathbf{R}),\ K \in \mathcal{K}_{\mathbf{R}}\} . \quad (1.3)$$

Proof: The algebra $\mathcal{A}$ is a "comparison algebra", as defined in [2] . Indeed, $\mathcal{A}$ is generated by

$$s(D) = \frac{1}{i}\frac{d}{dt}(1 - \frac{d^2}{dt^2})^{-1/2}$$

and by operators of type (i) and (ii). By Lemma V.1.1 of [2] , we conclude that the compact ideal $\mathcal{K}_{\mathbf{R}}$ is contained in the commutator ideal of $\mathcal{A}$, hence:

$$\mathcal{K}_{\mathbf{R}} \subset \hat{\mathcal{E}_A} . \quad (1.4)$$

Because of (1.2), it is clear that all operators of the form $b(D)\nabla_j a(\tau)T_j$ are in $\hat{\mathcal{E}_A}$, for $b \in \mathbf{CS}(\mathbf{R})$, $j \in \mathbf{Z}$ and $\nabla_j a(\tau) := a(\tau + j) - a(\tau)$, $a \in \mathbf{CO}(\mathbf{R})$. By the Stone-Weierstrass Theorem, the algebra generated by all such $\nabla_j a$ is dense in $\mathbf{CO}(\mathbf{R})$, hence $\hat{\mathcal{E}}_{A,O} \subseteq \hat{\mathcal{E}_A}$.

On the other hand, using (1.1), (1.2) and (1.4), we see that $\hat{\mathcal{E}}_{A,O}$ is a subalgebra of $\hat{\mathcal{A}}$ that contains the commutators of all operators of types $(\hat{i})$, $(\hat{ii})$ and $(\hat{iii})$ and that, furthermore, $\hat{\mathcal{E}}_{A,O}$ is invariant under left or right multiplication by those operators. Taking limits, it follows that the closure of $\hat{\mathcal{E}}_{A,O}$ is a two-sided ideal of $\hat{\mathcal{A}}$ containing all its commutators. Hence, $\hat{\mathcal{E}_A}$ is contained in the closure of $\hat{\mathcal{E}}_{A,O}$. q.e.d.

In the rest of this section, we define a unitary map

$$W : L^2(\mathbf{R}) \to L^2(\mathbb{S}^1; L^2(\mathbf{Z}))$$

and find a useful description for $W\hat{\mathcal{E}_A}W^{-1}$.

Given $u \in L^2(\mathbf{R})$, denote:

$$u^\diamond(\varphi) := (u(\varphi - j))_{j\in\mathbf{Z}} ,$$

for each $\varphi \in \mathbf{R}$. The sequence $u^\diamond(\varphi)$ belongs to $L^2(\mathbf{Z})$ for almost every φ, by Fubini's Theorem, since $L^2(\mathbf{R})$ can be identified with $L^2([0,1) \times \mathbf{Z})$. Let

$$F_d : L^2(\mathbb{S}^1, d\theta) \to L^2(\mathbf{Z}) , \quad \mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbf{R}\} ,$$

denote the discrete Fourier transform:

$$(F_d u)_j = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(\theta) e^{-ij\theta}\, d\theta \quad , \quad j \in \mathbb{Z} \, . \tag{1.5}$$

For each $\varphi \in \mathbb{R}$, define

$$Y_\varphi := F_d e^{-i\varphi\theta} F_d^{-1} \quad . \tag{1.6}$$

The operators Y_φ define a smooth function on $\mathbb{R}$, taking values in the unitary operators on $L^2(\mathbb{Z})$ and satisfying $(Y_k u)_j = u_{j+k}$, for $k \in \mathbb{Z}$ and $u \in L^2(\mathbb{Z})$, and $Y_\varphi Y_\omega = Y_{\varphi+\omega}$, for $\varphi, \omega \in \mathbb{R}$.

We now define the unitary map (with $\mathbb{S}^1 = \{e^{2\pi i\varphi}; \varphi \in \mathbb{R}\}$)

$$\begin{aligned} W:\ L^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{S}^1, d\varphi; L^2(\mathbb{Z})) \\ u &\longmapsto (Wu)(\varphi) = Y_\varphi u^\diamond(\varphi) \ . \end{aligned} \tag{1.7}$$

Let $\mathbf{CS}(\mathbb{Z})$ denote the set of sequences $b(j)$, $j \in \mathbb{Z}$, with limits as $j \to +\infty$ and $j \to -\infty$ and let $b(D_\theta)$ denote $F_d^{-1} b(M) F_d$, where $b(M)$ denotes the operator multiplication by b on $L^2(\mathbb{Z})$. We then denote by $\mathcal{SL}$ the C*-subalgebra of $\mathcal{L}_{\mathbb{S}^1}$ generated by $b(D_\theta)$, $b \in \mathcal{CS}(\mathbb{Z})$, and by the multiplications by smooth functions on $\mathbb{S}^1$. It is easy to check that, with $\Lambda_{\mathbb{S}^1} := (1 - \Delta_{\mathbb{S}^1})^{-1/2}$ ($\Delta_{\mathbb{S}^1} = d^2/d\theta^2$ is essentially self-adjoint on $\mathbf{C}^\infty(\mathbb{S}^1)$),

$$\frac{1}{i}\frac{d}{d\theta}\Lambda_{\mathbb{S}^1} = s(D_\theta) \ , \ s(j) = (1 + j^2)^{-1/2} \ .$$

Since the polynomials in s are dense in $\mathbf{CS}(\mathbb{Z})$, $\mathcal{SL}$ coincides with the C*-subalgebra of $\mathcal{L}_{\mathbb{S}^1}$ generated by $-i\frac{d}{d\theta}\Lambda_{\mathbb{S}^1}$ and $\mathbf{C}^\infty(\mathbb{S}^1)$. In other words, $\mathcal{SL}$ is the unique comparison algebra over $\mathbb{S}^1$. It therefore contains $\mathcal{K}_{\mathbb{S}^1}$ and all its commutators are compact. (c. f. [2], Chapters V and VI)

Theorem 1.2 *With the above notation, we have:*

$$W\hat{\mathcal{E}_A}W^{-1} = \mathcal{SL}\bar{\otimes}\mathcal{K}_{\mathbb{Z}} \ . \tag{1.8}$$

Furthermore, for $b \in \mathbf{CS}(\mathbb{R})$, $a \in \mathbf{CO}(\mathbb{R})$ *and* $j \in \mathbb{Z}$, *we have :*

$$A^\diamond(e^{2\pi i\varphi}) := Y_\varphi a(\varphi - M) Y_{-\varphi} \in \mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}})$$

and

$$W(b(D)aT_j)W^{-1} = b(D_\theta) Y_\varphi a(\varphi - M) Y_{-\varphi - j} + K \ , \quad K \in \mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z}} \ . \tag{1.9}$$

Proof Let us first consider the subalgebra $\mathcal{F} \subset \hat{\mathcal{E}_A}$ defined as the closure of

$$\mathcal{F}^\circ := \{ \sum_{j=-N}^{N} a_j T_j + K; \quad N \in \mathbb{N}, \; K \in \mathcal{K}_{\mathbb{R}}, \; a_j \in \mathbf{CO}(\mathbb{R})\} \ .$$

Noting that, for $j \in \mathbb{Z}$ and $\varphi \in \mathbb{R}$,

$$(Y_\varphi \, a(\varphi - M) \, Y_{-j} Y_{-\varphi})(Y_\varphi u^\circ(\varphi)) = Y_\varphi \, (a(\varphi - k) \, u(\varphi - k + j))_{k \in \mathbb{Z}} \ ,$$

we conclude that

$$W a T_j W^{-1} = Y_\varphi a(\varphi - M) \, Y_{-\varphi - j} \ , \tag{1.10}$$

for $j \in \mathbb{Z}$ and $a \in \mathbf{CO}(\mathbb{R})$. Using that

$$\lim_{k \to \pm\infty} a(\varphi - k) = 0 \ , \quad \text{for all} \quad \varphi \in \mathbb{R} \ ,$$

and that

$$Y_{(\varphi+1)} \, a(\varphi + 1 - M) \, Y_{-(\varphi+1)-j} = Y_\varphi Y_1 a(\varphi + 1 - M) Y_{-1} Y_{-\varphi - j} = Y_\varphi a(\varphi - M) Y_{-\varphi - j} \ ,$$

it follows that the right-hand side of (1.10) is a continuous compact-operator-valued function on $\mathbb{S}^1 = \{e^{2\pi i \varphi}; \; \varphi \in \mathbb{R}\}$:

$$Y_\varphi a(\varphi - M) \, Y_{-\varphi - j} \in \mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}}) \ .$$

Since the imbedding

$$\mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}}) \subset \mathcal{L}_{\mathbb{S}^1 \times \mathbb{Z}}$$

is an isometry (by Proposition A.3) and

$$\mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}}) = \mathbf{C}(\mathbb{S}^1) \bar{\otimes} \mathcal{K}_{\mathbb{Z}} \tag{1.11}$$

(by Proposition A.4) , we get :

$$W \mathcal{F} W^{-1} \subseteq \mathbf{C}(\mathbb{S}^1) \bar{\otimes} \mathcal{K}_{\mathbb{Z}} \ . \tag{1.12}$$

We postpone the proof of the following lemma.

Lemma 1.3 *Inclusion (1.12) is in fact an equality.*

We now need to describe what $Wb(D)GW^{-1}$ is, for $b \in \mathbf{CS}(\mathbb{R})$ and $G \in \mathcal{F}$. Let us consider the special case $b = s$, $s(t) = t(1+t^2)^{-1/2}$. It is well-known that the inverse Fourier transform of s is given by

$$(F^{-1}s)(\tau) = i\sqrt{\frac{2}{\pi}}\chi(\tau)\,\mathrm{p.v.}\frac{1}{\tau} + \psi(\tau)\,,$$

where ψ is continuous and integrable, $\chi \in \mathbf{C}_o^\infty(\mathbb{R})$ is even , equals 1 in a neighborhood of $\tau = 0$ and has support contained in $[-\frac{1}{4}, \frac{1}{4}]$, and p.v. denotes principal value. (See, for example, [18], [6], or apply [4], Theorem II.5.2.) Thus, we get, with $\omega = F\psi$,

$$(s(D)u)(t) = \frac{i}{\pi}\lim_{\epsilon\searrow 0}\int_{\epsilon\le|t-\tilde{t}|\le\frac{1}{2}}\frac{\chi(t-\tilde{t})}{t-\tilde{t}}u(\tilde{t})\,d\tilde{t} \;+\; (\omega(D)u)(t)\,, \tag{1.13}$$

where $\omega \in \mathbf{CO}(\mathbb{R})$ and the limit in the right-hand side exists for every $t \in \mathbb{R}$ and $u \in \mathbf{C}_o^\infty(\mathbb{R})$. Let us denote by S the bounded operator on $L^2(\mathbb{R})$ given by

$$(Su)(t) = \lim_{\epsilon\searrow 0}\int_{\epsilon\le|t-\tilde{t}|\le\frac{1}{2}}\frac{\chi(t-\tilde{t})}{t-\tilde{t}}u(\tilde{t})\,d\tilde{t}\,,\quad u \in \mathbf{C}_o^\infty(\mathbb{R})\,. \tag{1.14}$$

We will see next that, up to some integral operator with a smooth kernel, the conjugate of S with respect to W is the Hilbert transform on the circle.

Lemma 1.4 *Given $u \in \mathbf{C}_o^\infty(\mathbb{R})$ and defining $v := Wu$, we have, for every $\varphi \in \mathbb{R}$,*

$$(WSu)(e^{2\pi i\varphi}) = \lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde{\varphi}|\le\frac{1}{2}}\pi\cot\pi(\varphi-\tilde{\varphi})\,v(e^{2\pi i\tilde{\varphi}})\,d\tilde{\varphi} + \int_{|\varphi-\tilde{\varphi}|\le\frac{1}{2}}A(\varphi-\tilde{\varphi})\,v(e^{2\pi i\tilde{\varphi}})\,d\tilde{\varphi}\,. \tag{1.15}$$

Here, $A(\cdot)$ is a 1-periodic function in $\mathbf{C}^\infty(\mathbb{R}, \mathcal{L}_{\mathbb{Z}})$, the integrals are to be understood as Riemann integrals in $L^2(\mathbb{Z})$, and the limit exists in $L^2(\mathbb{Z})$-norm for every $\varphi \in \mathbb{R}$.

Proof: It is obvious that $v \in \mathbf{C}^\infty(\mathbb{S}^1; L^2(\mathbb{Z}))$. In order to simplify notation, let us write $w(\varphi)$ when we mean $w(e^{2\pi i\varphi})$, $w \in \mathbf{C}^\infty(\mathbb{S}^1; L^2(\mathbb{Z}))$. The function defined by

$$w_\varphi(\tilde{\varphi}) = \frac{v(\tilde{\varphi}) - v(\varphi)}{\varphi - \tilde{\varphi}}\quad ,\tilde{\varphi} \ne \varphi$$

and $w_\varphi(\varphi) = -v'(\varphi)$ is continuous. Since

$$\int_{\epsilon\le|\varphi-\tilde{\varphi}|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde{\varphi})}{\varphi-\tilde{\varphi}}v(\varphi)\,d\tilde{\varphi} = 0\,,$$

for all $\epsilon > 0$, we then have

$$\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}\,v(\tilde\varphi)\,d\tilde\varphi=\int_{|\varphi-\tilde\varphi|\le\frac{1}{2}}\chi(\varphi-\tilde\varphi)\,w_\varphi(\tilde\varphi)\,d\tilde\varphi\,. \tag{1.16}$$

It follows from the definition of Y_φ in (1.6) that the derivative of Y_φ at $\varphi = 0$ is the identity operator of $L^2(\mathbf{Z})$ and , hence,

$$\frac{\chi(\sigma)}{\sigma}Y_\sigma=\chi(\sigma)\,[\frac{1}{\sigma}+A_1(\sigma)]\,,\quad \sigma\in[-\frac{1}{2},\frac{1}{2}]\,,$$

where A_1 is a smooth $L^2(\mathbf{Z})$-valued function. Since

$$a(\sigma):=\frac{\chi(\sigma)}{\sigma}-\pi\cot\pi\sigma\in\mathbf{C}^\infty([-\frac{1}{2},\frac{1}{2}])$$

is 1-periodic and the limit in (1.16) exists, we obtain :

$$\begin{aligned}&\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}Y_{\varphi-\tilde\varphi}\,v(\tilde\varphi)\,d\tilde\varphi=\\ &\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}v(\tilde\varphi)\,d\tilde\varphi+\int_{|\varphi-\tilde\varphi|\le\frac{1}{2}}\chi(\varphi-\tilde\varphi)\,A_1(\varphi-\tilde\varphi)\,v(\tilde\varphi)\,d\tilde\varphi\\ &\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\pi\cot\pi(\varphi-\tilde\varphi)\,v(\tilde\varphi)\,d\tilde\varphi-\int_{|\varphi-\tilde\varphi|\le\frac{1}{2}}A(\varphi-\tilde\varphi)\,v(\tilde\varphi)\,d\tilde\varphi\,,\end{aligned} \tag{1.17}$$

where

$$A(\cdot)=-\chi(\cdot)\,A_1(\cdot)\,-a(\cdot)\,.$$

We have obtained, in particular, the existence of the limit on the last line of (1.17).

It is only left to be shown that the equality in (1.15) holds. Let us recall that, for $u\in L^2(\mathbf{Z})$ and $\varphi\in\mathbf{R}$, $u^\diamond(\varphi)$ denotes the vector $(u(\varphi-j))_{j\in\mathbf{Z}}\in L^2(\mathbf{Z})$ and $Wu(\varphi)=Y_\varphi u^\diamond(\varphi)$. We then have, after a simple change of integration variable,

$$(Su)^\diamond(\varphi)\;=\;\left(\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}\,u(\tilde\varphi-j)\,d\tilde\varphi\right)_{j\in\mathbf{Z}}$$

Since, for a fixed φ and $|\varphi-\tilde\varphi|\le\frac{1}{2}$, $u(\tilde\varphi-j)\neq 0$ only for j in a finite set, this limit exists also in $L^2(\mathbf{Z})$, i. e. ,

$$\begin{aligned}(Su)^\diamond(\varphi)\;&=\;\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}\,u^\diamond(\tilde\varphi)\,d\tilde\varphi\\ &=\;\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}\,Y_{-\tilde\varphi}\,v(\tilde\varphi)\,d\tilde\varphi\,.\end{aligned}$$

This finally implies that

$$\begin{aligned}(WSu)(\varphi)\;&=\;Y_\varphi\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}\,Y_{-\tilde\varphi}\,v(\tilde\varphi)\,d\tilde\varphi\\ &=\;\lim_{\epsilon\searrow 0}\int_{\epsilon\le|\varphi-\tilde\varphi|\le\frac{1}{2}}\frac{\chi(\varphi-\tilde\varphi)}{\varphi-\tilde\varphi}\,Y_{\varphi-\tilde\varphi}\,v(\tilde\varphi)\,d\tilde\varphi\,,\end{aligned}$$

what proves the lemma. q.e.d.

At this point, we are able to calculate $Wb(D)aT_jW^{-1}$ for $b \in \mathbf{CS}(\mathbb{R})$, $a \in \mathbf{CO}(\mathbb{R})$ and $j \in \mathbb{Z}$. We can write :

$$b = \frac{b(+\infty) - b(-\infty)}{2} s + \frac{b(+\infty) + b(-\infty)}{2} + b_\circ , \tag{1.18}$$

with $b_\circ \in \mathbf{CO}(\mathbb{R})$. Defining

$$b_s = \frac{b(+\infty) - b(-\infty)}{2} \quad \text{and} \quad b_1 = \frac{b(+\infty) + b(-\infty)}{2} ,$$

we then have :

$$\begin{aligned} Wb(D)aT_jW^{-1} &= b_sWs(D)aT_jW^{-1} + b_1WaT_jW^{-1} + Wb_\circ(D)aT_jW^{-1} \\ &= (\frac{i}{\pi} b_sWSW^{-1} + b_1)\, WaT_jW^{-1} + W[b_\circ(D) + \omega(D)]aT_jW^{-1} \\ &= (\frac{i}{\pi} b_sH^\diamond + b_1)\, (Y_\varphi a(\varphi - M)Y_{-\varphi-j}) + K . \end{aligned} \tag{1.19}$$

with $K \in \mathcal{L}_{\mathbb{S}^1 \times \mathbb{Z}}$. Here, $H^\diamond$ denotes the bounded operator on $L^2(\mathbb{S}^1; L^2(\mathbb{Z}))$ given by

$$(H^\diamond u)(\varphi) := \lim\nolimits_{\epsilon \searrow 0} \int_{\epsilon \le |\varphi - \tilde\varphi| \le \frac{1}{2}} \pi \cot \pi(\varphi - \tilde\varphi)\, v(\tilde\varphi)\, d\tilde\varphi , \quad v \in \mathbf{C}^\infty(\mathbb{S}^1, \mathcal{L}_{\mathbb{Z}}) .$$

For the second equality in (1.19) we needed (1.13) and (1.14), and for the last equality we used (1.10), (1.15) and two facts: the operator $a(D)b$ is compact for $a \in \mathbf{CO}(\mathbb{R})$ and $b \in \mathbf{CO}(\mathbb{R})$, and the mapping

$$u \in \mathbf{C}^\infty(\mathbb{S}^1; L^2(\mathbb{Z})) \longmapsto \int_{|\varphi - \tilde\varphi| \le \frac{1}{2}} B(\varphi - \tilde\varphi)\, u(\tilde\varphi)\, d\tilde\varphi$$

defines a compact operator on $L^2(\mathbb{S}^1; L^2(\mathbb{Z}))$ if B is a smooth 1-periodic $\mathcal{K}_{\mathbb{Z}}$-valued function. Taking limits in (1.19), we get, for every $G \in \mathcal{F}$,

$$W(b(D)G)W^{-1} = (\frac{i}{\pi} b_sH^\diamond + b_1)\, WGW^{-1} + K , \quad K \in \mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z}} . \tag{1.20}$$

It is straightforward to verify that

$$\lim\nolimits_{\epsilon \searrow 0} \int_{\epsilon \le |\varphi - \tilde\varphi| \le \frac{1}{2}} \pi \cot \pi(\varphi - \tilde\varphi)\, v(\tilde\varphi)\, d\tilde\varphi = [-i \,\mathrm{sgn}(D_\theta) v](\varphi)$$

for $v \in \mathbf{C}^\infty(\mathbb{S}^1)$, with $\mathrm{sgn}(j) = \frac{j}{|j|}$ for $j \neq 0$ and $\mathrm{sgn}(0) = 0$. That means that we have

$$H^\diamond = \frac{\pi}{i} \mathrm{sgn}(D_\theta) \otimes I_{\mathbb{Z}} ,$$

where $I_{\mathbb{Z}}$ denotes the identity operator on $L^2(\mathbb{Z})$. Ommiting "$\otimes I_{\mathbb{Z}}$" from our notation, we get

$$\frac{i}{\pi} b_s H^{\diamond} + b_1 = \frac{1}{\pi} b_s \mathrm{sgn}(D_\theta) + b_1 = b(D_\theta) + b^{\circ}(D_\theta) ,$$

with $b^{\circ}(j) := b_s\mathrm{sgn}(j) + b_1 - b(j)$, and $b(D_\theta)$ meaning $b_{|\mathbb{Z}}(D_\theta)$.

Since $b^{\circ}(D_\theta) \in \mathcal{K}_{\mathbb{S}^1}$, $WGW^{-1} \in \mathbf{C}(\mathbb{S}^1)\bar{\otimes}\mathcal{K}_{\mathbb{Z}}$ and $\mathcal{K}_{\mathbb{S}^1\times\mathbb{Z}} = \mathcal{K}_{\mathbb{S}^1}\bar{\otimes}\mathcal{K}_{\mathbb{Z}}$, we obtain from equation (1.20):

$$Wb(D)GW^{-1} = b(D_\theta)\,(WGW^{-1}) + K , \quad K \in \mathcal{K}_{\mathbb{S}^1\times\mathbb{Z}} . \tag{1.21}$$

This proves (1.9) and the inclusion "$\subseteq$" in (1.8), by (1.10). Given any $G^{\diamond} \in \mathbf{C}(\mathbb{S}^1) \otimes \mathcal{K}_{\mathbb{Z}}$, we can find, by Lemma 1.3 , some $G \in \mathcal{F}$ such that $WGW^{-1} = G^{\diamond}$ and, by (1.21) then:

$$b(D_\theta)G^{\diamond} \in W\hat{\mathcal{E}_A}W^{-1} \quad \text{for all} \quad b \in \mathbf{CS}(\mathbb{R}) \quad \text{and} \quad G^{\diamond} \in \mathbf{C}(\mathbb{S}^1)\bar{\otimes}\mathcal{K}_{\mathbb{Z}} .$$

Since all commutators of $\mathcal{SL}$ are compact and $\mathcal{K}_{\mathbb{S}^1\times\mathbb{Z}} \subset \mathcal{SL}$, this proves the equality in (1.8). q.e.d.

Proof of Lemma 1.3 : The C*-algebra $\mathbf{C}(\mathbb{S}^1)\bar{\otimes}\mathcal{K}_{\mathbb{Z}}$ is postliminal (c. f. [10] , 11.1.8.) and we can then apply the Stone-Weierstrass Theorem for such algebras (c. f. [10] , 10.4.5.). It says that a C*-subalgebra $\mathcal{B}$ of a postliminal C*-algebra $\mathcal{C}$ equals the whole algebra if the "pure states" of $\mathcal{C}$ are separated by $\mathcal{B}$. The pure states of $\mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}}) = \mathbf{C}(\mathbb{S}^1)\bar{\otimes}\mathcal{K}_{\mathbb{Z}}$ are the linear functionals of the form

$$f_{z,v}(A^{\diamond}) = (v, A^{\diamond}(z)v) , \quad A^{\diamond} \in \mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}}) , \tag{1.22}$$

for fixed $z \in \mathbb{S}^1$ and unit-norm $v \in L^2(\mathbb{Z})$ (c. f. [10] , 2.5.2 and 4.1.4, and [25] , Theorem IV.4.14). Accordingly, it is enough to show that, given two functionals of the form (1.22), $f_{z_1,v_1} \neq f_{z_2,v_2}$, we can find $A^{\diamond} \in W\mathcal{F}W^{-1}$ such that $f_{z_1,v_1}(A^{\diamond}) \neq f_{z_2,v_2}(A^{\diamond})$.

We need the two following properties of $W\mathcal{F}W^{-1}$:

(i) For each fixed $e^{2\pi i\varphi} \in \mathbb{S}^1$, $\{WGW^{-1}(\varphi); G \in \mathcal{F}\} = \mathcal{K}_{\mathbb{Z}}$. Indeed, it is easy to find $a \in \mathbf{CO}(\mathbb{R})$ and $j \in \mathbb{Z}$ such that $Y_{-\varphi}[WaT_jW^{-1}](\varphi)Y_\varphi$ equals the operator $(e_k,\cdot)e_l$, given a pair of indices $k, l \in \mathbb{Z}$ and $\{e_k;\ k \in \mathbb{Z}\}$ denoting the canonical basis of $L^2(\mathbb{Z})$. The algebra generated by all operators $(e_k,\cdot)e_l$ is dense in $\mathcal{K}_{\mathbb{Z}}$, what means that the map

$$\begin{array}{rcl} \mathcal{F} & \longrightarrow & \mathcal{K}_{\mathbb{Z}} \\ A & \longmapsto & Y_{-\varphi}WAW^{-1}(\varphi)Y_\varphi \end{array}$$

has dense image and, hence, is onto. (Here we used that a *-homomorphism between C*-algebras has closed image.) Since Y_φ is unitary, our claim (i) follows.

(ii) If $b \in \mathbf{C}(\mathbb{S}^1)$ and $A \in W\mathcal{F}W^{-1}$, we have $bA \in W\mathcal{F}W^{-1}$. Indeed, it is easy to verify that, for A of the form $A = aT_j$, $j \in \mathbb{Z}$, we have:

$$W[b(e^{2\pi i(\cdot)})a(\cdot)T_j]W^{-1} = b\,WAW^{-1} .$$

By definition of $\mathcal{F}$, our claim is true for all $A \in \mathcal{F}$.

In order to prove the separation property, suppose first that $z_1 \neq z_2$ for given $f_{z_1,v_1} \neq f_{z_2,v_2}$. By (i) we can find $A^\diamond \in W\mathcal{F}W^{-1}$ such that $(v_1, A^\diamond(z_1)v_1) \neq 0$. Let b denote a continuous function on $\mathbb{S}^1$ such that $b(z_1) = 1$ and $b(z_2) = 0$. We then have, by (ii), that $bA^\diamond \in W\mathcal{F}W^{-1}$ and

$$f_{z_1,v_1}(bA^\diamond) = (v_1, b(z_1)A^\diamond(z_1)v_1) \neq 0 ,$$

while

$$f_{z_2,v_2}(bA^\diamond) = (v_2, b(z_2)A^\diamond(z_2)v_2) = 0 .$$

Next suppose that $f_{z_0,v_1} \neq f_{z_0,v_2}$. Since v_1 and v_2 both have unit norm, it must then be true that $|(v_1, v_2)| \neq 1$. We therefore must have, for $A^\diamond \in W\mathcal{F}W^{-1}$ such that $A^\diamond(z_0) = (v_1, \cdot)v_1$,

$$f_{z_0,v_1}(A^\diamond) = (v_1, A^\diamond(z_0)v_1) = 1 \neq |(v_1, v_2)| = (v_2, A^\diamond(z_0)v_2) = f_{z_0,v_2}(A^\diamond) .$$

q.e.d.

1.2 Definition of two symbols on $\mathcal{A}$

Let us first find the maximal ideal space $\mathbf{M}_\natural$ of the commutative C*-algebra $\mathcal{A}^\natural$ generated by operators of types (i) and (ii) on page 6 . As for all C*-subalgebras of $\mathbf{CB}(\mathbb{R})$, one gets $\mathbb{R} \subset \mathbf{M}_\natural$, with $t \in \mathbb{R}$ being identified with the multiplicative linear functional $\omega_t(f) = f(t)$, $f \in \mathcal{A}^\natural$. It is easy to see that, for $\theta \in \mathbb{R}$, the limits

$$\lim_{k\to\infty} f(\theta \pm 2\pi k) := f_\theta^\pm$$

exist for every $f \in \mathcal{A}^\natural$ and that $f^\pm_{\theta+2\pi j} = f^\pm_\theta$, for $j \in \mathbb{Z}$. To each point $e^{i\theta} \in \mathbb{S}^1$, we can therefore associate the multiplicative linear functionals $\omega_{\theta,\pm}(f) = f^\pm_\theta$, so that we get two disjoint copies of $\mathbb{S}^1$, that are denoted by $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$, contained in $\mathbf{M}_\natural$.

Proposition 1.5 *The functionals ω_t and $\omega_{\theta,\pm}$, $t, \theta \in \mathbb{R}$, defined above are all the multiplicative linear functionals on $\mathcal{A}^\natural$, i. e. ,*

$$\mathbf{M}_\natural = \mathbb{R} \cup \mathbb{S}^1_- \cup \mathbb{S}^1_+ \ . \tag{1.23}$$

Moreover, $\mathbb{R}$ is dense in $\mathbf{M}_\natural$ endowed with its weak-topology.*

Proof: Suppose that ω is a multiplicative linear functional on $\mathcal{A}^\natural$ and let $s \in \mathbf{CS}(\mathbb{R})$ denote the function $s(t) = t(1+t^2)^{-1/2}$. As $-1 \leq s \leq 1$, it follows that $-1 \leq \omega(s) \leq 1$. We will prove that, if $|\omega(s)| < 1$, $\omega = \omega_a$, where $a \in \mathbb{R}$ solves $s(a) = \omega(s)$. Indeed, as the polynomials in s are dense in $\mathbf{CS}(\mathbb{R})$, ω and ω_a coincide on $\mathbf{CS}(\mathbb{R})$. Furthermore, if $\chi \in \mathbf{CO}(\mathbb{R})$ satisfies $\chi(a) = 1$, we get $\chi e^{i(\cdot)} \in \mathbf{CS}(\mathbb{R})$ and then

$$e^{ia} = \omega(\chi e^{i(\cdot)}) = \omega(\chi)\omega(e^{i(\cdot)}) = \omega(e^{i(\cdot)}) \ ,$$

which proves that ω and ω_a coincide on a set of generators of $\mathcal{A}^\natural$. For the case $\omega(s) = \pm 1$, let θ solve $\omega(e^{i(\cdot)}) = e^{i\theta}$. We can easily see that $\omega(s) = \omega_{\theta,\pm}(s)$ and $\omega(e^{i(\cdot)}) = \omega_{\theta,\pm}(e^{i(\cdot)})$. Hence: $\omega = \omega_{\theta,\pm}$, what proves (1.23).

We may obtain a neighborhood basis at each point of $\mathbb{S}^1_\pm$ and verify that $\mathbb{R}$ is dense in $\mathbf{M}_\natural$. q.e.d.

Let $\mathcal{A}^\natural_\circ$ denote the algebra (finitely) generated by $\mathbf{CS}(\mathbb{R})$, and let $\mathbf{P}_{2\pi}$ denote the 2π-periodic continuous functions. Let us fix $\chi_\pm \in \mathbf{CS}(\mathbb{R})$ such that $\chi_\pm(\pm\infty) = 1$, $\chi_+ + \chi_- = 1$ and $\chi_\pm(t) = 0$ if $\mp t > 1$. It is easy to see that every $a \in \mathcal{A}^\natural_\circ$ is of the form

$$a = a_+\chi_+ + a_-\chi_- + a_\circ \ , \quad a_\pm \in \mathbf{P}_{2\pi} \ , \quad a_\circ \in \mathbf{CO}(\mathbb{R}) \ , \tag{1.24}$$

where the choice of a_+, a_- and $a_\circ$ is unique. It follows then that $\mathcal{A}^\natural_\circ$ is closed and, hence, $\mathcal{A}^\natural = \mathcal{A}^\natural_\circ$. We have proven:

Proposition 1.6 *Given $a \in \mathcal{A}^\natural$ and fixed χ_+ and χ_- as above, there are uniquely determined $a_\circ \in \mathbf{CO}(\mathbb{R})$ and $a_\pm \in \mathbf{P}_{2\pi}$ such that (1.24) holds.*

The Gelfand map gives us a unique extension $\tilde{a}$ of $a \in \mathcal{A}^\natural$ to $\mathbf{C}(\mathbf{M}_\natural)$ by the assignment $\tilde{a}(\omega) := \omega(a)$, $\omega \in \mathbf{M}_\natural$. Using the descriptions of $\mathcal{A}^\natural$ and $\mathbf{M}_\natural$ given in (1.24) and (1.23), we obtain:

$$\tilde{a}(t) = a(t) \ , \quad t \in \mathbb{R} \quad \text{and} \quad \tilde{a}(e^{i\theta}_\pm) = a_\pm(\theta) \ , \quad e^{i\theta}_\pm \in \mathbb{S}^1_\pm \ . \tag{1.25}$$

Theorem 1.7 *The symbol space $\mathbf{M}_A$ of the algebra $\mathcal{A}$ can be given as*

$$\mathbf{M}_A = \mathbf{M}_\natural \times \{-\infty, +\infty\} \ ,$$

with $\mathbf{M}_\natural$ as in (1.23). The σ-symbol is given by:

$$\sigma_a(m, \pm\infty) = \tilde{a}(m) \ , \quad m \in \mathbf{M}_\natural \ , \quad a \in \mathcal{A}^\natural \ ,$$

with $\tilde{a}$ as in (1.25), and

$$\sigma_{b(D)}(m, \pm\infty) = b(\pm\infty) \ , \quad m \in \mathbf{M}_\natural \ , \quad b \in \mathbf{CS}(\mathbb{R}) \ .$$

Proof: We are going to use an argument known as Herman's Lemma (c. f. [4]). Let us consider the two following canonically defined inclusions:

$$i_1 : \mathcal{A}^\natural \longrightarrow \frac{\mathcal{A}}{\mathcal{E}_A}$$

and

$$i_2 : \mathbf{CS}(\mathbb{R}) \longrightarrow \frac{\mathcal{A}}{\mathcal{E}_A}$$

Let ι denote the product of the dual-maps of i_1 and i_2 :

$$\iota : \mathbf{M}_A \longrightarrow \mathbf{M}_\natural \times [-\infty, +\infty] \ ,$$

with a multiplicative linear functional ω on $\mathcal{A}/\mathcal{E}_\mathcal{A}$ being taken to

$$\iota(\omega) = (\omega \circ i_1, \omega \circ i_2) \ .$$

It is easy to see that ι is an injective map (the images of i_1 and i_2 generate $\mathcal{A}/\mathcal{E}_\mathcal{A}$) and, hence, a homeomorphism onto its image ($\mathbf{M}_A$ is compact). Using this homeomorphism as an identification, it is clear that

$$\sigma_a(m,\xi) = \tilde{a}(m) \quad \text{and} \quad \sigma_{b(D)}(m,\xi) = b(\xi) \ ,$$

for $a \in \mathcal{A}^\natural$, $b \in \mathbf{CS}(\mathbb{R})$ and $(m,\xi) \in \mathbf{M}_A \subset \mathbf{M}_\natural \times [-\infty,+\infty]$. The "cosphere-bundle" of a manifold is always contained in the symbol-spaces of its comparison algebras (c. f. [2], Theorems VII-1-5 and VI-2-2). In particular, we have $\mathbb{R} \times \{-\infty,+\infty\}$ contained in $\mathbf{M}_A$ and, thus,

$$\mathbf{M}_\natural \times \{-\infty,+\infty\} \subset \mathbf{M}_A \ ,$$

since $\mathbb{R}$ is dense in $\mathbf{M}_\natural$ (Proposition 1.5).

Now we need only to prove that, if $|\,\xi\,| \neq \infty$, $(m,\xi) \notin \text{image}\,\iota = \mathbf{M}_A$. Given $\xi_\circ \in \mathbb{R}$, we can choose $a \in \mathbf{CO}(\mathbb{R})$ such that $a(\xi_\circ) \neq 0$. It follows from Proposition 1.1 that $a(D) \in \mathcal{E}_\mathcal{A}$. We therefore must have

$$\sigma_{a(D)}(m,\xi) = a(\xi) = 0 \quad \text{for all} \quad (m,\xi) \in \mathbf{M}_A$$

and, hence, $(m,\xi_\circ) \notin \mathbf{M}_A$, $m \in \mathbf{M}_\natural$. q.e.d.

Next we define an operator-valued symbol $\mathcal{A}$, the "γ-symbol", which arises from the description of the quotient $\mathcal{E}_\mathcal{A}/\mathcal{K}_\mathbb{R}$ as an operator-valued function algebra.

We have proven in Theorem 1.2 that $\mathcal{E}_\mathcal{A}$ is *-isomorphic to $\mathcal{SL}\bar{\otimes}\mathcal{K}_\mathbb{Z}$. It is well known and it follows, for example, from [2], Theorem VI-2-2, that $\mathcal{SL}/\mathcal{K}_{\mathbb{S}^1}$ is *-isomorphic to the complex-valued continuous functions on $\mathbf{M}_{SL} = \mathbb{S}^1 \times \{-1,+1\}$. The isomorphism is given by $A \mapsto \sigma_A^{SL}$, where :

$$\begin{aligned} \sigma_a^{SL}(e^{2\pi i\varphi},\pm 1) &= a(e^{2\pi i\varphi}) \ , \quad a \in \mathbf{C}^\infty(\mathbb{S}^1) \quad \text{and} \\ \sigma_{b(D_\theta)}^{SL}(e^{2\pi i\varphi},\pm 1) &= b(\pm\infty) \ , \quad b \in \mathbf{CS}(\mathbb{Z}) \ . \end{aligned}$$

It follows now from [1] that

$$A \otimes K \longmapsto \sigma_A^{SL} K \tag{1.26}$$

extends to a *-isomorphism

$$\frac{\mathcal{SL}\bar{\otimes}\mathcal{K}_\mathbb{Z}}{\mathcal{K}_{\mathbb{S}^1\times\mathbb{Z}}} \longrightarrow \mathbf{C}(\mathbf{M}_{SL},\mathcal{K}_\mathbb{Z}) \ .$$

Proposition 1.8 *There is a *-isomorphism*

$$\Psi : \frac{\mathcal{E}_{\mathcal{A}}}{\mathcal{K}_{\mathbf{R}}} \longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}})$$

such that; if $\tilde{\gamma}$ *denotes the composition of* Ψ *with the canonical projection* $\mathcal{E}_{\mathcal{A}} \to \mathcal{E}_{\mathcal{A}}/\mathcal{K}_{\mathbf{Z}}$, *and* $E \in \mathcal{E}_{\mathcal{A}}$ *satisfies* $F^{-1}EF = b(D)aT_j$, $b \in \mathbf{CS}(\mathbf{R})$, $a \in \mathbf{CO}(\mathbf{R})$, $j \in \mathbf{Z}$; *we have :*

$$\tilde{\gamma}_E(\varphi, \pm 1) = b(\pm\infty)\, Y_\varphi\, a(\varphi - M)\, Y_{-\varphi-j} \ , \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL} \ .$$

Proof: Let us define Ψ by :

$$\frac{\mathcal{E}_{\mathcal{A}}}{\mathcal{K}_{\mathbf{R}}} \longrightarrow \frac{\hat{\mathcal{E}}_{\mathcal{A}}}{\mathcal{K}_{\mathbf{R}}} \longrightarrow \frac{\mathcal{SL}\bar{\otimes}\mathcal{K}_{\mathbf{Z}}}{\mathcal{K}_{\mathbf{S}^1\times\mathbf{Z}}} \longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}}) \ ,$$

where, in the first step, we take $A + \mathcal{K}_{\mathbf{R}}$ to $F^{-1}AF + \mathcal{K}_{\mathbf{R}}$, next to $WF^{-1}AFW^{-1} + \mathcal{K}_{\mathbf{S}^1\times\mathbf{Z}}$ and, last, we use the isomorphism in (1.26). By Theorem 1.2, equation (1.11) and the observations before the statement of this proposition, Ψ has the desired properties. q.e.d.

We will now extend $\tilde{\gamma}$, defined on $\mathcal{E}_{\mathcal{A}}$, to the whole algebra $\mathcal{A}$. Since $\mathcal{E}_{\mathcal{A}}/\mathcal{K}_{\mathbf{R}}$ is an ideal of $\mathcal{A}/\mathcal{K}_{\mathbf{R}}$, every $A \in \mathcal{A}$ defines an operator in $\mathcal{L}(\mathcal{E}_{\mathcal{A}}/\mathcal{K}_{\mathbf{R}})$ by the assignment

$$E + \mathcal{K}_{\mathbf{R}} \longmapsto AE + \mathcal{K}_{\mathbf{R}}\cdot,$$

thus defining :

$$T : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{E}_{\mathcal{A}}/\mathcal{K}_{\mathbf{R}}) \ .$$

It is clear that $\| T_A \| \leq \| A \|$. Let us define

$$\begin{aligned} \gamma : \ \mathcal{A} &\longrightarrow \mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}})) \\ A &\longmapsto \Psi\, T_A\, \Psi^{-1} \end{aligned} \tag{1.27}$$

For $E \in \mathcal{E}_{\mathcal{A}}$, γ_E is the multiplication by $\tilde{\gamma}_E \in \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}})$ of Proposition 1.8. Identifying functions of $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}})$ with the corresponding multiplication operator in $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}}))$, we can then say that γ extends $\tilde{\gamma}$.

Proposition 1.9 *There is a *-homomorphism*

$$\gamma : \mathcal{A} \longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}})$$

extending $\tilde{\gamma}$ *of Proposition 1.8 . On the generators of* $\mathcal{A}$, γ *is given by (with* $\mathbf{M}_{SL} = \{e^{2\pi i\varphi}; \varphi \in \mathbf{R}\} \times \{+1, -1\}$ *):*

$$\begin{aligned} &\gamma_a(\varphi, \pm 1) = a(\pm\infty) \ , \quad a \in \mathbf{CS}(\mathbf{R}) \ , \\ &\gamma_{b(D)}(\varphi, \pm 1) = Y_\varphi\, b(M - \varphi)\, Y_{-\varphi} \ , \quad b \in \mathbf{CS}(\mathbf{R}) \ , \quad \textit{and} \\ &\gamma_{e^{ij(\cdot)}}(\varphi, \pm 1) = Y_{-j} \ , \quad j \in \mathbf{Z} \ . \end{aligned} \tag{1.28}$$

Proof: It is enough to prove (1.28) for the map γ defined in (1.27). By continuity, the image of γ will then be contained in $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z}})$ regarded as a closed subalgebra of $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z}}))$.

Given $a \in \mathbf{CS}(\mathbb{R})$, we need to calculate $\tilde{\gamma}_{aE}$ in terms of $\tilde{\gamma}_E$, for $E \in \mathcal{E}_{\mathcal{A}}$. By Proposition 1.1 , it is enough to consider E such that

$$F^{-1}EF = d(D)cT_k\ , \quad d \in \mathbf{CS}(\mathbb{R})\ , \quad c \in \mathbf{CO}(\mathbb{R})\ , \quad \text{and} \quad k \in \mathbb{Z}\ .$$

We get

$$F^{-1}\,(aE)\,F\ =\ (ad)(D)\,c\,T_j$$

and therefore, by Proposition 1.8 ,

$$\tilde{\gamma}_{aE}(\varphi, \pm 1)\ =\ a(\pm\infty)\,d(\pm\infty)\,Y_\varphi\,c(\varphi - M)\,Y_{-\varphi-j}\ =\ a(\pm\infty)\,\tilde{\gamma}_E(\varphi, \pm 1)\ .$$

For E as above and $b \in \mathbf{CS}(\mathbb{R})$,

$$F^{-1}(b(D)E)F = b(-(\cdot))d(D)cT_j = d(D)\,b(-(\cdot))c\,T_j + K\ , \quad K \in \mathcal{K}_{\mathbb{R}}\ ,$$

and, hence,

$$\tilde{\gamma}_{aE}(\varphi, \pm 1)\ =\ d(\pm\infty)\,Y_\varphi\,b(-\varphi + M)\,c(\varphi - M)\,Y_{-\varphi-j}\ =\ Y_\varphi\,b(M - \varphi)\,Y_{-\varphi}\,\tilde{\gamma}_E(\varphi, \pm 1)\ ,$$

what proves the second equation in (1.28).

For that same E we have

$$F^{-1}e^{ij(\cdot)}EF = T_jd(D)cT_k = d(D)c(\cdot + j)T_{j+k}$$

and, then, using that $Y_{-j}c(\varphi - M) = c(\varphi - M + j)Y_{-j}$,

$$\begin{aligned}
\tilde{\gamma}_{e^{ij(\cdot)}E}(\varphi, \pm 1)\ &=\ d(\pm\infty)\,Y_\varphi\,c(\varphi - M + j)\,Y_{-\varphi-j-k}\\
&=\ Y_{-j}\,d(\pm\infty)\,Y_\varphi\,c(\varphi - M)\,Y_{-\varphi-k}\\
&=\ Y_{-j}\,\tilde{\gamma}_E(\varphi - M)\ .
\end{aligned}$$

This finishes the proof. q.e.d.

Proposition 1.10 *The map γ of Proposition 1.9 restricted to the closed sub-algebra $\mathcal{A}^\diamond \subset \mathcal{A}$ generated by the operators of type (ii) and (iii) on page 6 is an isometry.*

Proof: Analogously to (1.10), it is easy to show that

$$W\,bT_j\,W^{-1} \;=\; Y_\varphi\, b(\varphi - M)\, Y_{-\varphi-j}\;,$$

for $b \in \mathbf{CS}(\mathbb{R})$ and $j \in \mathbb{Z}$. It follows by (1.28) that

$$\gamma_{b(D)e^{ij\cdot}} \;=\; Y_\varphi\, b(M-\varphi)\, Y_{-\varphi-j} \;=\; Wb(-(\cdot))T_jW^{-1} \;=\; WF^{-1}b(D)\, e^{ij\cdot}\, FW^{-1}\;,$$

what proves that

$$\gamma_A \;=\; (WF^{-1})\, A\, (WF^{-1})^{-1}\;,$$

for all $A \in \mathcal{A}^\diamond$. q.e.d.

To end this section, we describe how the kernels of the two symbols we have defined relate. Let us denote by $\mathbf{W}_A$ the set $\mathbb{R} \times \{-\infty, +\infty\}$, which is dense in $\mathbf{M}_A$, by Proposition 1.5 .

Proposition 1.11 *For every $A \in \mathcal{A}$, $\| \sigma_A|_{\mathbf{M}_A \setminus \mathbf{W}_A} \| \leq \| \gamma_A \|$, i. e.* ,

$$\sup\{\ |\sigma_A(m,\xi)|;\ m \in \mathbb{S}^1_- \cup \mathbb{S}^1_+\ ,\ \xi \in \{-\infty,+\infty\}\ \} \leq \sup\{\ \|\ \gamma_A(e)\ \|_{\mathcal{L}_{\mathbb{Z}}};\ e \in \mathbf{M}_{SL}\ \} \quad (1.29)$$

Proof: The set of all operators of the form

$$\sum_{j=-N}^{N} a_j\, b_j(D) e^{ij(\cdot)} \;+\; K\;, \quad N \in \mathbb{N},\ \ a_j \in \mathbf{CS}(\mathbb{R}),\ \ b_j \in \mathbf{CS}(\mathbb{R}),\ \ K \in \mathcal{K}_{\mathbb{R}}\;, \qquad (1.30)$$

is dense in $\mathcal{A}$. Indeed, by (1.1) and by the Fourier-transform conjugate of (1.2), the commutator of operators of type (1.30) is still of type (1.30). Since $\mathcal{K}_{\mathbb{R}} \subset \mathcal{E}_{\mathcal{A}} = \ker\sigma$ and also, by definition of γ, $\mathcal{K}_{\mathbb{R}} \subset \ker\gamma$, it is enough to prove (1.29) for A of type (1.30) with $K = 0$.

For such an A, we define

$$A^{\pm} = \sum_{j=-N}^{N} a_j(\pm\infty)\, b_j(D)\, e^{ij(\cdot)}\;.$$

We then have:

$$\sigma_A(m, \pm\infty) \;=\; \sigma_{A^+}(m, \pm\infty)\;, \quad m \in \mathbb{S}^1_+$$

and

$$\sigma_A(m, \pm\infty) \;=\; \sigma_{A^-}(m, \pm\infty)\;, \quad m \in \mathbb{S}^1_-\;.$$

Here we are using Theorem 1.7 and $\tilde{a}_j(m) = a_j(\pm\infty)$ for $m \in \mathbb{S}^1_+ \cup \mathbb{S}^1_-$. Again by Theorem 1.7 and using that the extension to $\mathbb{S}^1_+ \cup \mathbb{S}^1_- \subset \mathbf{M}_\natural$ of $e^{ij(\cdot)}$ is obtained by assigning to both $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$ the function $e^{ij\theta}$, we get :

$$\sigma_{A^+}(m, \pm\infty) = \sum_{j=-N}^{N} a_j(+\infty)\, b_j(\pm\infty)\, e^{ij\theta} \tag{1.31}$$

and

$$\sigma_{A^-}(m, \pm\infty) = \sum_{j=-N}^{N} a_j(-\infty)\, b_j(\pm\infty)\, e^{ij\theta} \,. \tag{1.32}$$

(Formulas (1.31) and (1.32) hold for $m = e_+^{ij\theta} \in \mathbb{S}^1_+$ and for $m = e_-^{ij\theta} \in \mathbb{S}^1_-$.)

We then have

$$\begin{aligned} \| \sigma|_{\mathbf{M}_A \setminus \mathbf{W}_A} \| &= \max_{+\text{or}-} \sup\{ \, |\sigma_A(m, \pm\infty)| \, , \; m \in \mathbb{S}^1_+ \cup \mathbb{S}^1_- \, \} \\ &= \max\{ \| \sigma_{A^+}(\cdot, +\infty) \|, \| \sigma_{A^+}(\cdot, -\infty) \|, \| \sigma_{A^-}(\cdot, +\infty) \|, \| \sigma_{A^-}(\cdot, -\infty) \| \} \\ &= \max\{ \, \| \sigma_{A^+} |_{\mathbf{M}_A \setminus \mathbf{W}_A} \| \, , \; \| \sigma_{A^-} |_{\mathbf{M}_A \setminus \mathbf{W}_A} \| \, \} \\ &= \max\{ \, \| \sigma_{A^+} \|_{L^\infty(\mathbf{M}_A)} \, , \; \| \sigma_{A^-} \|_{L^\infty(\mathbf{M}_A)} \, \} \, . \end{aligned} \tag{1.33}$$

The norms on the second line of (1.33) are $L^\infty(\mathbb{S}^1)$-norms and in the last equality we use that $\sigma_{A^+}(t, \pm\infty) = \sigma_{A^-}(m, \pm\infty)$ for $t \in \mathbf{R}$ and $m = e^{ijt} \in \mathbb{S}^1_+ \cup \mathbb{S}^1_-$ and the analogous fact for σ_{A^-}.

Since σ is defined as the composition of the Gelfand map $\mathcal{A}/\mathcal{E}_A \to \mathbf{C}(\mathbf{M}_A)$ with the canonical projection $\mathcal{A} \to \mathcal{A}/\mathcal{E}_A$, we have

$$\| \sigma_{A^\pm} \|_{L^\infty(\mathbf{M}_A)} \leq \| A^\pm \| \quad . \tag{1.34}$$

By Proposition 1.10 ,

$$\| A^\pm \| = \| \gamma_{A^\pm} \| \quad . \tag{1.35}$$

From Proposition 1.9 , it follows that

$$\gamma_A(\varphi, +1) = \gamma_{A^+}(\varphi, +1) = \gamma_{A^+}(\varphi, -1)$$

and

$$\gamma_A(\varphi, -1) = \gamma_{A^-}(\varphi, +1) = \gamma_{A^-}(\varphi, -1) \,,$$

for $(e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}$. Hence:

$$\| \gamma_A \| = \max\{ \| \gamma_{A^+} \| , \| \gamma_{A^-} \| \} \,. \tag{1.36}$$

Inequality (1.29) follows from (1.33), (1.34), (1.35) and (1.36). q.e.d.

Proposition 1.12 *An operator $A \in \mathcal{A}$ is in the kernel of γ if and only if σ_A vanishes on $\mathbf{M}_A \backslash \mathbf{W}_A$. Furthermore, we have :*

$$\ker \sigma \cap \ker \gamma = \mathcal{K}_{\mathbb{R}} . \tag{1.37}$$

Proof: Let $\mathcal{J}_\circ$ denote the C*-algebra generated by $a \in \mathbf{CO}(\mathbb{R})$ and by $a \,.\, b(D)$, $a \in \mathbf{CO}(\mathbb{R})$ and $b \in \mathbf{CS}(\mathbb{R})$. It is clear from Proposition 1.9 that $\mathcal{J}_\circ \subseteq \ker \gamma$. Using the nomenclature of [2] , $\mathcal{J}_\circ$ is the minimal comparison algebra associated with the triple $\{\mathbb{R},\ dt,\ -\triangle + 1\}$. It can be easily conclude from [2] , Lemma VII-1-2, that $A \in \mathcal{C}_{\mathcal{P}}$ belongs to $\mathcal{J}_\circ$ if and only if σ_A vanishes on $\mathbf{M}_A \backslash \mathbf{W}_A$. This shows, by Proposition 1.11 , that $\mathcal{J}_\circ \supseteq \ker \gamma$.

Since $\ker \sigma = \mathcal{E}_{\mathcal{A}}$ and $\ker \gamma = \mathcal{J}_\circ$, (1.37) follows from [2], TheoremVII-1-3. q.e.d.

1.3 A Fredholm criterion and an improved γ-symbol

Let us denote by $L^2(\mathbb{R})^N$, $N \geq 1$, the Hilbert space $L^2(\mathbb{R};\mathbb{C}^N)$ and $\mathcal{A}^N$ the C*-subalgebra of $\mathcal{L}(L^2(\mathbb{R})^N)$ consisting of the $N \times N$-matrices whose entries are operators in $\mathcal{A}$. In this section we give a necessary and sufficient condition for operators in $\mathcal{A}^N$ to be Fredholm. There are natural extensions of the two symbols defined for the case $N = 1$:

$$\sigma_A^N = ((\sigma_{A_{jk}}))_{1\leq j,k\leq N} \quad \text{and} \quad \gamma_A^N = ((\gamma_{A_{jk}}))_{1\leq j,k\leq N} \,,$$

where $A = ((A_{jk}))_{1\leq j,k\leq N} \in \mathcal{A}^N$.

The following proposition follows immediately from Proposition 1.12 and the fact that the compact ideal of $\mathcal{L}(L^2(\mathbb{R})^N)$ coincides with the set of the $N \times N$-matrices whose entries are in $\mathcal{K}_{\mathbb{R}}$.

Proposition 1.13 *The kernel of γ^N consists precisely of those $A \in \mathcal{A}$ such that σ_A^N vanish on $\mathbf{M}_A \backslash \mathbf{W}_A$. Furthermore, we have :*

$$\ker \sigma^N \cap \ker \gamma^N = \mathcal{K}(L^2(\mathbb{R})^N) \tag{1.38}$$

Theorem 1.14 *An operator $A = ((A_{jk}))_{1\leq j,k\leq N} \in \mathcal{A}$ is Fredholm if and only if:*

(i) σ_A^N is invertible, i. e. , for all $m \in \mathbf{M}_A$, the $N \times N$-matrix $((\sigma_{A_{jk}}(m)))$ is invertible; and

(ii) γ_A^N is invertible, i. e. , for all $m \in \mathbf{M}_A$, the $N \times N$-matrix $((\gamma_{A_{jk}}(m)))$ with entries in $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z}})$ is invertible.

Proof: Suppose that A is Fredholm and let B be such that $1-AB$ and $1-BA$ are compact. We have $B \in \mathcal{A}^N$, since $\mathcal{A}^N/\mathcal{K}(L^2(\mathbb{R})^N)$ is a C*-subalgebra of $\mathcal{L}(L^2(\mathbb{R})^N)/\mathcal{K}(L^2(\mathbb{R})^N)$. We then get

$$\sigma_{1-AB}^N = \sigma_{1-BA}^N = 0 \quad \text{and} \quad \gamma_{1-AB}^N = \gamma_{1-BA}^N = 0$$

and, hence,

$$1 = \sigma_A^N \sigma_B^N = \sigma_B^N \sigma_A^N \quad \text{and} \quad 1 = \gamma_A^N \gamma_B^N = \gamma_B^N \gamma_A^N \,.$$

Conversely, suppose that (i) and (ii) above are satisfied. Since

$$\gamma^N : \mathcal{A}^N \longrightarrow \mathbf{C}(\mathbf{M}_{SL},\ N \times N\text{-matrices with entries in } \mathcal{L}_{\mathbb{Z}}\,)$$

is a *-homomorphism (by Proposition 1.9), its range is a C*-algebra. There must be then a $B \in \mathcal{A}^N$ such that $\gamma^N_B = (\gamma^N_A)^{-1}$. Since $1 - AB \in \ker \gamma^N$, $1 - \sigma^N_A \sigma^N_B$ vanishes on $\mathbf{M}_A \backslash \mathbf{W}_A$, by Proposition 1.13. As the map σ is surjective, so is σ^N. An operator $Q \in \mathcal{A}^N$ can therefore be found such that its symbol σ^N_Q equals the continuous function vanishing on $\mathbf{M}_A \backslash \mathbf{W_A}$

$$(\sigma^N_A)^{-1} - \sigma^N_B \ .$$

By Proposition 1.13 again, $Q \in \ker \gamma^N$ and, then,

$$\gamma^N_{1-A(B+Q)} = \gamma^N_{1-(B+Q)A} = 0 \ .$$

Since we also have

$$\sigma^N_{1-A(B+Q)} = 1 - \sigma^N_A \sigma^N_B - \sigma^N_A \sigma^N_Q = 0 = \sigma^N_{1-(B+Q)A} \ ,$$

the operator $B + Q$ is an inverse for A, modulo a compact operator, by (1.38). q.e.d.

It will prove useful for applications to differential operators to redefine the γ-symbol by conjugating it with the discrete Fourier transform F_d. Accordingly, let

$$\Gamma : \mathcal{A} \longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{S}^1})$$

be defined by

$$\Gamma_A(m) := F_d^{-1} \gamma_A(m) F_d \ , \quad m \in \mathbf{M}_{SL} \ . \tag{1.39}$$

It follows from (1.28) and (1.6) that the symbols of the generators of $\mathcal{A}$ are given by (with $\mathbf{M}_{SL} = \{e^{2\pi i \varphi}; \varphi \in \mathbb{R}\} \times \{-1, +1\}$) :

$$\begin{aligned} \Gamma_a(\varphi, \pm 1) &= a(\pm\infty) \ , \quad a \in \mathbf{CS}(\mathbb{R}) \\ \Gamma_{b(D)}(\varphi, \pm 1) &= e^{-i\varphi\theta} b(D_\theta - \varphi) e^{i\varphi\theta} \ , \quad b \in \mathbf{CS}(\mathbb{R}) \\ \Gamma_{e^{ij(\cdot)}}(\varphi, \pm 1) &= e^{ij\theta} \ , \quad j \in \mathbb{Z} \ . \end{aligned} \tag{1.40}$$

(See page 8 for the definition of $b(D_\theta)$.)

By continuity of Γ, we get, for $a \in \mathcal{A}^\natural$ of the form

$$a = a_+ \chi_+ + a_- \chi_- + a_\circ$$

(with the notation of Proposition 1.6),

$$\Gamma_a(\varphi, \pm 1) = a_\pm(\theta) \ . \tag{1.41}$$

Here we regard a_+ and a_- as defined on $\mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbb{R}\}$.

Remark 1.15 *Because of (1.39), it is obvious that condition (ii) of Theorem 1.14 can be replaced by :*

(*ii'*) $\Gamma_A^N(m) := ((\Gamma_{A_{jk}}(m)))_{1\leq j,k\leq N}$ *is invertible for all* $m \in \mathbf{M}_{SL}$.

Now we consider the differential operator

$$\begin{aligned} L : H^M(\mathbb{R},\mathbb{C}^N) &\longrightarrow L^2(\mathbb{R},\mathbb{C}^N) , \\ L &= \sum_{j=0}^{M} A_j(t)\,(\frac{1}{i}\frac{d}{dt})^j , \end{aligned} \tag{1.42}$$

where A_j, $j = 0,\ldots,M$, are $N \times N$-matrices with entries in $\mathcal{A}^\natural$ and H^M denotes the M-th Sobolev space. We denote by $\Lambda_{\mathbb{R}}$ the operator $(1+D^2)^{-1/2} \in \mathcal{L}_{\mathbb{R}}$ and also the matrix in $\mathcal{L}(L^2(\mathbb{R},\mathbb{C}^N))$ consisting of $\Lambda_{\mathbb{R}}$ on the diagonal and zero elsewhere. It is clear that

$$L\Lambda_{\mathbb{R}}^M = \sum_{j=0}^{M} A_j(t)\, s(D)^j\, \Lambda_{\mathbb{R}}^{M-j} \in \mathcal{A}^N .$$

Since $\Lambda_{\mathbb{R}}^M$ is an isomorphism from $L^2(\mathbb{R},\mathbb{C}^N)$ onto $H^M(\mathbb{R},\mathbb{C}^N)$, the operator L in (1.42) is a Fredholm operator if and only if $L\Lambda_{\mathbb{R}}^M$ is Fredholm as a bounded operator on $L^2(\mathbb{R},\mathbb{C}^N)$.

Since the entries of A_j are in $\mathcal{A}^\natural$, we have :

$$A_j = \chi_+ A_j^+ + \chi_- A_j^+ + A_j^\circ .$$

Here, χ_+ and χ_- are as in (1.24), $A_j^\pm$ are 2π-periodic and $A_j^\circ(\pm\infty) = 0$. Regarding $A_j^\pm$ as defined on $\mathbb{S}^1 = \{e^{i\theta};\theta \in \mathbb{R}\}$, we get :

Theorem 1.16 *The operator* L *in (1.42) is Fredholm if and only if:*

(*i*) L *is uniformly elliptic, i. e. ,* $A_M(t)$ *is invertible for all* $t \in \mathbb{R}$ *and* $\| A_M(t) \|^{-1}$ *is bounded, and*

(*ii*) *the differential operators*

$$\begin{aligned} L^\pm(\varphi) : H^M(\mathbb{S}^1,\mathbb{C}^N) &\longrightarrow L^2(\mathbb{S}^1,\mathbb{C}^N) , \\ L^\pm(\varphi) &= \sum_{j=0}^{M} A_j^\pm(\theta)\,(\frac{1}{i}\frac{d}{d\theta} - \varphi)^j , \end{aligned}$$

are invertible for every $\varphi \in [0,1]$.

Proof: We apply Theorem 1.14 with Remark 1.15 for the operator $A := L\Lambda^M$. For condition (*i*) of Theorem 1.14 to hold, it is necessary and sufficient that σ_A have a bounded inverse on $\mathbf{R} \times \{-\infty, +\infty\}$, which is a dense subset of $\mathbf{M}_A$ (see Proposition 1.5). By Theorem 1.23, we have $\sigma_A(t, \pm\infty) = \pm A_M(t)$ and we have therefore obtained equivalence of condition (i) of this theorem and condition (i) of Theorem 1.14 .

By (1.40) and (1.41), we get:

$$e^{i\varphi\theta}\,\Gamma_A^N(\varphi, \pm 1)\,e^{-i\varphi\theta} = \sum_{j=0}^{M} A_j^{\pm}(\theta)\,(D_\theta - \varphi)^j\,\lambda(D_\theta - \varphi)\ , \tag{1.43}$$

where $\lambda(s) = (1+s^2)^{-1/2}$. As, for each $\varphi \in \mathbf{R}$, $\lambda(D_\theta - \varphi)$ is an isomorphism from $L^2(\mathbb{S}^1, \mathbb{C}^N)$ onto $H^M(\mathbb{S}^1, \mathbb{C}^N)$, the equivalence of conditions (ii) of this theorem and (ii') of Remark 1.15 has been established. q.e.d.

1.4 An index formula for first-order systems

For the case of first-order operators of the type (1.42), we can give a more explicit Fredholm criterion and even find a formula for the index of L, defined as

$$\text{index } L = \text{dimension kernel } L - \text{ co-dimension range } L \ .$$

Without loss of generality, we can assume that L is of the form

$$L = \frac{1}{i}\frac{d}{dt} + A(t): \ H^1(\mathbb{R}, \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}, \mathbb{C}^N) \ , \tag{1.44}$$

where A is an $N \times N$-matrix with entries in $\mathcal{A}^\natural$. Indeed, condition (i) of Theorem 1.16 implies that, if L is Fredholm, multiplication by the inverse of the coefficient of its first-order term is a bounded operator. Multiplying $L\Lambda_{\mathbb{R}}$ by an invertible operator does not change its being or not Fredholm, or its index. Theorem 1.16 applied to L as in (1.44) has the following consequence.

Proposition 1.17 *Let $\Phi^{\pm}(\theta)$ denote the $N \times N$-matrix-valued functions that solve the Cauchy-problems:*

$$\begin{cases} \frac{1}{i}\frac{d\Phi^{\pm}}{d\theta}(\theta) + A^{\pm}(\theta)\Phi^{\pm}(\theta) = 0 \\ \Phi^{\pm}(0) = I \end{cases} , \tag{1.45}$$

where $A^{\pm}$ are the 2π-periodic $N \times N$-matrix-valued functions such that

$$A(t) - \chi_+(t)A^+(t) - \chi_-(t)A^-(t)$$

vanishes at $\pm\infty$. (See Proposition 1.6 for the definition of χ_+ and χ_-)

Then L in (1.44) is Fredholm if and only if $\Phi^+(2\pi)$ and $\Phi^-(2\pi)$ have no eigenvalue of absoluted value one.

Proof: We know, by Theorem 1.16 that L is Fredholm if and only if $L^+(\varphi)$ and $L^-(\varphi)$, defined by

$$L^{\pm}(\varphi) = \frac{1}{i}\frac{d}{d\theta} + A^{\pm}(\theta) + \varphi: \ H^1(\mathbb{S}^1, \mathbb{C}^N) \longrightarrow L^2(\mathbb{S}^1, \mathbb{C}^N) \ ,$$

are invertible, for all $\varphi \in [0,1]$. It can be easily verified that the problem

$$\begin{cases} \frac{1}{i}\frac{du}{d\theta} + A^{\pm}(\theta)u + \varphi u = 0 \\ u(0) = u(2\pi) \end{cases}$$

is solved by

$$u(\theta) = e^{-i\varphi\theta}\,\Phi^{\pm}(\theta)\,u(0)\ , \tag{1.46}$$

with u(0) satisfying

$$\Phi^{\pm}(2\pi)\,u(0) = e^{2\pi i\varphi}\,u(0)\ .$$

This shows that $L^{+}(\varphi)$ and $L^{-}(\varphi)$ are injective for all $\varphi \in [0,1]$ if and only if $\Phi^{+}(2\pi)$ and $\Phi^{-}(2\pi)$ have no eigenvalues of absolute value one.

The differential operators $L^{\pm}(\varphi)$ are invertible if and only if $L^{\pm}(\varphi)\Lambda_{\mathbb{S}^1}$ are invertible, where $\Lambda_{\mathbb{S}^1} := (1-\Delta_{\mathbb{S}^1})^{-1/2}$. We know that all operators $L^{\pm}(\varphi)\,\Lambda_{\mathbb{S}^1}$ have closed image, since they are in fact Fredholm (c. f. , for example, Theorem VI-2-2 of [2]). It remains to be decided if and when their adjoints are injective. It is easy to see that this will happen if and only if the differential operators

$$L^{\pm}(\varphi)^* = \frac{1}{i}\frac{d}{d\theta} + A^{\pm}(\theta)^* + \varphi\ : H^1(\mathbb{S}^1,\mathbb{C}^N) \to L^2(\mathbb{S}^1,\mathbb{C}^N)$$

are injective, where $A^{\pm}(\theta)^*$ denotes the transposed complex conjugate of $A^{\pm}(\theta)$. Since the matrices $(\Phi^{\pm}(\theta)^{-1})^*$ solve (1.45) with $A^{\pm}(\theta)$ replaced by $A^{\pm}(\theta)^*$, we conclude that $L^{\pm}(\varphi)^*$ are injective for all $\varphi \in [0,1]$ if and only if $\Phi^{+}(2\pi)$ and $\Phi^{-}(2\pi)$ have no eigenvalue of absolute value one. q.e.d.

Theorem 1.18 *Let L^{+} and L^{-} denote the differential operators on $\mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbb{R}\}$*

$$L^{\pm} = \frac{1}{i}\frac{d}{d\theta} + A^{\pm}(\theta)\ .$$

The eigenvalues of $L^{\pm}$ are then of the form:

$$\{\xi_j^{\pm} + k + i\eta_j^{\pm}\,;\ k \in \mathbb{Z},\, j = 1,\ldots,R\}\ ,\quad R \le N\ ,$$

for some fixed real numbers $\xi_1^{\pm},\ldots,\xi_R^{\pm},\eta_1^{\pm},\ldots,\eta_R^{\pm}$. The dimensions of the eigenspaces associated with $\xi_j^{\pm} + k + i\eta_j^{\pm}$, j fixed, are finite and independent of k.

The operator L in (1.44) is Fredholm if and only if $\eta_j^{\pm} \ne 0$ for all $j \in \{1,\ldots,R\}$. Moreover we have:

$$\text{index}\, L = \#\{\eta_j^{+}\,;\ \eta_j^{+} < 0\} - \#\{\eta_j^{-}\,;\ \eta_j^{-} < 0\}\ , \tag{1.47}$$

counting the multiplicity of the eigenvalues $\xi_j^{\pm} + i\eta_j^{\pm}$.

Proof: The properties of the eigenvalues of $L^{\pm}$ and the Fredholm criterion follow from Proposition 1.17 and its proof. Indeed, allowing φ assume non-real values, there we showed that the eigenvalues of $\Phi^{\pm}(2\pi)$ are $e^{2\pi i\zeta}$, where ζ is an eigenvalue of $L^{\pm}$. Furthermore, (1.46) defines an isomorphism between the corresponding eigenspaces. It suffices therefore to prove the index-formula (1.47), assuming that L is Fredholm.

Proof of the index formula for $N = 1$: Let us assume that L is of the form

$$L = \frac{1}{i}\frac{d}{dt} + a(t) : \ H^1(\mathbf{R}) \to L^2(\mathbf{R}) \ ,$$

with $a(t) = a_+(t)\chi_+(t) + a_-(t)\chi_-(t)$, $a_\pm$ 2π-periodic. Note that adding $a_\circ \in \mathbf{CO}(\mathbf{R})$ corresponds to compactly perturbing $L\Lambda_{\mathbf{R}}$, since $a_\circ\Lambda \in \mathcal{K}_{\mathbf{R}}$.

Let us define

$$b(t) := \frac{1}{2\pi}\int_t^{t+2\pi} a(y)\,dy$$

and

$$c(t) := a(t) - b(t) \ \ , \ \ d(t) := \int_0^t c(y)\,dy \ . \tag{1.48}$$

Notice that we have

$$b(t) = \frac{1}{2\pi}\int_0^{2\pi} a_\pm(y)\,dy \ , \quad \text{for} \quad \pm\, t > 2\pi + 1$$

and

$$c(t) = a_\pm(t) - \frac{1}{2\pi}\int_0^{2\pi} a_\pm(y)\,dy \ , \quad \text{for} \quad \pm\, t > 2\pi + 1 \ . \tag{1.49}$$

From (1.48) and (1.49), it follows that d is periodic for large $|\, t \,|$ and , thus, d and its derivative c are bounded continuous functions on $\mathbf{R}$. Multiplication by $e^{id(\cdot)}$ defines therefore an isomorphism of $H^1(\mathbf{R})$ onto itself. Now defining

$$M := e^{id(t)}Le^{-id(t)} = \frac{1}{i}\frac{d}{dt} + b(t) \ ,$$

it is clear that $M : H^1(\mathbf{R}) \to L^2(\mathbf{R})$ is Fredholm if and only if so is L, and, moreover,

$$\text{index } L = \text{index } M \ . \tag{1.50}$$

Since the function b is in $\mathbf{CS}(\mathbf{R})$, it follows from Theorem 36 of [7] (see also [20]) that the index of M can be found by the following procedure. The symbol

$$\sigma_{M\Lambda}(t,\tau) = \frac{\tau + b(t)}{(1+\tau^2)^{1/2}}$$

can be continuously extended to $\mathsf{R} := [-\infty,+\infty]\times[-\infty,+\infty]$. The boundary of R,

$$\partial\mathsf{R} \;=\; \{(t,\tau)\in\mathsf{R};\ |t|+|\tau|=\infty\}\ ,$$

is homeomorphic to the circle and is given here the positive orientation. The result cited above states that M is Fredholm if and only if $\sigma_{M\Lambda}|_{\partial R}$ never vanishes. If that is the case, we get

$$\text{index}\, M \;=\; \text{winding number of } \sigma_{M\Lambda}|_{\partial R} \text{ with respect to the origin.}$$

It is easy to see then that

$$\text{index}\, M \;=\; \begin{cases} +1 & \text{if} \quad \operatorname{Im} b(+\infty) < 0 < \operatorname{Im} b(-\infty) \\ -1 & \text{if} \quad \operatorname{Im} b(-\infty) < 0 < \operatorname{Im} b(+\infty) \\ 0 & \text{if} \quad \operatorname{Im} b(+\infty)\,.\,\operatorname{Im} b(-\infty) > 0 \end{cases} \tag{1.51}$$

The eigenvalues of $L^{\pm} = -id/d\theta \,+\, a_{\pm}(\theta)$ on $\mathbb{S}^1$ are

$$\lambda^{\pm} \;=\; \frac{1}{2\pi}\int_0^{2\pi} a_{\pm}(y)\,dy \;-\; k\ , \quad k\in\mathbb{Z}\ .$$

Since $b(\pm\infty) = (2\pi)^{-1}\int_0^{2\pi} a_{\pm}$, formula (1.47) has been proven for $N=1$, because of (1.50) and (1.51).

Let us postpone the proof of the two following lemmas and first conclude the proof of Theorem 1.18 .

Lemma 1.19 *Assume that L in (1.44) is Fredholm and that A is of the form*

$$A(t) \;=\; \chi_{+}(t)A^{+}(t) + \chi_{-}(t)A^{-}(t)\ , \tag{1.52}$$

with $\chi_{\pm}$ as in Proposition 1.6 , and $A^{\pm}$ 2π-periodic. We can then find a continuous curve A_s of $N\times N$-matrices of the form (1.52) such that: $A_0 = A$, the operators

$$L_s \;=\; \frac{1}{i}\frac{d}{dt} + A_s(t):\ H^1(\mathbb{R},\mathbb{C}^N)\to L^2(\mathbb{R},\mathbb{C}^N)$$

are Fredholm for all $s\in[0,1]$, and

$$A_1(t) \;=\; 0\ , \quad \textit{for} \quad t\in\bigcup_{k\in\mathbb{Z}}[2\pi k-\delta, 2\pi k+\delta]\ , \tag{1.53}$$

for some $\delta>0$.

Lemma 1.20 *Let $Gl(N)$ denote the set of all invertible complex $N \times N$-matrices and let $\Phi \in \mathbf{C}^1([0,2\pi], Gl(N))$ be given . Assume that $\Phi(0)$ equals the identity I and that Φ is constant on $[0,\epsilon] \cup [2\pi-\epsilon, 2\pi]$. There exists then a one-parameter family of curves*

$$\Phi_s \in \mathbf{C}^1([0,2\pi], Gl(N)) , \quad s \in [0,1] ,$$

with $\Phi_0 = \Phi$ and $\Phi_1(\theta)$ diagonal for all $\theta \in [0,2\pi]$, such that :

(i) $s \mapsto \Phi_s$ and $s \mapsto \Phi_s'$ are continuous with respect to the supremum-norm ,

(ii) $\Phi_s(0) = I$, for all $s \in [0,1]$,

(iii) Φ_s is constant on $[0, \frac{\epsilon}{2}] \cup [2\pi - \frac{\epsilon}{2}, 2\pi]$, for all $s \in [0,1]$, and

(iv) the absolute value of the eigenvalues of $\Phi(2\pi)$ do not change with s.

Proof of the index formula for the general case: It is no loss of generality to assume that A of (1.44) has the form (1.52), since adding $A_\circ\Lambda$ to $L\Lambda$, with $A_\circ(\pm\infty) = 0$, corresponds to a compact perturbation. By Lemma 1.19 , we can assume that A also satisfies (1.53), since $L_1\Lambda$ and $L\Lambda$ can be connected by a continuous curve of Fredholm operators. The periodic matrix-valued $A^\pm$ also satisfy (1.53) and, consequently, the corresponding $\Phi^\pm$ of (1.45) are constant on $[0,\epsilon] \cup [2\pi-\epsilon, 2\pi]$. Solutions of (1.45) are necessarily invertible for all θ and satisfy $\Phi^\pm(0) = I$. We can therefore apply Lemma 1.20 to Φ^+ and Φ^-.

Let $\Phi_s^\pm$ denote the deformations thus obtained and define:

$$A_s^\pm \;:=\; -i\Phi_s^{\pm\prime}(\theta)\,\Phi_s^\pm(\theta)^{-1}$$

All $A_s^\pm$ vanish on $[0, \frac{\epsilon}{2}] \cup [2\pi - \frac{\epsilon}{2}, 2\pi]$, since $\Phi_s^\pm$ are constant there. We can then periodically extend $A_s^\pm$ to $\mathbf{R}$ and define A_s of the form (1.52) by

$$A_s(t) \;=\; A_s^+(t)\chi_+(t) + A_s^-(t)\chi_-(t)$$

From (i) of Lemma 1.20 we get the continuity of $s \mapsto A_s$. Since the solutions of (1.45) for A_s are $\Phi_s^\pm$, we conclude that the operators

$$L_s \;=\; \frac{1}{i}\frac{d}{dt} + A_s(t)$$

are Fredholm for all $s \in [0,1]$. Here we have used Proposition 1.17 , (iv) of Lemma 1.20 and that $L = L_0$ is Fredholm. We have therefore obtained a curve $L_s\Lambda$ of Fredholm operators in $\mathcal{L}(L^2(\mathbf{R}, \mathbf{C}^N))$ with $L_0 = L$. Hence:

$$\text{index } L = \text{ index } L_1 \; . \tag{1.54}$$

Having proven the index formula (1.47) for $N = 1$, it is clear that it also holds for the case when $A(t)$ is diagonal for every $t \in \mathbb{R}$. The index of L_1 is therefore given by (1.47). It follows from Lemma 1.20 that the eigenvalues of $\Phi_s^{\pm}(2\pi)$ do not cross the unit circle, what means that the imaginary parts of the eigenvalues of $L^{\pm}$ never change sign as s varies, by our observation at the begining of the proof of the theorem. This, together with (1.54), shows that the formula (1.47) also holds for L. q.e.d.

Proof of Lemma 1.19 : Let χ_δ denote a 2π-periodic function such that $\chi_\delta(t) = 1$ for $|t| \leq \delta$ and $\chi_\delta(t) = 0$ for $2\delta \leq |t| \leq 2\pi - 2\delta$, where δ is positive. We claim that

$$A_s^\delta := (1 - s\chi_\delta(t))\, A(t)\,, \quad s \in [0,1]\,,$$

has the desired properties if δ is chosen sufficiently small. By Theorem 1.14, Remark 1.15 and (1.43), it is enough to show that

$$e^{i\varphi\theta}\, \Gamma_{L_s^\delta \Lambda}(\varphi, \pm 1)\, e^{-i\varphi\theta} = (-i\frac{d}{d\theta} + A_s^{\delta,\pm}(\theta) - \varphi)\,(1 + (-i\frac{d}{d\theta} - \varphi)^2)^{-1/2} \tag{1.55}$$

is invertible for all $\varphi \in [0,1]$. Because of our assumption that L_0 is Fredholm, we know that the right-hand side of (1.55) is invertible for $s = 0$ and all $\varphi \in [0,1]$.

Now we use that the invertible operators are an open set of $\mathcal{L}(L^2(\mathbb{S}^1, \mathbb{C}^N))$ and that the curve obtained by varying $\varphi \in [0,1]$, with $s = 0$ in (1.55), has compact image. Because of these facts, it is enough to show that

$$\lim_{\delta \to 0} \| \, e^{i\varphi\theta}(\Gamma_{L_o\Lambda}(\varphi, \pm 1) - \Gamma_{L_s^\delta\Lambda}(\varphi, \pm 1))e^{-i\varphi\theta} \, \| = 0\,, \quad \text{uniformly in } \varphi \text{ and } s\,. \tag{1.56}$$

The left-hand side of (1.56) equals

$$\lim_{\delta \to 0} \| \, s\, \chi_\delta(\theta)\, A^{\pm}(\theta)\, \Lambda_\varphi \, \|\,, \quad \text{where} \quad \Lambda_\varphi := (1 + (-i\frac{d}{d\theta} - \varphi)^2)^{-1/2}\,.$$

Since $\| \, A^{\pm}(\theta) \, \|$ is bounded on $[0, 2\pi]$, and $0 \leq s \leq 1$, all we have to show then is that

$$\| \, \chi_\delta(\theta)\, \Lambda_\varphi \, \|_{\mathcal{L}_{\mathbb{S}^1}} \longrightarrow 0 \quad \text{as} \quad \delta \to 0\,, \quad \text{uniformly in } \varphi \in [0,1]\,. \tag{1.57}$$

Using that the discrete Fourier transform diagonalizes the self-adjoint operator $-i\, d/d\theta$, we get :

$$\chi_\delta(\theta)\,(\Lambda_\varphi u)(\theta) = \frac{1}{2\pi}\chi_\delta(\theta) \sum_{j=-\infty}^{+\infty} e^{ij\theta}(1 + (j - \varphi)^2)^{-1/2} \int_0^{2\pi} e^{-ij\tilde{\theta}} u(\tilde{\theta})\, d\tilde{\theta}\,.$$

Using that the support of χ_δ is contained in an interval of length 4δ, we obtain:

$$\sum_{j=-\infty}^{+\infty} \frac{1}{4\pi^2} \| \chi_\delta(\theta)e^{ij(\theta-\tilde{\theta})}(1+(j-\varphi)^2)^{-1/2} \|^2_{L^2(\mathbb{S}^1\times\mathbb{S}^1)} \leq \frac{2\delta}{\pi} \sum_{j=-\infty}^{+\infty} \frac{1}{1+(j-\varphi)^2} \leq C\delta ,$$

where $C < \infty$ is independent of $\varphi \in [0,1]$. Defining then $K_{\delta,\varphi} \in L^2(\mathbb{S}^1 \times \mathbb{S}^1)$ by

$$K_{\delta,\varphi}(\theta,\tilde{\theta}) = \frac{1}{2\pi} \sum_{j=-\infty}^{+\infty} \chi_\delta(\theta)e^{ij(\theta-\tilde{\theta})}(1+(j-\varphi)^2)^{-1/2} ,$$

we get

$$\chi_\delta(\theta)\,(\Lambda_\varphi u)(\theta) = \int_0^{2\pi} K_{\delta,\varphi}(\theta,\tilde{\theta})u(\tilde{\theta})\,d\tilde{\theta}$$

and, hence,

$$\| \chi_\delta(\theta)\Lambda_\varphi \|_{\mathcal{L}_{\mathbb{S}^1}} \leq \| K_{\delta,\varphi} \|_{L^2(\mathbb{S}^1\times\mathbb{S}^1)} \leq \sqrt{C\delta} .$$

This proves (1.57). q.e.d.

Proof of Lemma 1.20 : Let $U \in Gl(N)$ be such that $U^{-1}\Phi(2\pi)U$ is Jordan and let $U(s)$, $s \in [0,1]$, be a smooth path connecting U to I. Define

$$\Phi_s(\theta) := U^{-1}(s)\Phi(\theta)U(s) .$$

It is evident that Φ_s is constant on $[0,\epsilon] \cup [2\pi-\epsilon, 2\pi]$ for every s and that Φ_s has all the desired properties except $\Phi_1(\theta)$ being diagonal. This shows that, without loss of generality, we can assume that $\Phi(2\pi)$ is Jordan. We can go further and assume that $\Phi(2\pi)$ is diagonal, after the following deformation. We define

$$\Phi_s(\theta) := \Phi(\theta) , \quad \text{for} \quad \theta \in [0, 2\pi-\epsilon] \quad \text{and} \quad s \in [0,1] ,$$

and, for $\theta \in [2\pi-\epsilon, 2\pi]$, we substitute the 1's outside the diagonal by $\omega_s(\theta)$, where $\omega_s(\theta) = 1$ for $\theta \in [2\pi-\epsilon, 2\pi-\frac{3\epsilon}{4}]$, $\omega_s(\theta) = 1-s$ for $\theta \in [2\pi-\frac{\epsilon}{2}, 2\pi]$, and $(s,\theta) \mapsto \omega_s(\theta)$ is smooth.

We can therefore assume that

$$\Phi(2\pi) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix}$$

Let us define $\Psi : [0, 2\pi] \to Gl(N)$ by

$$\Psi(\theta) = \Phi(2\pi) \begin{pmatrix} 1 + \omega(\theta)(\frac{1}{\lambda_1} - 1) & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 + \omega(\theta)(\frac{1}{\lambda_N} - 1) \end{pmatrix}, \tag{1.58}$$

where ω is smooth, $0 \leq \omega \leq 1$, $\omega(\theta) = 0$ for $\theta \in [0, \epsilon]$ and $\omega(\theta) = 1$ for $\theta \in [2\pi - \epsilon, 2\pi]$. Let $m \in \mathbf{Z}$ be the winding number with respect to the origin of the closed curve $c : [0, 4\pi] \to \mathbf{C}$,

$$c(\theta) = \begin{cases} \det\ \Phi(\theta) & \text{if} \quad \theta \in [0, 2\pi] \\ \det\ \Psi(\theta - 2\pi) & \text{if} \quad \theta \in [2\pi, 4\pi] \end{cases}.$$

Let $\Xi : [0, 2\pi] \to Gl(N)$ be defined by

$$\Xi(\theta) = \begin{pmatrix} e^{-2\pi i m \omega(\theta)} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix},$$

with ω as in (1.58). Let Υ be the smooth closed path in $Gl(N)$ obtained by going through Φ, followed by Ξ and, then, Ψ. It is obvious that the winding number with respect to the origin of $\det \Upsilon$ is zero and, hence, Υ is homotopically equivalent to the curve $\tilde{\Phi}$ obtained by going backwards through Ψ and Ξ. This homotopy Φ_s can be chosen differentiable, having both ends fixed and satisfying (iii). By construction, $\Phi_0 = \Phi$, Φ_1 is diagonal at every point and Φ_s satisfies (i), (ii), (iii) and (iv). q.e.d.

Chapter 2

A Comparison Algebra on a cylinder with semi-periodic multiplications

2.1 Definition of the algebra $\mathcal{C}_\mathcal{P}$ and a description of its commutator ideal

Throughout this chapter Ω denotes the Riemannian manifold $\mathbf{R} \times \mathbf{B}$, where $\mathbf{B}$ denotes an n-dimensional compact manifold with metric tensor locally given by h_{jk}, and $\mathcal{H}$ denotes the Hilbert space $L^2(\Omega)$, with Ω being given the surface measure

$$dS = \sqrt{h}\, dt\, dx^1 \ldots dx^n,$$

where h is the determinant of the $n \times n$-matrix $((h_{jk}))_{1 \le j,k \le n}$. The metric on Ω is given by $ds^2 = dt^2 + h_{jk}\, dx^j\, dx^k$, and the Laplace operator is locally given by

$$\Delta_\Omega = \Delta_{\mathbf{R}} + \Delta_{\mathbf{B}} = \frac{d^2}{dt^2} + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^j} \sqrt{h} h^{jk} \frac{\partial}{\partial x^k} \, ,$$

where $((h^{jk})) = ((h_{jk}))^{-1}$, and the summation convention from 1 to n is being used.

The symmetric operator Δ_Ω with domain $\mathrm{C}_o^\infty(\Omega)$ is essentially self-adjoint, since Ω is complete (c.f.[2],IV). We denote by H the closure of $1 - \Delta_\Omega$ and by Λ its inverse square root, $\Lambda = H^{-1/2}$. Since $H \geq 1$, we have $\Lambda \in \mathcal{L}(\mathcal{H})$. The algebra $\mathcal{C}_\mathcal{P}$ is defined as the smallest C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ containing the following operators (or classes of operators):

$$a \in \mathbf{C}^\infty(\mathbf{B});\ b \in \mathbf{CS}(\mathbf{R});\ e^{ijt}, j \in \mathbf{Z};\ \Lambda;\ \frac{1}{i}\frac{\partial}{\partial t}\Lambda \text{ and } D_x\Lambda, \tag{2.1}$$

D_x being a first order differential operator on $\mathbf{B}$, locally given by $-ib^j(x)\partial/\partial x^j$, where $b^j(x)$, $j = 1, \ldots, n$, are the components of a smooth vector field on $\mathbf{B}$. The operators $\frac{\partial}{\partial t}\Lambda$ and $D_x\Lambda$, defined on the dense subspace $\Lambda^{-1}(\mathbf{C}_o^\infty(\Omega))$, can be extended to bounded operators of $\mathcal{L}(\mathcal{H})$ (c.f. [2], for example). Bounded functions on Ω have been identified with the corresponding multiplication operators in $\mathcal{L}(\mathcal{H})$.

Our first objective in this chapter is to obtain a necessary and sufficient criterion for an operator in $\mathcal{C}_\mathcal{P}$ to be Fredholm. Such a criterion has been found by Cordes [3] for the algebra generated by the operators in (2.1) except e^{ijt}, $j \in \mathbf{Z}$.

Taking advantage of the tensor product structure of $\mathcal{H}$,

$$\mathcal{H} = L^2(\mathbf{R})\bar{\otimes}L^2(\mathbf{B})$$

(see A.1), we consider the conjugate of $\mathcal{C}_\mathcal{P}$ with respect to the unitary operator $F \otimes I_\mathbf{B}$, where $I_\mathbf{B}$ denotes the identity operator on $L^2(\mathbf{B})$ and F the Fourier transform on $L^2(\mathbf{R})$. In order to simplify notation, $A \otimes I_\mathbf{B}$ is denoted by A and $I_\mathbf{R} \otimes B$ by B, whenever $A \in \mathcal{L}(L^2(\mathbf{R}))$ or $B \in \mathcal{L}(L^2(\mathbf{B}))$.

We seek to describe what are $B_k := F^{-1}A_kF$, where A_k, $k = 1, \ldots, 6$, denote the operators listed in (2.1), in that order. The operator-valued functions $\tilde{\Lambda}(\tau) := (1 + \tau^2 - \triangle_\mathbf{B})^{-1/2}$, $\tau\tilde{\Lambda}(\tau)$ and $D_x\tilde{\Lambda}(\tau)$, $\tau \in \mathbf{R}$, are all in $\mathbf{CB}(\mathbf{R}, \mathcal{L}_\mathbf{B})$, as proven in [3], page 220, and thus determine operators in $\mathcal{L}(\mathcal{H})$, as defined in Proposition A.3. Here $\mathcal{L}_\mathbf{B}$ denotes the algebra of bounded operators on $L^2(\mathbf{B})$ and $\mathbf{CB}(\mathbf{R}, \mathcal{L}_\mathbf{B})$ the bounded continuous $\mathcal{L}_\mathbf{B}$-valued functions on $\mathbf{R}$. With this interpretation, we get B_k, $k = 1, \ldots, 6$, respectively given by

$$a \in \mathbf{C}^\infty(\mathbf{B});\ b(D),\ b \in \mathbf{CS}(\mathbf{R});\ T_j, j \in \mathbf{Z};\ \tilde{\Lambda}(\tau); -\tau\tilde{\Lambda}(\tau) \text{ and } D_x\tilde{\Lambda}(\tau), \tag{2.2}$$

where $b(D) := F^{-1}bF$ and T_j denotes the translation $(T_ju)(\tau) = u(\tau + j)$.

Let $\mathcal{K}_\mathbf{B}$ denote the ideal of compact operators on $L^2(\mathbf{B})$ and $\mathbf{CO}(\mathbf{R}, \mathcal{K}_\mathbf{B})$ denote the $\mathcal{K}_\mathbf{B}$-valued continuous functions on $\mathbf{R}$ that vanish at infinity. All commutators $[B_k, B_l]$, $k, l \neq 3$, are contained in the algebra

$$\mathcal{CK} := \mathbf{CO}(\mathbf{R}, \mathcal{K}_\mathbf{B}) + \mathcal{K}(\mathcal{H}),$$

where $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators of $\mathcal{L}(\mathcal{H})$, as proven in [3], Proposition 1.2. Next we investigate what are the commutators $[B_3, B_k]$, $k = 1, \ldots, 6$. We easily get $[B_3, B_1] = [B_3, B_2] = 0$. It is also clear that, for any $K(\tau) \in \mathbf{CB}(\mathbb{R}, \mathcal{L}_{\mathbf{B}})$, we have

$$[T_k, K(\tau)] = (K(\tau + k) - K(\tau))T_k, \; k \in \mathbb{Z}. \tag{2.3}$$

Proposition 2.1 *The commutators of the generators in (2.2) — and of their adjoints — of the algebra* $\hat{\mathcal{C}_{\mathcal{P}}} := F^{-1}\mathcal{C}_{\mathcal{P}}F$ *are contained in*

$$\mathcal{CKT} = \{ \sum_{j=-N}^{N} K_j(\tau)T_j + K; \; N \in \mathbf{N}, \; K_j \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbf{B}}), \; K \in \mathcal{K}(\mathcal{H})\}.$$

Proof: Let us first prove that $K(\tau + j) - K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbf{B}})$, for $K(\tau) = \tilde{\Lambda}(\tau)$, $\tau\tilde{\Lambda}(\tau)$ or $D_x\tilde{\Lambda}(\tau)$. It follows from the fact that $-\Delta_{\mathbf{B}}$ on $L^2(\mathbf{B})$ has an orthonormal basis of eigenfunctions, with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \ldots$, $\lambda_n \to \infty$ as $n \to \infty$, that, for each $\tau \in \mathbb{R}$, $\tilde{\Lambda}(\tau)$ is unitarily equivalent to the multiplication operator $(1 + \tau^2 + \lambda_n)^{-1/2}$ on $L^2(\mathbf{N})$. Hence: $\tilde{\Lambda}(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbf{B}})$,

$$\| \tau[\tilde{\Lambda}(\tau + j) - \tilde{\Lambda}(\tau)] \|_{L^2(\mathbf{B})} \le \max_{s \in [1,\infty)} | \tau[(s + (\tau + j)^2)^{-1/2} - (s + \tau^2)^{-1/2}] |$$

and

$$\| \tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1 \|_{L^2(\mathbf{R})} \le \max_{s \in [1,\infty)} | (\tau^2 + s)^{1/2}((\tau + j)^2 + s)^{-1/2} - 1 | \; .$$

Note that the right-hand sides of the two previous inequalities go to zero as $\tau \to \pm\infty$. Furthermore, as

$$\lim_{n \to \infty} (1 + \tau^2 + \lambda_n)^{1/2}(1 + (\tau + j)^2 + \lambda_n)^{-1/2} - 1 = 0,$$

we have that $\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1 \in \mathcal{K}_{\mathbf{B}}$, for each $\tau \in \mathbb{R}$. We then get:

$$(\tau + j)\tilde{\Lambda}(\tau + j) - \tau\tilde{\Lambda}(\tau) = \tau(\tilde{\Lambda}(\tau + j) - \tilde{\Lambda}(\tau)) + j\tilde{\Lambda}(\tau + j) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbf{B}})$$

and

$$D_x\tilde{\Lambda}(\tau + j) - D_x\tilde{\Lambda}(\tau) = D_x\tilde{\Lambda}(\tau)[\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1] \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbf{B}}).$$

By the remarks preceding the statement of the proposition, this proves that the commutators of the generators (2.2) are indeed contained in $\mathcal{CKT}$. Concerning the adjoints, let us note that the classes of B_k's, $k = 1, \ldots, 5$, are self-adjoint and that, as proven in [3], $D_x\tilde{\Lambda} - \tilde{\Lambda}D_x \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$, hence:

$$(D_x\tilde{\Lambda})^* - D_x^*\tilde{\Lambda} = \tilde{\Lambda}D_x^* - D_x^*\tilde{\Lambda} \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}}). \tag{2.4}$$

Here, D_x^* denotes the formal adjoint of D_x. The commutators of any $K(\tau) \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$ with the generators B_k, $k = 1, 3, 4, 5, 6$, are clearly contained in $\mathcal{CKT}$. For $K(\tau)$ of the special form $K(\tau) = a(\tau)\tilde{K}$, $a \in \mathbf{CO}(\mathbf{R})$ and $\tilde{K} \in \mathcal{K}_{\mathbf{B}}$, the commutator $[b(D), K(\tau)] = [b(D), a(\tau)] \otimes \tilde{K}$ is compact, since $[b(D), a(\tau)]$ is compact (c.f. [4], Chapter III, for example), for $b \in \mathbf{CS}(\mathbf{R})$. The vector space generated by all $K(\tau) = a(\tau)\tilde{K}$ as above is dense in $\mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$ (see Proposition A.4) and thus we have

$$[b(D), K(\tau)] \in \mathcal{K}(\mathcal{H}), \text{ for } b \in \mathbf{CS}(\mathbf{R}),\ K(\tau) \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}}). \tag{2.5}$$

This conclude the proof. q.e.d.

Denoting by $\mathcal{E}_{\mathcal{P}}$ the commutator ideal of $\mathcal{C}_{\mathcal{P}}$ and by $\hat{\mathcal{E}_{\mathcal{P}}}$ the commutator ideal of $\hat{\mathcal{C}_{\mathcal{P}}}$, it is obvious that $\hat{\mathcal{E}_{\mathcal{P}}} = F^{-1}\mathcal{E}_{\mathcal{P}}F$.

Proposition 2.2 *The commutator ideal $\hat{\mathcal{E}_{\mathcal{P}}}$ of the algebra $\hat{\mathcal{C}_{\mathcal{P}}}$ is obtained by closing the set of operators:*

$$\hat{\mathcal{E}}_{P,O} := \{ \sum_{j=-N}^{N} K_j(\tau)T_j + K;\ b \in \mathbf{CS}(\mathbf{R}),\ N \in \mathbf{N},\ K_j \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}}),\ K \in \mathcal{K}(\mathcal{H})\}.$$

Proof: The algebra $\mathcal{C}_{\mathcal{P}}$ is a "Comparison Algebra", in the sense of [2] , Chapter V, with "generating classes":

$$\mathcal{A}^\sharp := \mathbf{C}_o^\infty(\Omega)\ \cup\ \mathbf{C}^\infty(\mathbf{B})\ \cup\ \{e^{ijt}; j \in \mathbf{Z}\}\ \cup\ \{s(t) = t(1+t^2)^{-1/2}\} \tag{2.6}$$

and $\mathcal{D}^\sharp$ equal to the vector space generated by the first order linear partial differential expressions on $\mathbf{B}$ with smooth coefficients and by the expression $\partial/\partial t$. Indeed, $\mathcal{C}_{\mathcal{P}}$ can alternatively be defined as the C^*-algebra generated by all multiplications by functions in $\mathcal{A}^\sharp$ and by all $D\Lambda$, $D \in \mathcal{D}^\sharp$. It follows then from Lemma V-1-1 of [2] that $\mathcal{K}(\mathcal{H}) \subset \mathcal{E}_{\mathcal{P}}$ and therefore: $\mathcal{K}(\mathcal{H}) \subset \hat{\mathcal{E}_{\mathcal{P}}}$. Moreover, it was proven in [3] , Proposition 1.5, that $\mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$ is contained in the commutator ideal of the C^*-algebra generated by B_4, B_5 and B_6. Thus we get $\hat{\mathcal{E}}_{P,O} \subset \hat{\mathcal{E}_{\mathcal{P}}}$.

All commutators of the generators (2.2) and their adjoints are contained in $\hat{\mathcal{E}}_{P,O}$, by Proposition 2.1. Again using (2.3), (2.4) and (2.5), it is easy to verify that $\hat{\mathcal{E}}_{P,O}$ is invariant under right or left multiplication by the operators in (2.2) and their adjoints. Two facts then follow: (i) all commutators of the algebra (finitely) generated by the operators in (2.2) and their adjoints are contained in $\hat{\mathcal{E}}_{P,O}$ and therefore all commutators of $\hat{\mathcal{C}_{\mathcal{P}}}$ are contained in the closure of $\hat{\mathcal{E}}_{P,O}$, and (ii) the closure of $\hat{\mathcal{E}}_{P,O}$ is an ideal of $\hat{\mathcal{C}_{\mathcal{P}}}$. By definition of commutator ideal, $\hat{\mathcal{E}_{\mathcal{P}}}$ is contained in the closure of $\hat{\mathcal{E}}_{P,O}$. q.e.d.

Corollary 2.3 *With $\hat{\mathcal{E}_A}$ denoting the closure of $\hat{\mathcal{E}}_{A,O}$, defined in (1.3), we have:*

$$\hat{\mathcal{E}_{\mathcal{P}}} = \hat{\mathcal{E}_{\mathcal{A}}} \bar{\otimes} \mathcal{K}_{\mathbf{B}}$$

Proof: The vector-space generated by

$$\{(b(D)a(\tau)T_j + K) \otimes \tilde{K};\ b \in \mathbf{CS}(\mathbf{R}),\ a \in \mathbf{CO}(\mathbf{R}),\ j \in \mathbf{Z},\ K \in \mathcal{K}_{\mathbf{R}},\ \tilde{K} \in \mathcal{K}_{\mathbf{B}}\}$$

is dense in $\hat{\mathcal{E}}_{P,O}$ and in $\hat{\mathcal{E}_{\mathcal{A}}} \bar{\otimes} \mathcal{K}_{\mathbf{B}}$, by Proposition A.4 . q.e.d.

In order to give a better description of $\mathcal{E}_{\mathcal{P}}$, we consider the conjugate of $\hat{\mathcal{E}_{\mathcal{P}}}$ with respect to W, where $W : L^2(\mathbf{R}) \to L^2(\mathbb{S}^1; L^2(\mathbf{Z}))$ was defined in (1.7). Here we allow a slight abuse of notation, since by W we mean

$$W \otimes I_{\mathbf{B}} : L^2(\mathbf{R}) \bar{\otimes} L^2(\mathbf{B}) \to L^2(\mathbb{S}^1) \bar{\otimes} L^2(\mathbf{Z}) \bar{\otimes} L^2(\mathbf{B})$$

(see (1.7)).

Proposition 2.4 *The map*

$$\hat{\mathcal{E}_{\mathcal{P}}} \longrightarrow \mathcal{SL} \bar{\otimes} \mathcal{K}_{\mathbf{Z}} \bar{\otimes} \mathcal{K}_{\mathbf{B}}$$
$$A \longmapsto WAW^{-1}$$

*is an onto *-isomorphism, where $\mathcal{SL}$ denotes the algebra of bounded operators on $L^2(\mathbb{S}^1)$ defined in Section 1.1 . For $A \in \hat{\mathcal{E}_{\mathcal{P}}}$ of the form $A = b(D)K(\tau)T_j$, with $b \in \mathbf{CS}(\mathbf{R})$, $K(\tau) \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$ and $j \in \mathbf{Z}$, we have:*

$$WAW^{-1} = b(D_\theta) Y_\varphi K(\varphi - M) Y_{-\varphi-j} + K, \ \text{with } K \in \mathcal{K}_{\mathbb{S}^1 \times \mathbf{Z} \times \mathbf{B}} \ . \tag{2.7}$$

For each $\varphi \in \mathbf{R}$ *here,* $K(\varphi - M)$ *denotes the compact operator on* $L^2(\mathbf{Z})\bar{\otimes}L^2(\mathbf{B})$ *defined by the sequence* $K(\varphi - j) \in \mathcal{K}_{\mathbf{B}}$, $j \in \mathbf{Z}$. *The first term of the right-hand side of (2.7) defines therefore a* $\mathcal{K}_{\mathbf{Z}\times\mathbf{B}}$*-valued continuous function on* $\mathbf{S}^1 = \{e^{2\pi i\varphi}; \varphi \in \mathbf{R}\}$. *(See Appendix for details.) The operators* Y_φ *have been defined on page 8.*

Proof: By Corollary 2.3 and (1.8),

$$W\hat{\mathcal{E}_\mathcal{P}}W^{-1} = \mathcal{SL}\bar{\otimes}\mathcal{K}_{\mathbf{Z}}\bar{\otimes}\mathcal{K}_{\mathbf{B}}$$

and, by (1.9), formula 2.7 holds for $K(\tau)$ of the form $a(\tau)\otimes\tilde{K}$, $a \in \mathbf{CO}(\mathbf{R})$ and $\tilde{K} \in \mathcal{K}_{\mathbf{B}}$. By Proposition A.4, we can then find a sequence $K_m(\tau) \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$ such that $K_m(\tau) \to K(\tau)$, uniformly in $\tau \in \mathbf{R}$, and (2.7) is valid for each $K_m(\tau)$. Then

$$Y_\varphi K_m(\varphi - M)Y_{-\varphi-j} \to Y_\varphi K(\varphi - M)Y_{-\varphi-j}$$

in $\mathcal{K}_{\mathbf{Z}\times\mathbf{B}}$, uniformly in $e^{2\pi i\varphi} \in \mathbf{S}^1$. By Proposition A.3, the convergence also holds in $\mathcal{L}_{\mathbf{S}^1\times\mathbf{Z}\times\mathbf{B}}$. q.e.d.

We recall that $\mathbf{M}_{SL} = \mathbf{S}^1\times\{-1,+1\}$ denotes the symbol-space of $\mathcal{SL}$ (see page 17).

Theorem 2.5 *There exists an onto *-isomorphism*

$$\Psi : \frac{\mathcal{E}_\mathcal{P}}{\mathcal{K}(\mathcal{H})} \longrightarrow \mathrm{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}})$$

such that; if $\tilde{\gamma}$ *denotes the composition of* Ψ *with the canonical projection* $\mathcal{E}_\mathcal{P} \to \mathcal{E}_\mathcal{P}/\mathcal{K}(\mathcal{H})$ *and* $A \in \mathcal{E}_\mathcal{P}$ *is such that* $B = F^{-1}AF$ *is of the form* $B = b(D)K(\tau)T_j$, *where* $b \in \mathbf{CS}(\mathbf{R})$, $K(\tau) \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_{\mathbf{B}})$ *and* $j \in \mathbf{Z}$, *we then have :*

$$\tilde{\gamma}_A(e^{2\pi i\varphi}, \pm 1) = b(\pm\infty)Y_\varphi K(\varphi - M)Y_{-\varphi-j}\ .$$

Proof: Let Ψ be given by

$$\frac{\mathcal{E}_\mathcal{P}}{\mathcal{K}(\mathcal{H})} \to \frac{\hat{\mathcal{E}_\mathcal{P}}}{\mathcal{K}(\mathcal{H})} \to \frac{\mathcal{SL}\bar{\otimes}\mathcal{K}_{\mathbf{Z}}\bar{\otimes}\mathcal{K}_{\mathbf{B}}}{\mathcal{K}_{\mathbf{S}^1\times\mathbf{Z}\times\mathbf{B}}} \to \mathrm{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}})\ ,$$

where in the first step we take $A + \mathcal{K}(\mathcal{H}) \in \mathcal{E}_\mathcal{P}/\mathcal{K}(\mathcal{H})$ to $F^{-1}AF + \mathcal{K}(\mathcal{H})$, next to

$$WF^{-1}AFW^{-1} + \mathcal{K}_{\mathbf{S}^1\times\mathbf{Z}\times\mathbf{B}}\ ,$$

and in the last step we use the onto *-isomorphism(by page 17 and [1]):

$$\frac{\mathcal{SL}\bar{\otimes}\mathcal{K}_{\mathbf{Z}}\bar{\otimes}\mathcal{K}_{\mathbf{B}}}{\mathcal{K}_{\mathbf{S}^1\times\mathbf{Z}\times\mathbf{B}}} \longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}})$$

$$A\otimes K_1\otimes K_2 + \mathcal{K}_{\mathbf{S}^1\times\mathbf{Z}\times\mathbf{B}} \longmapsto \sigma_A^{SL}(\varphi,\pm 1)K_1\otimes K_2\ .$$

Defined this way, Ψ has the desired properties, by Proposition 2.4 and its proof. q.e.d.

2.2 Definition of two symbols on $\mathcal{C}_\mathcal{P}$

Our first task in this section is to give a precise description of the symbol space of $\mathcal{C}_\mathcal{P}$, i. e. , the maximal-ideal space of the commutative C*-algebra $\mathcal{C}_\mathcal{P}/\mathcal{E}_\mathcal{P}$. The symbol space of $\mathcal{C}$, the C*-algebra generated by the operators listed in (2.1) except the periodic functions e^{ijt}, was described in [3] :

Theorem 2.6 *(Theorem 2.3 of [3]) The symbol space* $\mathbf{M}$ *of* $\mathcal{C}$ *can be identified with the bundle of unit spheres of the cotangent bundle of the compact manifold with boundary* $[-\infty,+\infty]\times\mathbb{B}$*, where* $[-\infty,+\infty]$ *denotes the compactification of* $\mathbb{R}$ *obtained by adding the points* $-\infty$ *and* $+\infty$*. The* σ*-symbols of the generators* A_1*,* A_2*,* A_4*,* A_5 *and* A_6 *are given below as functions of the local coordinates* $(t,x;\tau,\xi)$*, where* $(t,\tau)\in[-\infty,+\infty]\times\mathbb{R}^*$*,* $(x,\xi)\in T^*\mathbb{B}$ *and* $\tau^2+h^{jk}\xi_j\xi_k=1$*:*

$$\sigma_{A_1}=a(x)\quad \sigma_{A_2}=b(t)\quad \sigma_{A_4}=0\quad \sigma_{A_5}=\tau\quad \sigma_{A_6}=b^j(x)\xi_j\ .$$

When periodic functions are adjoined to the algebra, the points over $|t|=\infty$ become circles. More precisely, we have:

Theorem 2.7 *The symbol space* $\mathbf{M}_P$ *of* $\mathcal{C}_\mathcal{P}$ *is homeomorphic to the closed subset of* $\mathbf{M}\times\mathbb{S}^1$ *described in local coordinates by*

$$\{((t,x;\tau,\xi),e^{i\theta});\ (t,x;\tau,\xi)\in\mathbf{M},\ \theta\in\mathbb{R}\ \textit{and}\ \theta=t\ \textit{if}\ \mid t\mid<\infty\}\ .$$

Using this description of $\mathbf{M}_P$*, the* σ*-symbols of the generators in (2.1) are respectively given by*

$$a(x)\quad b(t)\quad e^{ij\theta}\quad 0\quad \tau\quad \textit{and}\quad b^j(x)\xi_j\ .$$

Proof: Let $\mathbf{P}_{2\pi}$ denote the closed algebra generated by $\{e^{ijt};j\in\mathbb{Z}\}$, i. e. , the 2π-periodic continuous functions on $\mathbb{R}$. Its maximal-ideal space is $\mathbb{S}^1$, with $e^{i\theta}\in\mathbb{S}^1$ defining the multiplicative linear functional $f\to f(\theta)$.

With $\mathcal{E}$ denoting the commutator ideal of $\mathcal{C}$, the maximal-ideal space of $\mathcal{C}/\mathcal{E}$ is $\mathbf{M}$, as described in Theorem 2.6 . By definition of the Gelfand map, a point $(t,x;\tau,\xi)$ defines the multiplicative linear functional

$$A+\mathcal{E}\to\sigma_A(t,x;\tau,\xi)$$

The following maps are canonically defined:

$$i_1 : \frac{\mathcal{C}}{\mathcal{E}} \longrightarrow \frac{\mathcal{C}_{\mathcal{P}}}{\mathcal{E}_{\mathcal{P}}} \tag{2.8}$$

and

$$i_2 : \mathbf{P}_{2\pi} \longrightarrow \frac{\mathcal{C}_{\mathcal{P}}}{\mathcal{E}_{\mathcal{P}}} \ . \tag{2.9}$$

(It is obvious that $\mathcal{E} \subseteq \mathcal{E}_{\mathcal{P}}$)

Let us denote by ι the product of the dual maps of i_1 and i_2, i. e. ,

$$\begin{aligned} \iota :\ & \mathbf{M}_P \longrightarrow \mathbf{M} \times \mathbb{S}^1 \\ & \omega \longmapsto (\omega \circ i_1, \omega \circ i_2) \ , \end{aligned} \tag{2.10}$$

where ω denotes a multiplicative linear functional on $\mathcal{C}_{\mathcal{P}}/\mathcal{E}_{\mathcal{P}}$.

As the images of i_1 and i_2 generate $\mathcal{C}_{\mathcal{P}}/\mathcal{E}_{\mathcal{P}}$, ι is an injective map, clearly continous, what proves that $\mathbf{M}_P$ is homeomorphic to a compact subset of $\mathbf{M} \times \mathbb{S}^1$. Now we proceed to investigate which points of $\mathbf{M} \times \mathbb{S}^1$ belong to the image of ι. This dual-map argument is essentially "Herman's Lemma"(c. f. [4]).

As in the proof of Proposition 2.2 , here again we use general results on comparison algebras. It follows from Theorem VII-1-5 of [2] that for every point of the cosphere-bundle of Ω, $(t, x; \tau, \xi) \in S^*\Omega$, there is a multiplicative linear functional on $\mathcal{C}_{\mathcal{P}}/\mathcal{E}_{\mathcal{P}}$ that takes any function a, belonging to the closed algebra generated by $A^\sharp$ in (2.6), to $a(x,t)$ and $D\Lambda$,

$$D = \frac{1}{i}\frac{\partial}{\partial t} + \frac{1}{i} b^j(x) \frac{\partial}{\partial x^j} + q(x) \in \mathcal{D}^\sharp \ ,$$

to $\tau + b^j(x)\xi_j$. This multiplicative linear functional must correspond to the point

$$((t, x; \tau, \xi), e^{it}) \in \mathbf{M} \times \mathbb{S}^1 \ ,$$

with $| \ t \ | < \infty$.

Suppose now that $((t, x; \tau, \xi), e^{i\theta})$ is in the image of ι and that $| \ t \ | < \infty$. Let ω denote the corresponding multiplicative linear functional on $\mathcal{C}_{\mathcal{P}}/\mathcal{E}_{\mathcal{P}}$ and χ denote a funtion in $\mathbf{C}_o^\infty(\Omega)$ with $\chi(t) = 1$. It is clear that $\chi(\cdot)e^{i(\cdot)} + \mathcal{E}_{\mathcal{P}}$ belongs to the image of i_1 and thus, by (2.10),

$$\omega(\chi(\cdot)e^{i(\cdot)} + \mathcal{E}_{\mathcal{P}}) = e^{it} \ .$$

On the other hand, since $e^{i(\cdot)} + \mathcal{E}_{\mathcal{P}}$ belongs to the image of i_2 , we get:

$$\omega(\chi(\cdot)e^{i(\cdot)} + \mathcal{E}_{\mathcal{P}}) = \omega(\chi(\cdot) + \mathcal{E}_{\mathcal{P}})\omega(e^{i(\cdot)} + \mathcal{E}_{\mathcal{P}}) = e^{i\theta} \ .$$

We then obtain $e^{i\theta} = e^{it}$.

For $t = \pm\infty$ and any $e^{i\theta} \in \mathbf{S}^1$, let us consider the sequence $t_m = \theta \pm 2\pi m$. Since $\mathbf{M}_P$ is closed and

$$((t_m, x; \tau, \xi), e^{it_m}) \to ((t, x; \tau, \xi), e^{i\theta}) \quad \text{as} \quad m \to \infty ,$$

we conclude that $((t, x; \tau, \xi), e^{i\theta}) \in \mathbf{M}_P$ q.e.d.

Remark 2.8 *We have just proven above that*

$$\mathbf{W}_P := \{((t, x; \tau, \xi), e^{i\theta}) \in \mathbf{M}_P;\ | t | < \infty\}$$

is dense in $\mathbf{M}_P$.

Next we define the γ-symbol.

The C*-algebra $\mathcal{C}_\mathcal{P}/\mathcal{K}(\mathcal{H})$ has the closed two-sided ideal $\mathcal{E}_\mathcal{P}/\mathcal{K}(\mathcal{H})$, which was proven to be *-isomorphic to $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}})$ in Theorem 2.5. Every $A \in \mathcal{C}_\mathcal{P}$ determines a bounded operator of $\mathcal{L}(\mathcal{E}_\mathcal{P}/\mathcal{K}(\mathcal{H}))$ by $E + \mathcal{K}(\mathcal{H}) \to AE + \mathcal{K}(\mathcal{H})$, thus defining

$$T : \mathcal{C}_\mathcal{P} \to \mathcal{L}(\mathcal{E}_\mathcal{P}/\mathcal{K}(\mathcal{H})) .$$

Let us define

$$\begin{array}{rccl} \gamma : & \mathcal{C}_\mathcal{P} & \longrightarrow & \mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}})) \\ & A & \longmapsto & \gamma_A = \Psi T_A \Psi^{-1} , \end{array} \tag{2.11}$$

for Ψ defined in Theorem 2.5.

For $E \in \mathcal{E}_\mathcal{P}$, γ_E is the operator multiplication by $\tilde{\gamma}_E \in \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}})$ (see Theorem 2.5). Identifying functions in $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$ with the corresponding multiplication operator of $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}}))$, we can say then that γ is an extension of $\tilde{\gamma}$.

Proposition 2.9 *There exists a *-homomorphism*

$$\gamma : \mathcal{C}_\mathcal{P} \longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}}) ,$$

where

$$\mathbf{M}_{SL} = \{ e^{2\pi i\varphi};\ \varphi \in \mathbf{R} \} \times \{ +1, -1 \} ,$$

given on the generators (2.1), according to notation established in Sections 1.1 and 2.1 and in Theorem 2.5 , by :

$$\begin{aligned} \gamma_{A_1} &= a(x)\,;\ \ \gamma_{A_2} = b(\pm\infty)\,;\ \ \gamma_{A_3} = Y_{-j}\,;\ \ \gamma_{A_4} = Y_\varphi\,\tilde{\Lambda}(\varphi - M)\,Y_{-\varphi}\,; \\ \gamma_{A_5} &= Y_\varphi\,K(\varphi - M)\,Y_{-\varphi}\,,\ \textit{where}\ K(\tau) = -\tau\tilde{\Lambda}(\tau)\,,\ \tau \in \mathbf{R}\ \ \textit{and} \\ \gamma_{A_6} &= Y_\varphi\,L(\varphi - M)\,Y_{-\varphi}\,,\ \textit{where}\ L(\tau) = D_x\tilde{\Lambda}(\tau)\,,\ \tau \in \mathbf{R}\,. \end{aligned} \tag{2.12}$$

Furthermore, γ restricted to the C^*-algebra $\mathcal{C_P}^\diamond$, generated by the operators in (2.1) except $b \in \mathbf{CS}(\mathbf{R})$, is an isometry.

Proof: Let us calculate γ, defined in (2.11), for the generators $A_1, \ldots, A_6$ of (2.1). By Proposition 2.2 , it is enough to calculate the result of a left multiplication by A_p, $p = 1, \ldots, 6$, on operators $E \in \mathcal{E_P}$ such that $F^{-1}EF$ are of the form $c(D)K(\tau)T_l$, $c \in \mathbf{CS}(\mathbf{R})$, $K \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_\mathbf{B})$ and $l \in \mathbf{Z}$. For such an E, we get $F^{-1}(A_pE)F$, $p = 1, \ldots, 6$, equal to, modulo compact operators,

$$\begin{aligned} &c(D)a(x)K(\tau)T_l\,,\quad (cb)(D)K(\tau)T_l\,,\quad c(D)K(\tau + j)T_{j+l}\,, \\ &c(D)\tilde{\Lambda}(\tau)K(\tau)T_l\,,\quad -c(D)\tau\tilde{\Lambda}(\tau)K(\tau)T_l \ \text{ and } \ c(D)D_x\tilde{\Lambda}(\tau)K(\tau)T_l\,, \end{aligned}$$

respectively. Here we have used (2.3) and

$$[c(D), B_k] \in \mathcal{K}(\mathcal{H})\ ,.\ k = 4,\, 5,\, 6$$

(c. f. [3], Proposition 1.2). By Theorem 2.5, we get :

$$\begin{aligned} \gamma_{A_1E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi\tilde{K}(\varphi - M)Y_{-\varphi-l} = a(x)\gamma_E(\varphi, \pm 1)\ \ (\tilde{K}(\tau) = a(x)K(\tau))\,, \\ \gamma_{A_2E}(\varphi, \pm 1) &= (cb)(\pm\infty)Y_\varphi K(\varphi - M)Y_{-\varphi-l} = b(\pm\infty)\gamma_E(\varphi, \pm\infty)\,, \\ \gamma_{A_3E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi K(\varphi + j - M)Y_{-\varphi-j-l} = Y_j\gamma_E(\varphi, \pm 1)\,, \\ \gamma_{A_4E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi(\tilde{\Lambda}K)(\varphi - M)Y_{-\varphi-l} = Y_\varphi\tilde{\Lambda}(\varphi - M)Y_{-\varphi}\gamma_E(\varphi, \pm 1) \end{aligned}$$

and analagously for $p = 5$ and 6. This proves formulas (2.12).

For any $A \in \mathcal{C_P}$ such that $F^{-1}AF = J(\tau) \in \mathbf{CO}(\mathbf{R}, \mathcal{K}_\mathbf{B})$, it is also clear, using (2.5), that

$$\gamma_A(\varphi, \pm 1) = Y_\varphi J(\varphi - M)Y_{-\varphi}\,.$$

Hence, by (2.4), $\gamma_{A_6^\bullet}$ also belongs to $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$.

The norm of the operator of $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}}))$ given by multiplication by a function in $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$ is equal to the sup-norm of this function. In other words, the C^*-algebra $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$ is isometrically imbedded in $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbf{Z}\times\mathbf{B}}))$. As the image of a dense subalgebra of $\mathcal{C}_\mathcal{P}$ is contained in $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$, we conclude that γ maps $\mathcal{C}_\mathcal{P}$ into $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$.

Using the identifcation

$$L^2(\mathbb{S}^1)\bar{\otimes}L^2(\mathbf{Z})\bar{\otimes}L^2(\mathbf{B}) = L^2(\mathbb{S}^1, L^2(\mathbf{Z}\times\mathbf{B}))$$

(see Proposition A.1), it can be straightforwardly verified that, for $A(\tau) \in \mathbf{CB}(\mathbf{R}, \mathcal{L}_\mathbf{B})$, $WA(\tau)W^{-1} \in \mathbf{C}(\mathbb{S}^1, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$ and it is given by $Y_\varphi A(\varphi - M)Y_{-\varphi}$. This means that for $k = 1, 4, 5, 6$, we have

$$\gamma_{A_k} = WF^{-1}A_kFW^{-1} \quad \text{and} \quad \gamma_{A_k^*} = WF^{-1}A_k^*FW^{-1} .$$

It is also clear that $WT_jW^{-1} = Y_{-j}$ and, hence,

$$\gamma_A = WF^{-1}A(WF^{-1})^{-1} , \quad \text{for} \quad A \in \mathcal{C}_\mathcal{P}{}^\diamond ,$$

proving that

$$\| \gamma_A \|_{\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})} = \| A \|_{\mathcal{L}(\mathcal{H})} \quad \text{and} \quad \gamma_{A^*} = (\gamma_A)^* \quad \text{for} \quad A \in \mathcal{C}_\mathcal{P}{}^\diamond .$$

This finishes the proof, since it is obvious that $\gamma_{A_2^*} = (\gamma_{A_2})^*$. q.e.d.

The σ-symbol and the γ-symbol, defined in Theorem 2.7 and Proposition 2.9 respectively, are related by:

Proposition 2.10 *For every $A \in \mathcal{C}_\mathcal{P}$, $\| \sigma_A |_{\mathbf{M}_P \setminus \mathbf{W}_P} \| \leq \| \gamma_A \|$, i. e.,*

$$\sup\{ \mid \sigma_A((t,x;\tau,\xi), e^{i\theta}) \mid ; \ \mid t \mid = \infty \} \leq \sup\{ \| \gamma_A(m) \|_{\mathcal{L}_{\mathbf{Z}\times\mathbf{B}}} ; \ m \in \mathbf{M}_{SL} \} .$$

Proof: Since the commutators of A_2 with the other generators in (2.1) and their adjoints are compact (c. f. [3], Proposition 1.2), the set of operators of the form

$$A = \sum_{j=1}^{N} b_j(t)A_j + K , \ b_j \in \mathbf{CS}(\mathbf{R}), \ A_j \in \mathcal{C}_\mathcal{P}{}^\diamond , \ K \in \mathcal{K}(\mathcal{H}), \ N \in \mathbf{N} , \tag{2.13}$$

is dense in $\mathcal{C}_\mathcal{P}$. As $\sigma_K = 0$ and $\gamma_K = 0$ for $K \in \mathcal{K}(\mathcal{H})$, it suffices to assume A of the form (2.13) with $K = 0$.

For such an A, Theorem 2.7 implies:

$$\sigma_A((t,x;\tau,\xi), e^{i\theta}) = \sum_{j=1}^{N} b_j(t)\sigma_{A_j}((t,x;\tau,\xi), e^{i\theta}) .$$

Letting $A^{\pm}$ denote the operators $\sum_{j=1}^{N} b_j(\pm\infty)A_j$, it is clear then that

$$\sigma_A((+\infty,x;\tau,\xi),e^{i\theta}) = \sigma_{A^+}((\pm\infty,x;\tau,\xi),e^{i\theta}) \quad \text{and}$$
$$\sigma_A((-\infty,x;\tau,\xi),e^{i\theta}) = \sigma_{A^-}((\pm\infty,x;\tau,\xi),e^{i\theta}) \ ,$$

hence:

$$\| \sigma_A |_{\mathbf{M}_P\setminus\mathbf{W}_P} \| \leq \max\{ \| \sigma_{A^+} \| \ , \ \| \sigma_{A^-} \| \} \ . \tag{2.14}$$

The map $\sigma : \mathcal{C}_{\mathcal{P}} \to \mathbf{C}(\mathbf{M}_P)$ was defined as the composition of the Gelfand map (an isometry) with the canonical projection $\mathcal{C}_{\mathcal{P}} \to \mathcal{C}_{\mathcal{P}}/\mathcal{K}(\mathcal{H})$. It then follows that

$$\| \sigma_{A^{\pm}} \| \leq \| A^{\pm} \| \ .$$

As $A^{\pm} \in \mathcal{C}_{\mathcal{P}}{}^{\diamond}$, where γ is an isometry,

$$\| \sigma_{A^{\pm}} \|_{\mathbf{C}(\mathbf{M}_P)} \leq \| \gamma_{A^{\pm}} \|_{\mathbf{C}(\mathbf{M}_{SL},\mathcal{L}_{2\times 2})} \ . \tag{2.15}$$

By Proposition 2.9 ,

$$\gamma_A(\varphi,+1) = \sum_{j=1}^{N} b_j(+\infty)\gamma_{A_j}(\varphi,+1) = \gamma_{A^+}(\varphi,+1) \quad \text{and} \quad \gamma_A(\varphi,-1) = \gamma_{A^-}(\varphi,-1) \ .$$

Furthermore, for any $A \in \mathcal{C}_{\mathcal{P}}{}^{\diamond}$, it is clear from (2.12) that $\gamma_A(\varphi,+1) = \gamma_A(\varphi,-1)$ and, therefore,

$$\| \gamma_A \| = \max \{ \| \gamma_{A^+} \| \ , \ \| \gamma_{A^-} \| \} \tag{2.16}$$

We are finished by (2.14), (2.15) and (2.16). q.e.d.

If $\gamma_A = 0$, then, $\sigma_A |_{\mathbf{M}_P\setminus\mathbf{W}_P} = 0$. The converse is also true:

Proposition 2.11 *An operator $A \in \mathcal{C}_{\mathcal{P}}$ belongs to the kernel of γ if and only if σ_A vanishes on $\mathbf{M}_P\setminus\mathbf{W}_P$. Furthermore, we have:*

$$\ker\gamma \cap \ker\sigma = \mathcal{K}(\mathcal{H}) \ . \tag{2.17}$$

Proof: Let $\mathcal{J}_o$ denote the C*-algebra generated by multiplications by functions in $\mathbf{C}_o^{\infty}(\Omega)$ and by the operators of the form $D\Lambda$, where D is a first order linear differential operator on Ω with smooth coefficients of compact support. Given A_o, one of these generators just described, we can find $\chi \in \mathbf{C}_o^\infty(\mathbf{R})$ such that $\chi A_\circ = A_\circ$ and then $\gamma_{A_\circ} = \gamma_\chi \gamma_{A_\circ} = 0$, by Proposition 2.9 . So, we have $\mathcal{J}_\circ \subseteq \ker \gamma$.

Using the nomenclature of [2], $\mathcal{J}_\circ$ is the minimal comparison algebra associated with the triple $\{\Omega, dS, H\}$. It can be easily concluded from [2] , Lemma VII-1-2, that $A \in \mathcal{C}_\mathcal{P}$ belongs to $\mathcal{J}_\circ$ if and only if σ_A vanishes on $\mathbf{M}_P \backslash \mathbf{W}_P$, proving that $\mathcal{J}_\circ \supseteq \ker \gamma$, by Proposition 2.10 .

Since $\ker \sigma = \mathcal{E}_\mathcal{P}$ and $\ker \gamma = \mathcal{J}_\circ$, the equality in (2.17) follows from [2], Theorem VII-1-3. q.e.d.

2.3 A Fredholm criterion and an application to differential operators

As we did for the algebra $\mathcal{A}$ in Chapter 1, we will now give a necessary and sufficient criterion for an $N \times N$-matrix whose entries are operators in $\mathcal{C_P}$, regarded as a bounded operator on $L^2(\Omega, \mathbf{C}^N)$, $N \geq 1$, to be Fredholm. Let us denote $L^2(\Omega, \mathbf{C}^N)$ by $\mathcal{H}^N$ and by ${\mathcal{C_P}}^N$ the C*-subalgebra of $\mathcal{L}(\mathcal{H}^N)$

$$ {\mathcal{C_P}}^N := \{\ ((A_{jk}))\ ;\ A_{jk} \in \mathcal{C_P},\ 1 \leq j,k \leq N\ \}\ . $$

It is easy to see that the compact ideal of $\mathcal{L}(\mathcal{H}^N)$ coincides with the matrices with entries in $\mathcal{K}(\mathcal{H})$, i. e. ,

$$ \mathcal{K}(\mathcal{H}^N) \ =\ \mathcal{K}^N \ :=\ \{\ ((K_{jk})\ ;\ K_{jk} \in \mathcal{K}(\mathcal{H}),\ 1 \leq j,k \leq N\}\ . $$

Let us define two symbols on ${\mathcal{C_P}}^N$:

$$ \sigma_A^N \ =\ ((\sigma_{A_{jk}}))_{1\leq j,k\leq N} \quad \text{and} \quad \gamma_A^N \ =\ ((\gamma_{A_{jk}}))_{1\leq j,k\leq N}\ , $$

where $A = ((A_{jk}))_{1\leq j,k\leq N}\ \in {\mathcal{C_P}}^N$. The following propoition follows immediately from the definitions above and Proposition 2.11.

Proposition 2.12 *The γ^N-symbol of an operator $A \in {\mathcal{C_P}}^N$ is identically zero if and only if its σ^N-symbol vanishes on $\mathbf{M}_P \backslash \mathbf{W}_P$. Furthermore, we have:*

$$ \ker \sigma^N \cap \ker \gamma^N \ =\ \mathcal{K}^N $$

Theorem 2.13 *For an operator $A = ((A_{jk}))_{1\leq j,k\leq N} \in {\mathcal{C_P}}^N$ to be Fredholm, it is necessary and sufficient that :*

(i) σ_A^N be invertible, i. e. , the $N \times N$-matrix $((\sigma_{A_{jk}}(m)))$ be invertible for all $m \in \mathbf{M}_P$, and

(ii) γ_A^N be invertible, i. e. , the $N \times N$-matrix, with entries in $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbf{Z}\times\mathbf{B}})$, $((\gamma_{A_{jk}}(m)))$ be invertible for all $m \in \mathbf{M}_{SL}$.

Proof: The proof of Theorem 1.14 applies here, substituting Proposition 1.9 by Proposition 2.9 and Proposition 1.13 by Proposition 2.12 . The other small changes required are obvious. q.e.d.

In order to apply this result to differential operators, it is convenient to conjugate the γ-symbol with the discrete Fourier transform. We define:

$$\begin{aligned} \Gamma: \ \mathcal{C}_{\mathcal{P}} &\longrightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{S}^1\times\mathbf{B}}) \\ A &\longmapsto \Gamma_A(m) = F_d^{-1}\gamma_A(m)F_d\ , \quad m \in \mathbf{M}_{SL}\ , \end{aligned} \tag{2.18}$$

where $F_d : L^2(\mathbb{S}^1) \to L^2(\mathbb{Z})$, $\mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbb{R}\}$, was defined in (1.5), and, as usual, F_d also denotes $F_d \otimes I_{\mathbf{B}}$ (see Appendix).

Next we calculate Γ_A for the generators of $\mathcal{C}_{\mathcal{P}}$. It is obvious that, for $a \in \mathbf{C}^\infty(\mathbf{B})$,

$$\Gamma_a(\varphi, \pm 1) = a\ , \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}\ , \tag{2.19}$$

and, for $b \in \mathbf{CS}(\mathbb{R})$,

$$\Gamma_b(\varphi, \pm 1) = b(\pm\infty)\ , \quad \text{independent of } \varphi\ . \tag{2.20}$$

For $j \in \mathbb{Z}$, $F_d^{-1}Y_{-j}F_d$ equals the operator multiplication by $e^{ij\theta}$ on $\mathbb{S}^1 = \{e^{i\theta}, \theta \in \mathbb{R}\}$, and then, by (2.18) and (2.12),

$$\Gamma_{e^{ijt}}(\varphi, \pm 1) = e^{ij\theta}\ , \quad \text{for all} \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}\ . \tag{2.21}$$

Let $a \in \mathbf{C}(\Omega)$ be of the form

$$a(t,x) = a_+(t,x)\chi_+(t,x) + a_-(t,x)\chi_-(t,x) + a_\circ(t,x)\ , \tag{2.22}$$

where $a_\pm$ are continuous and 2π-periodic in t, $a_\circ \in \mathbf{CO}(\Omega)$ and $\chi_\pm \in \mathbf{CS}(\mathbb{R})$ satisfy $\chi_\pm(\pm\infty) = 1$, $\chi_+ + \chi_- = 1$. By the continuity of Γ, (2.19), (2.20) and (2.21), it follows that

$$\Gamma_a(\varphi, \pm 1) = a_\pm(\theta, x)\ , \quad \text{for} \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}\ . \tag{2.23}$$

Note that (2.22) gives Γ_{A_1}, Γ_{A_2} and Γ_{A_3}, for A_p as defined on page 36 .

Now we calculate $F_d^{-1}K(\varphi - M)F_d$, for $\varphi \in \mathbb{R}$ and $K(\tau) = \tilde{\Lambda}(\tau)$, $-\tau\tilde{\Lambda}(\tau)$ or $D_x\tilde{\Lambda}(\tau)$, which is needed for obtaining Γ_{A_p}, $p = 4,5,6$. Let us use again that $-\Delta_{\mathbf{B}}$ has an orthonormal basis of eigenfunctions w_m, $m \in \mathbb{N}$, with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \ldots$, $\lambda_m \to \infty$ as $m \to \infty$, and define the unitary map

$$\begin{aligned} U:\ L^2(\mathbf{B}) &\longrightarrow L^2(\mathbb{N}) \\ u &\longmapsto (w_m, u)_{m\in\mathbb{N}}\ . \end{aligned}$$

By the spectral theorem, the conjugate $U(1+(\varphi-j)^2-\Delta_{\mathbf{B}})^{-1/2}U^{-1}$ equals the operator multiplication by $(1+(\varphi-j)^2+\lambda_m)^{-1/2}$ on $L^2(\mathbf{N})$, for each $j\in\mathbf{Z}$, $\varphi\in\mathbf{R}$. The operator $\tilde{\Lambda}(\varphi-M)\in\mathcal{L}_{\mathbf{Z}\times\mathbf{B}}$ acts on

$$\mathsf{u}=(u_j)_{j\in\mathbf{Z}}\in L^2(\mathbf{Z};L^2(\mathbf{B}))$$

by

$$\tilde{\Lambda}(\varphi-M)\mathsf{u}=(\,(1+(\varphi-j)^2+\Delta_{\mathbf{B}})^{-1/2})u_j\,)_{j\in\mathbf{Z}}$$

and, thus,

$$(I_{\mathbf{Z}}\otimes U)\tilde{\Lambda}(\varphi-M)(I_{\mathbf{Z}}\otimes U)^{-1}\;=\;(1+(\varphi-j)^2+\lambda_m)^{-1/2}\;, \tag{2.24}$$

where, by $(1+(\varphi-j)^2+\lambda_m)^{-1/2}$, we now mean the corresponding multiplication operator on $L^2(\mathbf{Z})\bar{\otimes}L^2(\mathbf{N})$.

Let us adopt the notation:

$$1+(\varphi-D_\theta)^2-\Delta_{\mathbf{B}}\;:=\;(F_d\otimes U)^{-1}(1+(\varphi-j)^2+\lambda_m)(F_d\otimes U)\;. \tag{2.25}$$

It is easy to see that $1+(\varphi-D_\theta)^2-\Delta_{\mathbf{B}}$ is the unique self-adjoint realization of the differential expression $1+(\varphi+i\frac{\partial}{\partial\theta})^2-\Delta_{\mathbf{B}}$ on $\mathbb{S}^1\times\mathbf{B}$ (see Lemma 2.14). By (2.24) and (2.25) then, we obtain:

$$(F_d\otimes I_{\mathbf{B}})^{-1}\tilde{\Lambda}(\varphi-M)(F_d\otimes I_{\mathbf{B}})\;=\;(1+(\varphi-D_\theta)^2-\Delta_{\mathbf{B}})^{-1/2}\;, \tag{2.26}$$

for every $\varphi\in\mathbf{R}$. Using that $Y_\varphi=F_d^{-1}e^{-i\varphi\theta}F_d$, $\varphi\in\mathbf{R}$ and (2.12), it follows that:

$$\Gamma_\Lambda(\varphi,\pm1)=e^{-i\varphi\theta}(1+(D_\theta-\varphi)^2-\Delta_{\mathbf{B}})^{-1/2}e^{i\varphi\theta}\;,\quad(e^{2\pi i\varphi},\pm1)\in\mathbf{M}_{SL}\;. \tag{2.27}$$

Since, for each $j\in\mathbf{Z}$ and each $\varphi\in\mathbf{R}$, $U(\varphi-j)(1+(\varphi-j)^2-\Delta_{\mathbf{B}})^{-1/2}U^{-1}$ equals the operator multiplication by $(\varphi-j)(1+(\varphi-j)^2+\lambda_m)^{-1/2}$ on $L^2(\mathbf{N})$, we obtain, in a way analogous to how (2.27) was obtained:

$$\Gamma_{A_4}(\varphi,\pm1)=e^{-i\varphi\theta}(D_\theta-\varphi)(1+(D_\theta-\varphi)^2-\Delta_{\mathbf{B}})^{-1/2}e^{i\varphi\theta}\;,\quad(e^{2\pi i\varphi},\pm1)\in\mathbf{M}_{SL}\;. \tag{2.28}$$

Here we have assumed the notation:

$$(\varphi-D_\theta)(1+(\varphi-D_\theta)^2-\Delta_{\mathbf{B}})^{-1/2}:=(F_d\otimes U)^{-1}(\varphi-j)(1+(\varphi-j)^2+\lambda_m)^{-1/2}(F_d\otimes U)\;.$$

For the last type of generator, we need the following lemma.

Lemma 2.14 *The subspace*

$$\{ u \in L^2(\mathbb{S}^1 \times \mathbf{B});\ (1 + (\varphi - D_\theta)^2 - \triangle_{\mathbf{B}})^{-1/2} u \ \in\ \mathbf{C}^\infty(\mathbb{S}^1 \times \mathbf{B}) \}$$

is dense in $L^2(\mathbb{S}^1 \times \mathbf{B})$*, for every* $\varphi \in \mathbf{R}$.

Proof: The statement is true for $\varphi = 0$, since

$$1 + {D_\theta}^2 - \triangle_{\mathbf{B}} = 1 + \triangle_{\mathbb{S}^1 \times \mathbf{B}}$$

is essentially self-adjoint on $\mathbf{C}^\infty(\mathbb{S}^1 \times \mathbf{B})$, by [2], Theorem IV-1-8, for example. For $\varphi \in \mathbf{R}$,

$$(1 + (\varphi - D_\theta)^2 - \triangle_{\mathbf{B}})^{-1/2}(1 + {D_\theta}^2 - \triangle_{\mathbf{B}})^{1/2}$$

is a Banach-space isomorphism, since it is unitarily equivalent to the multiplication by the function on $\mathbf{Z} \times \mathbf{N}$

$$(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}(1 + j^2 + \lambda_m)^{-1/2} ,$$

which is bounded and bounded away from zero. q.e.d.

For every $v \in \mathbf{C}^\infty(\mathbb{S}^1 \times \mathbf{B})$, it is clear that

$$D_x F_d v = F_d D_x v ,$$

where, on the right-hand side, D_x is regarded as a differential expression on $\mathbb{S}^1 \times \mathbf{B}$ and, on the left-hand side, D_x acts, as a differential operator on $\mathbf{B}$, on each component $w_j \in \mathbf{C}^\infty(\mathbf{B})$ of

$$w = (w_j)_{j \in \mathbf{Z}} = F_d v \in L^2(\mathbf{Z}; L^2(\mathbf{B})) .$$

By Lemma 2.14, it therefore follows that

$$F_d[D_x(1 + (\varphi - D_\theta)^2 - \triangle_{\mathbf{B}})^{-1/2}]F_d^{-1} = D_x[F_d(1 + (\varphi - D_\theta)^2 - \triangle_{\mathbf{B}})^{-1/2} F_d^{-1}] . \qquad (2.29)$$

The right-hand side of (2.29) equals $D_x\tilde{\Lambda}(\varphi - M)$, by (2.26) . We have, hence:

$$\Gamma_{A_6}(\varphi, \pm 1) = e^{-i\varphi\theta}[D_x(1 + (\varphi - D_\theta)^2 - \triangle_{\mathbf{B}})^{-1/2}]e^{i\varphi\theta} . \qquad (2.30)$$

Equations (2.23), (2.25), (2.26), (2.27), (2.28) and (2.30) prove:

Proposition 2.15 *The map Γ defined in (2.18) is given on the generators of $\mathcal{C}_{\mathcal{P}}$ (with $m = (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}$ and $\Gamma_A(\varphi, \pm 1) \in \mathcal{L}_{\mathbb{S}^1 \times \mathbf{B}}$, $\mathbb{S}^1 = \{e^{i\theta};\ \theta \in \mathbb{R}\}$) by:*

$$\Gamma_a(\varphi, \pm 1) = a_{\pm}(\theta, x)\ , \quad \textit{for } a \textit{ as in (2.22)}$$
$$\Gamma_\Lambda(\varphi, \pm 1) = e^{-i\varphi\theta}(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbf{B}})^{-1/2} e^{i\varphi\theta}$$
$$\Gamma_{-i\frac{\partial}{\partial t}\Lambda}(\varphi, \pm 1) = e^{-i\varphi\theta}(D_\theta - \varphi)(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbf{B}})^{-1/2} e^{i\varphi\theta}$$
$$\Gamma_{D_x\Lambda}(\varphi, \pm 1) = e^{-i\varphi\theta} D_x (1 + (D_\theta - \varphi)^2 - \Delta_{\mathbf{B}})^{-1/2} e^{i\varphi\theta}\ .$$

Remark 2.16 *Because of the way Γ was defined, it is obvious that condition (ii) of Theorem 2.13 can be replaced by*

(ii') The matrix $\Gamma_A^N(m) := ((\Gamma_{A_{jk}}(m)))_{1 \le j,k \le N}$ is invertible for all $m \in \mathbf{M}_{SL}$.

Our next and final objective is to find necessary and sufficient conditions for a differential operator with semi-periodic coefficients on Ω to be Fredholm. Most of the ideas and proofs in what follows are borrowed from [2], Sections VII.3 and IX.3, where the more general problem of finding differential expressions within reach of a Comparison Algebra is addressed.

Proposition 2.17 *Let L be an M-th order differential expression on $\mathbf{B}$, with smooth coefficients. The operator $L\Lambda^M$, defined initially on the dense subspace $\Lambda^{-M}(\mathbf{C}_o^\infty(\Omega))$, can be extended to a bounded operator A in $\mathcal{L}(\mathcal{H})$. Moreover, we have that $A \in \mathcal{C}_{\mathcal{P}}$, σ_A coincides with the principal symbol of L on $\mathbf{W}_P$ (points of $\mathbf{M}_P$ over $|t| < \infty$) and*

$$\Gamma_A(\varphi, \pm 1) = e^{-i\varphi\theta} L (1 + (D_\theta - \varphi)^2 - \Delta_{\mathbf{B}})^{-1/2} e^{i\varphi\theta}\ , \quad (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}\ .$$

Proof: It is easy to see that any M-th order differential expression on a compact manifold equals a sum of products of at most M first-order differential expressions. (See, for example, the proof of Proposition VI-3-1 of [2].) It is therefore enough to consider L of the form

$$L = D_1 D_2 \ldots D_M\ ,$$

where D_j, $j = 1, \ldots, M$, are first order expressions. For $M = 1$, the proposition is true by Theorem 2.7 and Proposition 2.15.

Using that $\Lambda^2 = H^{-1}$, $H = 1 - \Delta_{\mathbb{R}} - \Delta_{\mathbf{B}}$, it is easy to see that, for $u \in \Lambda^{-2}(\mathbf{C}_o^\infty(\Omega))$, and D_1 and D_2 first order expressions, we have:

$$D_1 D_2 \Lambda^2 u = D_1 \Lambda^2 D_2 u + D_1 \Lambda^2 [H, D_2] \Lambda^2 u\ . \qquad (2.31)$$

The commutator $[H, D_2]$ is a second order expression on $\mathbf{B}$ and can therefore be expressed as a sum of products of at most two first order differential expressions:

$$[H, D_2] = \sum_{j=1}^{p} F_j G_j \ .$$

This shows that, on the dense subspace $\Lambda^{-2}(\mathbf{C}_o^\infty(\Omega))$, $D_1 D_2 \Lambda^2$ equals the operator

$$(D_1\Lambda)(D_2^*\Lambda)^* + (D_1\Lambda)\sum_{j=1}^{p}(F_j^*\Lambda)^*(G_j\Lambda)\Lambda \ \in \ \mathcal{C}_{\mathcal{P}} \ ,$$

where D^* denotes the formal adjoint of a differential expression D.

Since $\sigma_\Lambda = 0$, we get:

$$\sigma_{D_1 D_2 \Lambda^2} = \sigma_{D_1\Lambda}\bar{\sigma}_{D_2^*\Lambda} \ ,$$

which, restricted to $\mathbf{W}_P$, coincides with the principal symbol of $D_1 D_2$, by Theorem 2.7. It also follows that:

$$\Gamma_{D_1 D_2 \Lambda^2} = \Gamma_{D_1\Lambda}\Gamma^*_{D_2^*\Lambda} + \Gamma_{D_1\Lambda}\sum_{j=1}^{p}\Gamma^*_{F_j^*\Lambda}\Gamma_{G_j\Lambda}\Gamma_\Lambda \ .$$

By Proposition 2.15, we get:

$$\begin{aligned} e^{i\varphi\theta}\Gamma_{D_1 D_2\Lambda^2}(\varphi, \pm 1)e^{-i\varphi\theta} &= (D_1\Lambda_\varphi)(D_2^*\Lambda_\varphi)^* + D_1\Lambda_\varphi\sum_{j=1}^{p}(F_j^*\Lambda_\varphi)^*(G_j\Lambda_\varphi)\Lambda_\varphi \\ &= D_1\Lambda_\varphi^2 D_2 + D_1\Lambda_\varphi^2\sum_{j=1}^{p}F_j G_j\Lambda_\varphi^2 \ , \end{aligned}$$

where $\Lambda_\varphi = H_\varphi^{-1/2}$, $H_\varphi = 1 + (D_\theta - \varphi)^2 - \Delta_{\mathbf{B}}$. Since $[H, D_2]$ and $[H_\varphi, D_2]$ are equal (as expressions on $\mathbf{B}$), we get :

$$e^{i\varphi\theta}\Gamma_{D_1 D_2\Lambda^2}(\varphi, \pm 1)e^{-i\varphi\theta} = D_1\Lambda_\varphi^2 D_2 + D_1\Lambda_\varphi^2[H_\varphi, D_2]\Lambda_\varphi^2 = D_1 D_2\Lambda_\varphi^2$$

proving the proposition for $L = D_1 D_2$.

Suppose now that the proposition is true for sums of products of at most M first order differential expressions and let $L = D_1 D_2 \ldots D_{M+1}$ be a product of first order expressions. Define: $F = D_1 D_2$ and $G = D_3 \ldots D_{M+1}$. Using the formula

$$L\Lambda^{M+1}u = F\Lambda^2 G\Lambda^{M-1}u + F\Lambda^2[H, G]\Lambda^{M+1}u \ , \quad u \in \Lambda^{-M-1}(\mathbf{C}_o^\infty(\Omega)) \ ,$$

the proposition follows for this L, by the same argument as above. q.e.d.

Let $\{U_\beta\}$ be a finite atlas on $\mathbb{B}$ and $\{\phi_\beta\}$ a subordinate partition of unity, i. e. support $\phi_\beta \subset U_\beta$. Let L be a differential operator on Ω, acting on $\mathbb{C}^N$-valued functions, locally given on U_β by

$$L = \sum_{j=0}^{\tilde{M}} \sum_{|\alpha|\le M_j} A_{\beta,j,\alpha}(t,x)(\frac{1}{i}\frac{\partial}{\partial x})^\alpha(\frac{1}{i}\frac{\partial}{\partial t})^j , \tag{2.32}$$

where $(\frac{1}{i}\frac{\partial}{\partial x})^\alpha := (-i\frac{\partial}{\partial x_1})^{\alpha_1}\dots(-i\frac{\partial}{\partial x_n})^{\alpha_n}$, for $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. We will say that L has *semi-periodic coefficients* if the matrices

$$\tilde{A}_{\beta,j,\alpha}(t,x) := \phi_\beta(x)A_{\beta,j,\alpha}(t,x) ,$$

regarded as functions on Ω, have as entries functions of the type (2.22). It is easy to see that this definition is independent of the choice of atlas on $\mathbb{B}$. We want to decide when

$$L : H^M(\Omega,\mathbb{C}^N) \longrightarrow L^2(\Omega,\mathbb{C}^N)$$

is a Fredholm operator, assuming that L has semi-periodic coefficients. Here M denotes the order of L, $M = \max\{M_j + j,\ j = 1,\dots,\tilde{M}\}$.

We also denote by Λ the operator $\Lambda\otimes I_N$ on $\mathcal{L}(L^2(\Omega,\mathbb{C}^N))$, where I_N denotes the $N\times N$ identity matrix. Since Λ commutes with $\frac{\partial}{\partial t}$ and $L = \sum L_\beta$, for $L_\beta := \phi_\beta L$, we get:

$$L\Lambda^M = \sum_{\beta,j,\alpha} \tilde{A}_{\beta,j,\alpha}(t,x)\,(\frac{1}{i}\frac{\partial}{\partial x})^\alpha\Lambda^{|\alpha|}\,(\frac{1}{i}\frac{\partial}{\partial t})^j\Lambda^j\,\Lambda^{M-|\alpha|-j} .$$

After multiplying $(\frac{1}{i}\frac{\partial}{\partial x})^\alpha$ above by $\chi_{\beta,j,\alpha} \in \mathbb{C}_o^\infty(U_\beta)$, $\chi_{\beta,j,\alpha}(x) = 1$ for x in the support of $\tilde{A}_{\beta,j,\alpha}$, we still get the same operator and $\chi_{\beta,j,\alpha}(x)(\frac{1}{i}\frac{\partial}{\partial x})^\alpha$ is now a differential expression defined on $\mathbb{B}$. We can therefore apply Proposition 2.17 and conclude that $L\Lambda^M \in \mathcal{C}_{\mathcal{P}}{}^N$. Using, moreover, that $\sigma_{\Lambda^{M-|\alpha|-j}} = 0$ for $|\alpha| + j < M$, we get:

$$\sigma_{L\Lambda^M}(t,x;\tau,\xi) = \sum_\beta \sum_{|\alpha|+j=M} \tilde{A}_{\beta,j,\alpha}(t,x)\,\xi^\alpha\tau^j , \quad |t| < \infty .$$

The right-hand side of the previous equation coincides with the principal symbol of L restricted to the co-sphere bundle of Ω. Invertibility of the σ-symbol is therefore equivalent to uniform ellipticity of L, by Remark 2.8.

The operator-valued symbol $\Gamma_{L\Lambda^M}$ is also given by Proposition 2.17 (and Proposition 2.15):

$$e^{-i\varphi\theta}\,\Gamma_{L\Lambda^M}(\varphi,\pm 1)\,e^{i\varphi\theta} = \sum_{\beta,j,\alpha} \tilde{A}^{\pm}_{\beta,j,\alpha}(\theta,x)\,(\frac{1}{i}\frac{\partial}{\partial x})^\alpha\,(\frac{1}{i}\frac{\partial}{\partial\theta} - \varphi)^j\,\Lambda^M_\varphi ,$$

where we have used that Λ_φ and $\frac{\partial}{\partial\theta}$ commute. We have denoted by $\tilde{A}^{\pm}_{\beta,j,\alpha}$ the 2π-periodic continuous functions such that

$$\tilde{A}_{\beta,j,\alpha}(t,x) - \chi_+(t)\tilde{A}^{+}_{\beta,j,\alpha}(t,x) - \chi_-(t)\tilde{A}^{-}_{\beta,j,\alpha}(t,x) \in \mathbf{CO}(\Omega) .$$

(See (2.22).)

Let $L^{\pm}_{\beta}(\varphi)$ denote the differential expressions on $\mathbb{S}^1 \times \mathbf{B}$

$$L^{\pm}_{\beta}(\varphi) := \sum_{j=0}^{\tilde{M}} \sum_{|\alpha|\leq M_j} \tilde{A}^{\pm}_{\beta,j,\alpha}(\theta,x)\,(\frac{1}{i}\frac{\partial}{\partial x})^{\alpha}\,(\frac{1}{i}\frac{\partial}{\partial\theta} - \varphi)^{j} ,$$

and define the operator

$$L^{\pm}(\varphi) := \sum_{\beta} L^{\pm}_{\beta}(\varphi) : H^M(\mathbb{S}^1 \times \mathbf{B}, \mathbf{C}^N) \longrightarrow L^2(\mathbb{S}^1 \times \mathbf{B}, \mathbf{C}^N) . \tag{2.33}$$

Since Λ_φ is an isomorphism from $L^2(\mathbb{S}^1 \times \mathbf{B}, \mathbf{C}^N)$ onto $H^M(\mathbb{S}^1 \times \mathbf{B}, \mathbf{C}^N)$, the above considerations, together with Theorem 2.13 and Remark 2.16 prove the following theorem.

Theorem 2.18 *Let L denote an M-th order differential operator on Ω of the form (2.32), with continuous semi-periodic coefficients, and let $L^{\pm}(\varphi)$ denote the differential operators on $\mathbb{S}^1 \times \mathbf{B}$ defined in (2.33). Then*

$$L : H^M(\Omega, \mathbf{C}^N) \longrightarrow L^2(\Omega, \mathbf{C}^N)$$

is Fredholm if and only if L is uniformly elliptic and $L^{\pm}(\varphi)$ are invertible for all $\varphi \in [0,1]$.

Appendix A

Review of basic facts

Here we establish notation and state without proofs basic facts that are used throughout this dissertation. (See also a list of symbols before Chapter 1.)

Our Hilbert spaces are always separable. Indeed, they are all L^2-spaces on a measure space X, where X is either a Riemmanian manifold with a finite atlas considered with its surface measure, or X is a countable set with the counting measure. By X or Y in this Appendix, we mean one of these spaces.

By $V \otimes W$, we denote the tensor product of two vector spaces V and W. By $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$, we denote the Hilbert-space tensor product of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, i. e. , the completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to its natural inner product. We also use the symbol $\otimes$ to denote the product $v \otimes w \in V \otimes W$ of $v \in V$ and $w \in W$.

Proposition A.1 *(c. f. [22] , for example) For X and Y measure spaces as above,*

$$L^2(X) \bar{\otimes} L^2(Y) = L^2(X \times Y) = L^2(X; L^2(Y)) \; , \tag{A.1}$$

where the natural identification $(u \otimes v)(x, y) = u(x)v(y)$ is assumed and $u \in L^2(X \times Y)$ is regarded as the $L^2(Y)$-valued function $x \mapsto u(x, \cdot)$.

We denote by $\mathcal{L}_X$ or $\mathcal{L}(L^2(X))$ the algebra of bounded operators on $L^2(X)$, and by $\mathcal{K}_X$ or $\mathcal{K}(L^2(X))$ its ideal of compact operators. Given $A \in \mathcal{L}_X$ and $B \in \mathcal{L}_Y$, $A \otimes B$ denotes the bounded operator of $\mathcal{L}_{X \times Y}$ defined by

$$(A \otimes B)(u \otimes v) = Au \otimes Bv \; .$$

Given two subalgebras $\mathcal{A} \subseteq \mathcal{L}_X$ and $\mathcal{B} \subseteq \mathcal{L}_Y$, $\mathcal{A} \otimes \mathcal{B}$ denotes the algebra generated by all $A \otimes B$ such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $\mathcal{A} \bar{\otimes} \mathcal{B}$ denotes the closure of $\mathcal{A} \otimes \mathcal{B}$.

Proposition A.2 *(c. f. [1], for example) We have:*

$$\mathcal{K}_{X \times Y} = \mathcal{K}_X \bar{\otimes} \mathcal{K}_Y \tag{A.2}$$

We denote by $\mathbf{CB}(X, \mathcal{A})$ the C*-algebra of bounded continuous functions on X taking values on a C*-subalgebra $\mathcal{A} \subseteq \mathcal{L}_Y$. By $\mathbf{CO}(X, \mathcal{A})$, we denote the C*-subalgebra of $\mathbf{CB}(X, \mathcal{A})$ consisting of the functions that vanish at infinity. For compact X, we simply write $\mathbf{C}(X, \mathcal{A})$.

Proposition A.3 *(See [3], for example) A function $A \in \mathbf{CB}(X, \mathcal{A})$ defines an operator in $\mathcal{L}_{X \times Y}$ by mapping $u \in L^2(X; L^2(Y))$ into $A(x)u(x)$. The norm of this operator equals the norm of A in $\mathbf{CB}(X, \mathcal{A})$.*

Proposition A.4 *The vector space generated by*

$$\{a(\tau)K; \quad a \in \mathbf{CO}(X) \quad \text{and} \quad K \in \mathcal{K}_Y\}$$

is dense in $\mathbf{CO}(X, \mathcal{K}_Y)$. Using Proposition A.3, this means that

$$\mathbf{CO}(X) \bar{\otimes} \mathcal{K}_Y = \mathbf{CO}(X, \mathcal{K}_Y) \ ,$$

where $\mathbf{CO}(X)$ is naturally interpreted as a subalgebra of $\mathcal{L}_X$.

Sketch of a Proof: Let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal basis of $L^2(Y)$ and let P_N denote the projection onto the subspace generated by $\{\phi_1, \ldots, \phi_N\}$. Then $A_N(x) := P_N A(x) P_N$ is in $\mathbf{CO}(X) \otimes \mathcal{K}_Y$ for $A \in \mathbf{CO}(X, \mathcal{K}_Y)$, and $A_N(x) \to A(x)$ in $\mathcal{K}_Y$, as $N \to \infty$, uniformly in $x \in X$. q.e.d.

Example: Suppose that $K_j(\varphi) \in \mathcal{K}_{\mathbf{B}}$, for every $\varphi \in \mathbb{R}$ and $j \in \mathbb{Z}$, where $\mathbf{B}$ is the compact manifold of Chapter 2. Furthermore, assume that, for each φ, the sequence $K^\diamond(\varphi) := (K_j(\varphi))_{j \in \mathbb{Z}}$ is bounded. By Proposition A.3, $K^\diamond(\varphi) \in \mathcal{L}_{\mathbb{Z} \times \mathbf{B}}$, for each φ. Suppose in addition that $K^\diamond$ defines an $\mathcal{L}_{\mathbb{Z} \times \mathbf{B}}$-valued continuous function on $\mathbb{S}^1 = \{e^{2\pi i \varphi}; \ \varphi \in \mathbb{R}\}$. We then get $K^\diamond \in \mathcal{L}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbf{B}}$ and

$$\| K^\diamond \|_{\mathcal{L}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbf{B}}} = \sup\{\| K_j(\varphi) \|_{\mathcal{L}_{\mathbf{B}}}; \ j \in \mathbb{Z}, \ \varphi \in \mathbb{R}\} \ .$$

It is easy to check, moreover, that, if $K_j(\varphi) \to 0$ as $j \to \infty$ for each $\varphi \in \mathbb{R}$, we get

$$K^\diamond \in \mathbf{C}(\mathbb{S}^1) \bar{\otimes} \mathcal{K}_{\mathbb{Z} \times \mathbf{B}} \ .$$

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