Multiple solutions for a higher order variational problem in conformal geometry

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Outiline of this talk

- **Q-curvature and the Paneitz operator**
  - Intro: best constant in Sobolev embeddings
  - Conformal invariance
  - Paneitz operator and Q-curvature in Riemannian manifolds
- **Multiplicity results**
  - Yamabe type invariants
  - Aubin inequality
  - A topological argument
  - Bifurcation
- **Examples (explicit computations)**
  - homogeneous fibrations
  - Hopf bundles, Berger spheres

Acknowledgements

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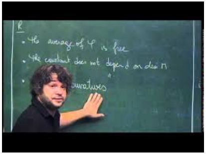

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FAPESP

Intro: Sobolev embeddings (M. Gursky)

- Sobolev embedding: $H^2(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-4}}(\mathbb{R}^n) \quad (n \geq 5)$

◇ $\|\Delta u\|^2_{L^2(\mathbb{R}^n)} \geq \lambda_n \cdot \|u\|^2_{L^{\frac{2n}{n-4}}}$

- Best λ_n known (Lions, Edmunds, Fortunato, Jannelli)
- it is attained at some radial function u satisfying:

$$\Delta^2 u = u^{\frac{n+4}{n-4}}$$

- minimizers are positive
- equation is *conformally invariant*: solutions u invariant by:
  - translations $u(x) \mapsto u(x+v)$
  - dilations $u(x) \mapsto t^{\frac{n-4}{2}} u(t \cdot x)$
  - inversions $u(x) \mapsto |x|^{4-n} \cdot u\left(\frac{x}{|x|^2}\right)$

Question

Does there exist an analogous operator in Riemannian manifolds? (conformally invariant, positive minimizers, ...)

The Paneitz operator

Theorem (Paneitz 1983)

Given $(M^n, \mathbf{g})$*,* $n \geq 5$*, there exists a differential operator* $P_{\mathbf{g}}$ *such that:*

- $P_{\mathbf{g}} = \Delta_{\mathbf{g}}^2 +$ *lower order terms of order* ≤ 2
- *if* $\widehat{\mathbf{g}} = u^{\frac{4}{n-4}} \cdot \mathbf{g}$*, then* $P_{\widehat{\mathbf{g}}}(\phi) = u^{\frac{n+4}{n-4}} P_{\mathbf{g}}(u \cdot \phi)$*.*

Examples.

- $P_{\mathbb{R}^n} = \Delta^2$
- $P_{\mathbb{S}^n} = \Delta_{\mathbb{S}^n}(\Delta_{\mathbb{S}^n} - c)$.

Definition of Q-curvature and Paneitz operator

$(M^n, \mathbf{g})$, $n \geq 5$

- Q-curvature:

$$Q_{\mathbf{g}} = c_n \cdot \Delta_{\mathbf{g}}(\mathrm{scal}_{\mathbf{g}}) + d_n \cdot \|\mathrm{Ric}_{\mathbf{g}}\|^2 + e_n \cdot \mathrm{scal}_{\mathbf{g}}^2$$

◇ $\boxed{c_n = \frac{1}{2(n-1)}}$ $\boxed{d_n = -\frac{2}{(n-2)^2}}$ $\boxed{e_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}}$

- Paneitz operator:

$$P_{\mathbf{g}}\psi = \Delta_{\mathbf{g}}^2\psi + \alpha_n \cdot \mathrm{div}_{\mathbf{g}}\big(\mathrm{Ric}_{\mathbf{g}}(\nabla\psi, e_i)e_i\big) - \beta_n \cdot \mathrm{div}_{\mathbf{g}}(\mathrm{scal}_{\mathbf{g}} \cdot \nabla\psi) + \gamma_n \cdot Q_{\mathbf{g}}\psi$$

◇ $\boxed{\alpha_n = \frac{4}{n-2}}$ $\boxed{\beta_n = \frac{n^2-4n+8}{2(n-1)(n-2)}}$ $\boxed{\gamma_n = \frac{n-4}{2}}$

Constant Q-curvature

Constant Q-curvature metric: $\mathbf{g} = u^{\frac{4}{n-4}} \cdot \mathbf{g}_0$

$$P_{\mathbf{g}_0} u = \lambda \cdot u^{\frac{n+4}{n-4}}, \qquad \lambda = \frac{n-4}{2} Q_{\mathbf{g}}.$$

Variational formulation. Solutions are critical points of associated quadratic functional:

$$E_{\mathbf{g}_0}(u) = \frac{1}{2} \int_M u \cdot P_{\mathbf{g}_0} u \, \mathrm{d}M$$

in the space:

$$\left\{ u \in H^2(M) : \|u\|_{L^{\frac{2n}{n-4}}} = \text{const.} \right\}$$

Problems:

- non-compact embedding $\Longrightarrow$ minimizing sequences may converge weakly to 0;
- minimizers may not be positive.

Yamabe problem
(constant scal curvature)

- Non-compact embedding: $H^1 \hookrightarrow L^{\frac{2n}{n-2}}$
- Conf. invariant operator: conformal Laplacian $L_{\mathbf{g}} = \Delta_{\mathbf{g}} + \frac{n-2}{4(n-1)}\mathrm{scal}_{\mathbf{g}}$
- scalar curvature $\mathrm{scal}_{\mathbf{g}}$: $\mathbf{g} = u^{\frac{4}{n-2}}\mathbf{g}_0$

  $\mathrm{scal}_{\mathbf{g}} = \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \cdot L_{\mathbf{g}_0}(u)$
- minimizers exist and have constant $\mathrm{scal}_{\mathbf{g}}$.

Constant Q-curvature

- Non-compact embedding: $H^2 \hookrightarrow L^{\frac{2n}{n-4}}$
- Conf. inv. operator: $P_{\mathbf{g}}$
- Q-curvature $Q_{\mathbf{g}}$: $\mathbf{g} = u^{\frac{4}{n-4}}\mathbf{g}_0$

  $Q_{\mathbf{g}} = \frac{2}{n-4} u^{-\frac{n+4}{n-4}} \cdot P_{\mathbf{g}_0}(u)$
- Positivity of minimizers? When so, they give constant Q-curvature.
- Very little known on the existence of positive minimizers: $\mathrm{scal}_{\mathbf{g}_0} > 0$ and $Q_{\mathbf{g}_0}$ almost positive.

Yamabe-type invariants

$$\mathcal{L}_{\mathbf{g}_0}(u) = \int_M u \cdot L_{\mathbf{g}_0} u \, \mathrm{d}M \qquad \mathcal{E}_{\mathbf{g}_0}(u) = \int_M u \cdot P_{\mathbf{g}_0} u \, \mathrm{d}M$$

Yamabe invariant: $Y(M, \mathbf{g}_0) = \inf_{u \neq 0} \frac{\mathcal{L}(u)}{\|u\|^2_{L^{\frac{2n}{n-2}}}}$

$$Y_4(M, \mathbf{g}_0) = \inf_{u \neq 0} \frac{\mathcal{E}(u)}{\|u\|^2_{L^{\frac{2n}{n-4}}}} \qquad Y_4^+(M, \mathbf{g}_0) = \inf_{u > 0} \frac{\mathcal{E}(u)}{\|u\|^2_{L^{\frac{2n}{n-4}}}}$$

Theorem (Gursky–Han–Lin, 2016)

If $\exists\, \mathbf{g} \in [\mathbf{g}_0]$ *with* $\mathrm{scal}_{\mathbf{g}} > 0$ *and* $Q_{\mathbf{g}} > 0$*, then:*

$$Y_4(M, \mathbf{g}_0) = \mathbf{Y}_4^+(M, \mathbf{g}_0)(\geq 0),$$

with infimum attained by some positive function.

Not known if $Y_4 > 0$ when $Y > 0$ and Q almost positive.

A new invariant and Aubin inequality

Aubin inequality: $\boxed{Y(M^n, \mathbf{g}_0) \leq Y(\mathbb{S}^n, \mathbf{g}_{\text{round}})}$

Theorem (Gursky–Han–Lin 2016)

Assume $Y(M, \mathbf{g}_0) > 0$ and $Q_{\mathbf{g}_0}$ almost positive. Then:

- $\mathrm{Ker}(P_{\mathbf{g}_0}) = \{0\}$ *and* $P_{\mathbf{g}_0} > 0$*;*
- *Green function* $G_{P_{\mathbf{g}_0}}$ *positive on* $M \times M$*.*

Inverse of $P_{\mathbf{g}_0}$*:* $G_{\mathbf{g}_0} f(p) = \int_M G_{P_{\mathbf{g}_0}}(p, q) f(q)\, \mathrm{d}q$

Quadratic functional: $\mathcal{G}_{\mathbf{g}_0}(f) = \int_M f \cdot G_{\mathbf{g}_0} f \, \mathrm{d}M$

New invariant: $\Theta_4(M, \mathbf{g}_0) = \sup_{f \in L^{\frac{2n}{n+4}}} \overline{\mathcal{G}}(f) = \sup_{f \in L^{\frac{2n}{n+4}}} \dfrac{\mathcal{G}_{\mathbf{g}_0}(f)}{\|f\|_{L^{\frac{2n}{n+4}}}}$

- *Supremum always attained at some smooth positive* f*,* $f^{\frac{4}{n-4}} \cdot \mathbf{g}_0$ *has constant* Q*-curvature.*
- $\Theta_4(M^n, \mathbf{g}_0) \geq \Theta_4(\mathbb{S}^n, \mathbf{g}_{round})$

Multiple constant Q-curvature metrics on spheres

Theorem

For $n \geq 5$ and $0 \leq k < \frac{n-4}{2}$, there are infinitely many pairwise nonhomothetic complete metrics with constant Q-curvature on $\mathbb{S}^n \setminus \mathbb{S}^k$ that are conformal to the round metric.

Proof.

- $\mathbb{S}^n \setminus \mathbb{S}^k$ conf. equivalent to $\mathbb{S}^{n-k-1} \times \mathbb{H}^{k+1}$
- $\mathrm{scal} = (n-2k-2)(n-1)$ $\quad Q = \frac{1}{8}n(n-2k)(n-2k-4)$ are positive when $k < \frac{n-4}{2}$;
- topological argument + Aubin inequality. □

The topological argument

1. $\mathbb{H}^{k+1}$ has compact quotients that give an infinite tower of finite-sheeted Riemannian coverings:

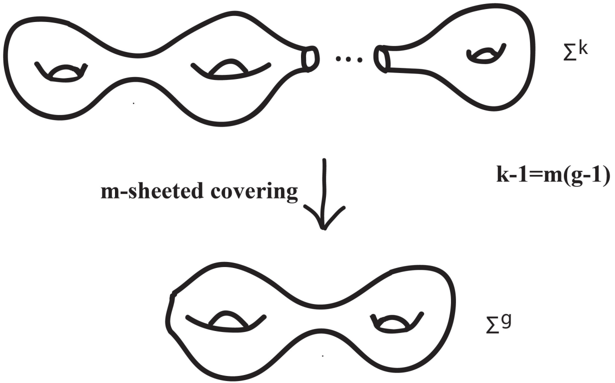
$$(\mathbb{H}^{k+1}, g_{\mathrm{hyp}}) \to \ldots \to (\Sigma_2, g_2) \to (\Sigma_1, g_1) \to (\Sigma_0, g_0)$$

2. Multiply by $(S^{n-k-1}, g_{\mathrm{round}})$, product metrics:

$$\ldots \to (S^{n-k-1} \times \Sigma_1, g_{\mathrm{round}} \oplus g_1) \to (S^{n-k-1} \times \Sigma_0, g_{\mathrm{round}} \oplus g_0)$$

3. pull-back Θ_4-metric in $\left[g_{\mathrm{round}} \oplus g_0\right]$: energy $\overline{\mathcal{G}}$ goes to 0! (uses $\mathrm{scal} > 0$ and $Q > 0$)
4. By Aubin inequality, maximum of $\overline{\mathcal{G}}$ must be attained at some other metric in the conformal class of the product.
5. Iterate.

Coverings of hyperbolic surfaces



Profinite completion and residually finite groups

Infinite tower of finite-sheeted coverings:

$$\ldots \to M_k \to M_{k-1} \to \ldots \to M_1 \to M_0$$

iff $G = \pi_1(M_0)$ has infinite *profinite completion* $\widehat{G}$.

Def. $\widehat{G} = \varprojlim G/\Gamma, \qquad \Gamma \trianglelefteq G, \quad \big[G : \Gamma\big] < +\infty.$

Canonical homomorphism $\iota : G \to \widehat{G}$

$$\mathrm{Ker}(i) = \bigcap_{\substack{\Gamma \trianglelefteq G \\ [G:\Gamma] < +\infty}} \Gamma$$

Def. G is *residually finite* if: $\displaystyle\bigcap_{\substack{\Gamma \trianglelefteq G \\ [G:\Gamma] < +\infty}} \Gamma = \{1\}$

Symmetric spaces

Theorem (Borel)

Symmetric spaces of noncompact type X admit irreducible compact quotients X/Γ.

X/Γ loc. symmetric $\implies$ constant scalar and Q-curvature

Selberg–Malcev lemma

Finitely generated linear groups are residually finite.

Corollary. $\Gamma = \pi_1(X/\Gamma)$ has infinite profinite completion.

Theorem

- $(M, \mathbf{g})$ closed, $(X, \mathbf{h})$ as above
- scal and Q-curvature of $\mathbf{g} \oplus \mathbf{h}$ positive.

Then, there are infinitely many complete constant Q-curvature metrics in $[\mathbf{g} \oplus \mathbf{h}]$.

Need a (compact) manifold M with a family $(\mathbf{g}_t)_{t\in[a,b]}$ of constant Q-curvature metrics with *computable* Morse index.

Ansatz: Riemannian submersions $\pi_t : (M, \mathbf{g}_t) \to (B, \mathbf{g}_B)$ with:

- $\mathrm{scal}_{\mathbf{g}_t}$ and $Q_{\mathbf{g}_t}$ constant;
- minimal fibers;
- horizontally Einstein: $\mathrm{Ric}_{\mathbf{g}_t} = \kappa_t \cdot \mathbf{g}_t$ on the horizontal distribution.

Typical example: Homogeneous fibration.

- $H \subset K \subset G$ compact Lie groups
- $K/H \to G/H \to G/K$ with bi-invariant metric $\mathbf{g}_1$ on G
- $\mathbf{g}_t$ obtained by rescaling metric of fibers.

A bifurcation criterion

Theorem

- *M^n closed manifold with $n \geq 5$*
- $\pi_t \colon (M, \mathbf{g}_t) \to (B, \mathbf{g}_B), \quad t \in [t_* - \varepsilon, t_* + \varepsilon]$, *a 1-parameter family of horizontally Einstein (κ_t) Riemannian submersions with minimal fibers*
- $\mathrm{scal}_{\mathbf{g}_t}$ *and $Q_{\mathbf{g}_t}$ constant for all t*
- $\alpha_t = \frac{(n^2-4n+8)\mathrm{scal}_{\mathbf{g}_t} - 8\,\kappa_t(n-1)}{4(n-1)(n-2)}, \ \beta_t = -2\,Q_{\mathbf{g}_t}.$

If for some $\lambda \in \mathrm{spec}(\Delta_{\mathbf{g}_B})$:

- $\frac{1}{2}\lambda^2 + \alpha_{t_*}\,\lambda + \beta_{t_*} = 0$
- $\alpha'_{t_*}\,\lambda + \beta'_{t_*} \neq 0,$

then t_* is a bifurcation instant for $(\mathbf{g}_t)_t$. If $\lambda \neq \frac{\mathrm{scal}_{\mathbf{g}_{t_*}}}{n-1}$, then bifurcation metrics do not have constant scalar curvature.

Berger spheres $\mathbb{S}^{2n+1}$

- Hopf bundle $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n$
- Berger metric $\mathbf{g}_t = t\,\mathbf{g}_{\mathcal{V}} + \mathbf{g}_{\mathcal{H}}$ ($\mathbf{g}_1$ unit round metric)
- homogeneous fibration corresponding to $\mathrm{U}(n) \subset \mathrm{U}(n)\mathrm{U}(1) \subset \mathrm{U}(n+1)$
- horizontal space $\mathcal{H} \cong \mathbb{C}^n$ irreducible $\mathrm{U}(n)$-representation,
- vertical space $\mathcal{V} \cong \mathbb{R}$ does not contain copies of this irreducible

Upshot

For $n \geq 6$, $\exists$ sequence $t_k \to +\infty$ bifurcation instants for $(\mathbb{S}^{2n+1}, \mathbf{g}_t)$.

If $6 \leq n \leq 9$, then infinitely many of these bifurcating branches issue from metrics on $\mathbb{S}^{2n+1}$ that have $\mathrm{scal} < 0$ and $Q < 0$.

The Berger spheres $\mathbb{S}^{4n+3}$

- Hopf bundle $\mathbb{S}^3 \to \mathbb{S}^{4n+3} \to \mathbb{H}P^n$
- Berger metric $\mathbf{g}_t = t\,\mathbf{g}_{\mathcal{V}} + \mathbf{g}_{\mathcal{H}}$ ($\mathbf{g}_1$ unit round metric)
- homogeneous fibration corresponding to $\mathrm{Sp}(n) \subset \mathrm{Sp}(n)\mathrm{Sp}(1) \subset \mathrm{Sp}s(n+1)$
- horizontal space $\mathcal{H} \cong \mathbb{H}^n$ irreducible $\mathrm{Sp}(n)$-representation,
- vertical space $\mathcal{V} \cong \mathbb{R}^3$ does not contain copies of this irreducible

Upshot

For $n \geq 2$, $\exists$ sequences $t_k \to +\infty$ and $t'_k \to 0$ of bifurcation instants for $(\mathbb{S}^{2n+3}, \mathbf{g}_t)$.

Infinitely many of these bifurcating branches issue from metrics on $\mathbb{S}^{2n+3}$ that have $\mathrm{scal} < 0$ and $Q < 0$.

Similar results for Hopf bundle $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$.

Thanks for your attention!!

See you at ICM2018!