BIFURCATION AND SYMMETRY BREAKING OF NODOIDS WITH FIXED BOUNDARY

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ABSTRACT. We prove bifurcation results for (compact portions of) nodoids in $\mathbb{R}^3$, whose boundary consists of two fixed coaxial circles of the same radius lying in parallel planes. Degeneracy occurs at an infinite discrete sequence of instants, that are divided into four classes. Different types of bifurcation and break of symmetry occur at each instant of three of the four classes; bifurcation does not occur at the degeneracy instants of the fourth class.

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1. Introduction

The classical Delaunay surfaces of revolution in $\mathbb{R}^3$ with constant mean curvature (CMC for brevity) are divided into six different types: planes, spheres, cylinders, catenoids, unduloids and nodoids, see [7]. The nodoids (Figure 2) are immersed surfaces in $\mathbb{R}^3$, obtained by rotating around a fixed line ℓ the curve traced by the focus of a hyperbola that rolls without slipping along ℓ. Such generatrix, the *nodary* (Figure 1) is a periodic immersed curve with non vanishing curvature, with loops toward the axis. The nodoids form a two-parameter family of immersions of the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$. In this paper we study bifurcation problems with symmetry breaking for CMC immersions in $\mathbb{R}^3$ of the compact cylinder $\mathcal{C} = \mathbb{S}^1 \times [-t_0, t_0]$ having fixed boundary. Let us consider two parallel horizontal planes Π_0 and Π_1 in $\mathbb{R}^3$, and let C_0, C_1 be coaxial circles of the same radius $r_* > 0$ on Π_0 and Π_1 respectively. Moreover, let Π be a plane parallel to Π_0 and Π_1 lying half way between Π_0 and Π_1 (see Figure 3). The nodoids whose intersection with Π_0 and Π_1 contain respectively C_0 and C_1 and that are symmetric with respect to reflections around Π form a real analytic 1-parameter family $\Sigma = (\Sigma_{t_0})$, with $t_0 > 0$. For all t_0, Σ_{t_0} is a CMC immersion of the cylinder in $\mathbb{R}^3$ whose symmetry group is the direct product $\mathbb{S}^1 \times \mathbb{Z}_2$, where $\mathbb{S}^1$ is the group of rotations, and the generator of $\mathbb{Z}_2$ is the reflection around the plane Π. The mean curvature $H(t_0)$ of Σ_{t_0} is a real analytic function satisfying $\lim\limits_{t_0 \to +\infty} H(t_0) = -\infty$, see Proposition 3.1. The parameterization of the family will be chosen in such a way that $\sin t_0$ is equal to the component of the unit normal to Σ_{t_0} on the boundary in the direction of the the rotation axis, see Section 3.1. Thus, when t_0 is of the form $\frac{\pi}{2} + k\pi$, with $k \in \mathbb{N}$, the corresponding nodoids Σ_{t_0} intersect Π_0 and Π_1 *tangentially* at C_0 and C_1. Figures 7 and 8 illustrate the situation. When t_0 is of the form $k\pi$, $k \geq 1$, the corresponding nodoids Σ_{t_0} intersect Π_0 and Π_1 *orthogonally* at C_0 and C_1, see for instance Figure 11. These two types of nodoids are *degenerate*, in the sense that they are degenerate critical points of the area functional subject to the constraint $\mathrm{volume} = \mathrm{const.}$ (Propositions 3.5, 3.6). The main result of the paper is that there is bifurcation of fixed boundary CMC immersions of the cylinder with *break of symmetry* at each one of these degenerate nodoids. Rotational symmetry breaks at the bifurcating branch at the instants $\frac{\pi}{2} + k\pi$, while at the instants $t_0 = k\pi$ the reflection symmetry is broken. Moreover, we determine a sequence of other degenerate instants where no bifurcation occurs. A complete statement of our result is as follows:

Theorem. *The set* $\mathrm{Deg}(\Sigma)$ *of degeneracy instants* t_0 *of the family* $\Sigma = (\Sigma_{t_0})$ *contains the set* $\left\{k\frac{\pi}{2} : k \in \mathbb{N}\right\}$ *and a sequence* $s_0 < s_1 < \ldots < s_k < \ldots$, *where* $k\pi < s_k < k\pi + \frac{\pi}{2}$ *for all* $k \geq 0$. *Moreover:*

(1) *There is a bifurcating branch of non axially symmetric CMC immersed cylinders with fixed boundary* $C_0 \bigcup C_1$ *issuing from the family* Σ *at each degenerate nodoid* Σ_{t_0}, *with* $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{N}$.

(2) *At the instants* $t_0 = k\pi$, *with* $k \geq 1$, *there is a bifurcating branch issuing from* Σ_{t_0} *that consists of fixed boundary nodoids (axially symmetric), that are not symmetric with respect to reflections around* Π.

(3) *There is no bifurcation at the degenerate instants* $t_0 = s_k$ *for all* $k \geq 0$.

(4) *There are other degenerate instants at large values of the parameter* t_0, *and at each of them bifurcation occurs by a branch of non axially symmetric CMC immersions. At some of these instants, the bifurcating branch is also not symmetric with respect to reflections around* Π. *In all cases, the bifurcating branch consists of CMC immersions of the cylinder that are invariant by rotation of an angle* $\frac{2\pi}{n}$, *with* n *arbitrarily large.*

(5) *For all nondegenerate instants, the nodoid family* Σ *is* locally rigid, *in the sense that every CMC immersion of the cylinder having boundary* $C_0 \bigcup C_1$ *which is*

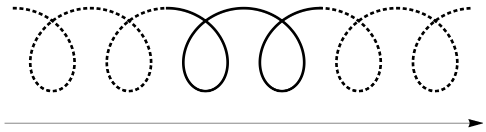

FIGURE 1. The nodary and the symmetry axis.

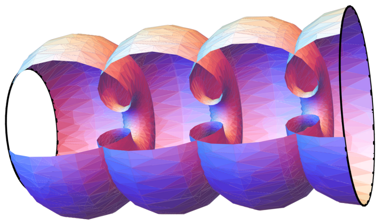

FIGURE 2. A portion of nodoid, with boundary on parallel planes orthogonal to the axis.

sufficiently close to a nodoid Σ_{t_0} with $t_0 \neq k\frac{\pi}{2}$ for all $k \geq 1$ must be isometrically congruent *to a nodoid of the family.*

There are some related results in the literature concerning bifurcation of constant mean curvature curvature submanifolds and nodoids in particular. In [17], the author uses a general bifurcation result of Crandall and Rabinowitz [6] to produce a symmetry breaking bifurcation from a family of stable nodoids. We wish to emphasize that this method only applies to bifurcation from a stable branch while here we consider a much larger class of bifurcations. In [16] the authors prove a result of bifurcation for infinite nodoids in $\mathbb{R}^3$, see also [18, 19]. The bifurcation of domains in nodoids has been studied in detail in the applied mathematics literature (see [20]) but not in a mathematically rigorous way. Again, only bifurcation from stable equilibria has been treated previously. Our objective here is to present a theoretical justification of these bifurcations in the framework of global analysis. In [2] it is proved a bifurcation result for embedded tori in spheres, the bifurcating branch issuing from the family of CMC Clifford tori with varying radius. More results on bifurcation and completeness of the bifurcating branch have been announced in [9].

The proof of our theorem uses an abstract bifurcation result of Smoller and Wasserman in [21]; for the proof of item (4) of our main theorem, we use the equivariant version of the result, see Subsection 2.8. We set up the appropriate variational framework in Section 2, where the CMC problem is cast in the language of constrained critical points. Bifurcation is then reduced to the study of the jumps of the Morse index of the nodoid immersions. Here, by Morse index we mean the index of the quadratic form given by the CMC Jacobi operator defined in the space of *all* variational vector fields vanishing on the boundary. This is what we call the *strong Morse index* of the CMC immersion.

The 2-dimensional space of horizontal translations determines two linearly independent Jacobi fields along the nodoids, that vanish on the boundary of the nodoid precisely when the nodoid is tangent to the planes Π_0 and Π_1 at its boundary. This corresponds to the instants $t_0 = \frac{\pi}{2} + k\pi$, $k \geq 0$. Thus, the corresponding CMC immersions are degenerate, i.e., the kernel of the second variation of the CMC variational problem is non trivial. We show that such degeneracy does in fact produce a jump of the Morse index. This, together with a further assumption of local injectivity of the mean curvature function, implies bifurcation, see Proposition 2.5. It is interesting to observe here that the mean curvature function *fails* to be locally injective at those degeneracy instant at which some Jacobi field has non zero

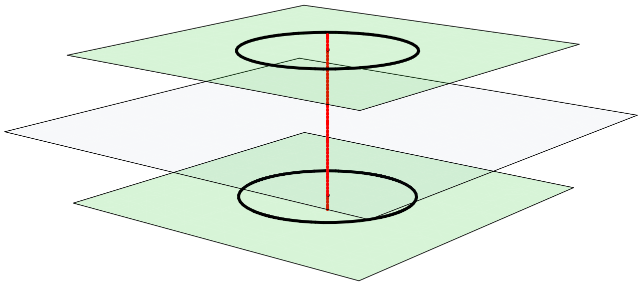

FIGURE 3. Circles C_0 and C_1 for the CMC fixed boundary problem, lying on the planes Π_0 and Π_1. In the middle, the symmetry plane Π.

average, see Subsection 2.6. This occurs, for instance, when the degeneracy is produced by the vanishing of the first eigenvalue of the Jacobi operator. Thus, bifurcation can only occur when some eigenvalue λ_k, with $k > 1$, vanishes.

As to the symmetry breaking property, this follows by a direct analysis of all possible axially invariant CMC immersions of the cylinder having fixed boundary and that are "close" to the family of nodoids. All such immersions must indeed belong to the family of nodoids, which proves that bifurcation must be realized by non axially symmetric immersions.

Similarly, the one-dimensional space of vertical translations determine a Jacobi field on each nodoid of the family, that vanishes on the boundary exactly when the normal to the surface is horizontal at the points of the boundary. This corresponds to the instants $t_0 = k\pi$, $k \geq 1$. In this case, the Jacobi field is axially symmetric, and that implies that there is a bifurcating branch of axially symmetric fixed boundary CMC immersions of the cylinders. Thus, the bifurcating branch consists of nodoids, and they are not symmetric with respect to the horizontal plane Π. We determine an explicit parameterization of each bifurcating branch in this case, see Subsection 3.5.

A third type of degeneracy instant of the nodoid family is given by those instants where the mean curvature function of the family has vanishing derivative. In this case, a non trivial Jacobi field vanishing on the boundary along the degenerate nodoid is obtained as the variational vector field of the family Σ itself, see Remark 2.11. Thus, there is no bifurcation at this type of degeneracy instants.

Finally, a fourth type of degeneracy instants is determined via Sturm–Liouville theory applied to the Jacobi equation, see Appendix A. We do not give a geometrical description of these instants, but we prove that bifurcation occurs, and that all types of symmetry break in the bifurcating branch.

The *local rigidity* of the nodoid family at nondegenerate instants is proved using the Implicit Function Theorem, see also [12, Theorem 1.1].

The paper is organized as follows. In Section 2 we discuss the abstract framework for the CMC variational problem, and we give criteria for the existence of bifurcations for families of CMC immersions. Special emphasis is given to equivariant bifurcation, which is employed in the study of axially symmetric surfaces.

In Section 3 we will describe the family of fixed boundary nodoids; the set of degenerate instants is divided into four families, and we study the existence of bifurcating branches and break of symmetry at instants of each class.

Appendix A contains a detailed study of the Jacobi field equation, using the method of separation of variables.

The bifurcation results for fixed boundary nodoids discussed in this paper can be extended to the more general case of anisotropic nodoids which arise as critical points for axially symmetric anisotropic surface energies. Assuming adequate symmetry of the functional, such an extension does not present significative differences from the standard CMC case. In Section 5 we will present briefly the construction of families of fixed boundary anisotropic nodoids, for which bifurcation results totally analogous to those for the standard nodoids hold.

2. THE VARIATIONAL FRAMEWORK

2.1. **The manifold of immersed submanifolds.** Let $\mathcal{C}$ denote the cylinder $\mathbb{S}^1 \times [0,1]$; this is a compact manifold with boundary $\partial\mathcal{C} = \mathbb{S}^1 \times \{0\} \bigcup \mathbb{S}^1 \times \{1\}$. For $k \geq 2$ and $\alpha \in \,]0,1[$, let X denote the Banach space $C^{k,\alpha}(\mathcal{C}, \mathbb{R}^3)$, consisting of all maps $x : \mathcal{C} \to \mathbb{R}^3$ of class $C^{k,\alpha}$. The set:

$$X_0 = \big\{x \in X : x \text{ is an immersion, and } x|_{\partial\mathcal{C}} \text{ is injective}\big\}$$

is an open subset of X.

Proposition 2.1. *Given any two $C^{k,\alpha}$-embeddings of the circle $c_i : \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$, $i = 0, 1$, then the subset:*

$$X_0(c_0, c_1) = \big\{x \in X_0 : x(\cdot, 0) = c_0,\ x(\cdot, 1) = c_1\big\}$$

is a non empty smooth embedded submanifold of X_0.

Proof. In order to see that the set $X_0(c_0, c_1)$ is non empty, choose any smooth map $D : \mathbb{S}^1 \times [0,1] \to \mathbb{R}^3$ such that $D(\cdot, i) = c_i$, for $i = 1, 2$, and such that D is an immersion at all points of $\mathbb{S}^1 \times \{0\} \bigcup \mathbb{S}^1 \times \{1\}$. Such a function exists, by the assumption that c_i is an embedding for $i = 1, 2$. Then, D can be approximated in the C^{k+1}-topology by smooth immersions that coincide with D in a neighborhood of $\mathbb{S}^1 \times \{0\} \bigcup \mathbb{S}^1 \times \{1\}$, see [10]. In particular, $X_0(c_0, c_1)$ is non empty.[1]

Denote by Y the Banach space $C^{k,\alpha}(\mathbb{S}^1, \mathbb{R}^3) \times C^{k,\alpha}(\mathbb{S}^1, \mathbb{R}^3)$. The map

$$E : X_0 \longrightarrow Y$$

that carries $x \in X_0$ to the pair of embeddings $\big(x|_{\mathbb{S}^1 \times \{0\}}, x|_{\mathbb{S}^1 \times \{1\}}\big)$ is a smooth submersion, and $X_0(c_0, c_1) = E^{-1}\big\{(c_0, c_1)\big\}$. $\square$

We want to study a bifurcation problem for constant mean curvature immersions in $X_0(c_0, c_1)$, and to this aim we need to identify immersions that differ by a change of parameterization. We introduce the following equivalence relation in $X_0(c_0, c_1)$: x is equivalent to y if there exists a diffeomorphism[2] $\phi : \mathcal{C} \to \mathcal{C}$ such that $x = y \circ \phi$. In this situation, we write $x \cong y$; the equivalence class of an immersion $x \in X_0(c_0, c_1)$ will be denoted by $\big[x\big]_{\cong}$. Let $\mathcal{X}_0(c_0, c_1)$ be the set of all equivalence classes $\big[x\big]_{\cong}$:

$$\mathcal{X}_0(c_0, c_1) = \Big\{\big[x\big]_{\cong} : x \in X_0(c_0, c_1)\Big\}.$$

Thus, $\mathcal{X}_0(c_0, c_1)$ is the set of all immersed submanifolds of class $C^{k,\alpha}(\mathbb{R}^3)$ that are diffeomorphic to $\mathcal{C}$, and whose boundary is the union of the images of c_0 and of c_1.

The geometrical structure of the set of $C^{k,\alpha}$-embeddings of a compact manifold M (without boundary) into a differentiable manifold N has been studied in [2]. The same analysis carries over *verbatim* to the case of immersions of a manifold with boundary; let us briefly recall the main facts here.

[1] Note that the argument does not show the existence of *injective* immersions of the cylinder with given boundary values. Of course, injective immersions (embeddings) of the cylinder with arbitrary boundary values may not exist.

[2] In fact, for our purposes it suffices to consider only diffeomorphisms of $\mathcal{C}$ that preserve each connected component of the boundary $\partial\mathcal{C}$.

(a) $\mathcal{X}_0(c_0, c_1)$ is endowed with a family of charts, denote by Φ_x, where $x \in X_0(c_0, c_1)$ is a smooth (i.e., C^∞) immersion. The domain of the chart Φ_x is a neighborhood of $[x]_\cong$ in $\mathcal{X}_0(c_0, c_1)$, and it takes values in a $C^{k,\alpha}$-neighborhood of the null section of the normal bundle of x vanishing on $\partial\mathcal{C}$. For $[y]_\cong \in \mathcal{X}(c_0, c_1)$ near $[x]_\cong$, $\Phi_x([y]_\cong)$ is the unique $C^{k,\alpha}$-map $V : \mathcal{C} \to \mathbb{R}^3$ such that:
- $V(\theta, t)$ is orthogonal to $x(\mathcal{C})$ at $x(\theta, t)$, for all $(\theta, t) \in \mathcal{C}$;
- $V(\theta, 0) = V(\theta, 1) = 0$ for all $\theta \in \mathbb{S}^1$,
- $x + V \cong y$.

(b) As x varies in the set of smooth immersions in $X(c_0, c_1)$, the maps Φ_x form an atlas of charts that make $\mathcal{X}(c_0, c_1)$ into an infinite dimensional *topological* manifold, modeled on the (non separable) Banach space $C^{k,\alpha}_0(\mathcal{C}, \mathbb{R})$.

(c) if $F : X_0(c_0, c_1) \to \mathbb{R}$ is a smooth function which is invariant by reparameterizations, i.e., $F(x) = F(x \circ \phi)$ for all diffeomorphism $\phi : \mathcal{C} \to \mathcal{C}$, then the induced map $\mathcal{F} : \mathcal{X}_0(c_0, c_1) \to \mathbb{R}$ is such that $\mathcal{F} \circ \Phi_x^{-1}$ is smooth in its domain. In this situation, a normal field V in the image of the chart Φ_x is a critical point of the function $\mathcal{F} \circ \Phi_x^{-1}$ if and only if $x + V$ is a critical point of F in $X_0(c_0, c_1)$.

It is observed in [2] that the transition maps of the atlas $\{\Phi_x\}_{x \in C^\infty \cap X_0(c_0,c_1)}$ are only continuous, and not differentiable. This is due to the fact that their expression involves the operation of taking the *inverse* of a $C^{k,\alpha}$-diffeomorphism of $\mathcal{C}$, which is not a differentiable map in the group of diffeomorphism of class $C^{k,\alpha}$. We observe however that, for the purposes of the present paper, a *global* differentiable structure for the manifold of unparameterized embeddings will not be needed. The local differentiable structure given in (c) above will suffice, since we will establish a local bifurcation result for smooth maps on $X_0(c_0, c_1)$ that are invariant by diffeomorphisms of $\mathcal{C}$.

Given a smooth immersion $x \in X_0(c_0, c_1)$, using the chart Φ_x centered at x (as described above), then the tangent space $T_{[x]_\cong}\mathcal{X}_0(c_0, c_1)$ will be identified with the Banach space of all $C^{k,\alpha}$-section of the normal bundle $x^\perp$ of x that vanish on $\partial\mathcal{C}$. Note that $\Phi_x(x) = \mathbf{0}_x$ is the null section of $x^\perp$. Let us assume fixed an orientation of $\mathcal{C}$; then one has a canonical choice of an orientation of $x^\perp$. Let $\vec{n}_x$ denote the unit normal vector field along x which is positively oriented. Sections of $x^\perp$ are of the form $f \cdot \vec{n}_x$, for some function $f : \mathcal{C} \to \mathbb{R}$; thus, when $x \in X_0(c_0, c_1)$ is smooth, the tangent space of $\mathcal{X}_0(c_0, c_1)$ at $[x]_\cong$ can be identified with the Banach space of real valued $C^{k,\alpha}$-maps on $\mathcal{C}$ vanishing on $\partial\mathcal{C}$.

2.2. **Area and volume of an immersion.** Let g_0 denote the Euclidean metric on $\mathbb{R}^3$, vol_3 the canonical volume form of $\mathbb{R}^3$, and let η be any primitive of vol_3, i.e., $\mathrm{d}\eta = \mathrm{vol}_3$. For $x \in X_0(c_0, c_1)$, we denote by $\mathrm{A}(x)$ the *area* of the immersed submanifold $X(\mathcal{C})$, given by:

$$\mathrm{A}(x) = \int_{\mathcal{C}} \mathrm{vol}_x,$$

where vol_x is the area form of the pull-back metric $x^*(g_0)$.

Similarly, we define the *volume* of an immersion $x \in X_0(c_0, c_1)$, denoted by $\mathrm{V}(x)$, the real number defined by[3] :

$$\mathrm{V}(x) = \int_{\mathcal{C}} x^*(\eta). \tag{2.1}$$

The reason for calling such a number “volume” is that, when x is an embedding in $\mathbb{R}^3$ of a closed orientable surface, then the above expression gives (up to a sign) the volume of the bounded subset of $\mathbb{R}^3$ whose boundary is the image of x.

[3]For instance, taking $\eta = \frac{1}{3}(x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_2 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2)$, then (2.1) is given by $\mathrm{V}(x) = \frac{1}{3}\int_{\mathcal{C}} x \cdot \vec{n}_x \,\mathrm{vol}_x$

Given an immersion $x \in X_0(c_0, c_1)$, we will denote by $\mathcal{S}_x$ its *second fundamental form*; recall that for $p \in \mathcal{C}$, $\mathcal{S}_x(p)$ is a symmetric operator on the tangent space at p to $x(\mathcal{C})$, whose trace $\mathrm{tr}\big(\mathcal{S}_x(p)\big)$ is twice of the *mean curvature of* x *at* p. An immersion $x \in X_0(c_0, c_1)$ is said to have *constant mean curvature* $H \in \mathbb{R}$ if $\mathrm{tr}\big(\mathcal{S}_x(p)\big) = 2H$ for all $p \in \mathcal{C}$. If $\nu_x : \mathcal{C} \to \mathbb{S}^2$ is the Gauss map of x, then for $p \in \mathcal{C}$ the second fundamental form $\mathcal{S}_x(p)$ can be identified with the differential $\mathrm{d}\nu_x(p)$.

Let us recall in the following proposition the main facts about the CMC variational problem, formulated in the language of the present paper:

Proposition 2.2. *The following statements hold.*

(1) *The functions* A *and* V *are smooth on* $X_0(c_0, c_1)$.

(2) *They are invariant by reparameterization, and thus by (c) above they define* smooth *functions on* $\mathcal{A}$ *and* $\mathcal{V}$ *on* $\mathcal{X}_0(c_0, c_1)$.

(3) *Given* $\big[x\big]_{\cong} \in \mathcal{X}_0(c_0, c_1)$*, and identifying the tangent space* $T_{[x]_{\cong}}\mathcal{X}_0(c_0, c_1)$ *with the space* $C_0^{k,\alpha}(\mathcal{C}, \mathbb{R})$*, then the differential* $\mathrm{d}\mathcal{V}\big([x]\big)$ *is given by the linear operator* $f \mapsto \int_{\mathcal{C}} f \,\mathrm{vol}_x$*. In particular,* $\mathcal{V}$ *has no critical point in* $\mathcal{X}_0(c_0, c_1)$*, and for all* c *in the image of* $\mathcal{V}$*, the set* $\Sigma_c = \mathcal{V}^{-1}(c)$ *is an embedded submanifold of* $\mathcal{X}_0(c_0, c_1)$*, and for all* $[x]_{\cong} \in \Sigma_c$*, the tangent space* $T_{[x]_{\cong}}\Sigma_c$ *is identified with the closed subspace of* $C_0^{k,\alpha}(\mathcal{C}, \mathbb{R})$ *consisting of functions* f *such that* $\int_{\mathcal{C}} f \,\mathrm{vol}_x = 0$.

(4) *Given* $x \in X_0(c_0, c_1)$ *and a real number* λ*, then* $\big[x\big]_{\cong}$ *is a critical point of the functional* $\mathfrak{f}_\lambda = \mathcal{A} + \lambda \cdot \mathcal{V}$ *in* $\mathcal{X}_0(c_0, c_1)$ *if and only if* x *has constant mean curvature* $H = \frac{\lambda}{2}$*, and in this case* $\big[x\big]_{\cong}$ *contains a smooth representative.*

(5) *If* $x \in X_0(c_0, c_1)$ *is a smooth immersions with constant mean curvature* H *and* $\lambda = 2H$*, then the second variation* $\mathrm{d}^2\big(\mathfrak{f}_\lambda \circ \Phi_x^{-1}\big)$ *at* $\mathbf{0}_x$ *is identified with the quadratic form:*

$$Q_x(f) = -\int_{\mathcal{C}} (J_x f) \cdot f \,\mathrm{vol}_x, \tag{2.2}$$

defined on the space $C_0^{k,\alpha}(\mathcal{C}, \mathbb{R})$ *of real valued* $C^{k,\alpha}$*-maps* $f : \mathcal{C} \to \mathbb{R}$ *vanishing on* $\partial\mathcal{C}$*. Here* $J_x : C_0^{k,\alpha}(\mathcal{C}, \mathbb{R}) \to C^{k-2,\alpha}(\mathcal{C}, \mathbb{R})$ *is the* Jacobi operator *along* x*, which is the strongly elliptic second order linear differential operator defined by:*

$$J_x f = \Delta_x f - \|\mathcal{S}_x\|^2 \cdot f, \tag{2.3}$$

being Δ_x *the (positive definite) Laplacian of the pull-back metric* $x^*(g_0)$, vol_x *the area form of this metric, and* $\|\cdot\|$ *is the Hilbert–Schmidt norm (*$\|L\|^2 = \mathrm{tr}(L^*L)$*).*

Proof. Using local coordinates in $X_0(c_0, c_1)$, the maps A and V are given as composition of a first order nonlinear differential operator (having smooth coefficients), with the linear operator of integration. This proves part (1). Part (2) follows immediately from the formula of change of variables in a double integral. Parts (3), (4) and (5) are standard in the classical literature, see for instance [3, 4]. ☐

2.3. **Degeneracy and Morse index.** Let $x \in X_0(c_0, c_1)$ be a CMC smooth immersion, with mean curvature equal to H_x. A function $f \in C^{k,\alpha}(\mathcal{C}, \mathbb{R})$ satisfying $J_x f = 0$ will be called a *Jacobi field* along x; by standard elliptic regularity (see [8]), a Jacobi field, is in fact smooth. Jacobi fields along a CMC immersion x are variational vector fields corresponding to variations of x by other immersions having the same constant mean curvature, up to infinitesimals of first order. More precisely, if $]-\varepsilon, \varepsilon[\ni s \mapsto x_s \in C^{k,\alpha}(\mathcal{C}, \mathbb{R}^3)$ is a C^1-variation of x by CMC immersions, with $x_0 = x$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} x_s = V$, then setting $f = V \cdot \vec{n}_x$, one has:

$$J_x f = 2\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} H(s), \tag{2.4}$$

where $H(s)$ is the mean curvature of x_s. If $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}H(s) = 0$, then f is a Jacobi field along x. In particular, if $\vec{v} \in \mathbb{R}^3$ is any fixed vector, then the function[4] $f = \vec{v} \cdot \vec{n}_x$ is a Jacobi field along x.

Definition 2.3. We will say that x is a *degenerate* CMC immersion in $X_0(c_0, c_1)$, if $\big[x\big]_{\cong}$ is a degenerate critical point of the functional $\mathfrak{f}_{2H_x}$, i.e., if there exists a non trivial Jacobi field f along x such that f vanishes on $\partial\mathcal{C}$.

There are two distinct notions of Morse index for solutions of the constrained variational problem above.

Definition 2.4. Let $x \in X_0(c_0, c_1)$ be a smooth immersion having constant mean curvature. The *strong Morse index of* x, denoted by $\mathrm{i_s}(x)$, is the index of the quadratic form Q_x of (2.2) in the space $C_0^{k,\alpha}(\mathcal{C}, \mathbb{R})$ of real valued $C^{k,\alpha}$-maps $f : \mathcal{C} \to \mathbb{R}$ vanishing on $\partial\mathcal{C}$. The *weak Morse index of* x, denoted by $\mathrm{i_w}(x)$, is the index of the restriction of the quadratic form Q_x to the closed subspace of $C_0^{k,\alpha}(\mathcal{C}, \mathbb{R})$ consisting of functions $f : \mathcal{C} \to \mathbb{R}$ with vanishing integral: $\int_{\mathcal{C}} f \,\mathrm{vol}_x = 0$.

Since the subspace of functions with vanishing integral has codimension 1 (it is the kernel of the bounded linear functional $f \mapsto \int_{\mathcal{C}} f \,\mathrm{vol}_x$), then one has the following inequalities:

$$\mathrm{i_w}(x) \le \mathrm{i_s}(x) \le \mathrm{i_w}(x) + 1, \tag{2.5}$$

for all constant mean curvature immersion $x \in X_0(c_0, c_1)$.

2.4. **Bifurcation of CMC immersions.** Let us now assume that $I \subset \mathbb{R}$ is an interval and $I \ni s \mapsto x_s \in X_0(c_0, c_1)$ is a continuous path of (smooth) constant mean curvature immersions. Let us also assume that the value of the mean curvature of x_s, denoted by $H(s)$, has non vanishing derivative in I. Let $\overline{s} \in I$ be fixed; we say that $\overline{s}$ is a *bifurcation instant* for the family $\{x_s\}_{s\in I}$ if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in I and a sequence $x_n \in X_0(c_0, c_1)$ such that:

i. $s_n \to \overline{s}$ as $n \to \infty$;
ii. x_n has constant mean curvature equal to $H(s_n)$ for all n;
iii. $x_n \to x_{\overline{s}}$ in $X_0(c_0, c_1)$ as $n \to \infty$;
iv. $x_n \notin \big[x_{s_n}\big]_{\cong}$ for all $n \in \mathbb{N}$.

In other words, $\overline{s}$ is a bifurcation instant for the family $(x_s)_{s\in I}$ if $x_{\overline{s}}$ is an accumulation of constant mean curvature immersions in $X_0(c_0, c_1)$ that are not isometrically congruent to any of the immersions of the family $(x_s)_{s\in I}$.

This is our first result that gives a sufficient condition for bifurcation:

Proposition 2.5. *Let $\varepsilon > 0$ and let $(x_s)_{s\in[\overline{s}-\varepsilon,\overline{s}+\varepsilon]} \subset X_0(c_0, c_1)$ be a C^1-path of constant mean curvature immersions, such that the mean curvature function $s \mapsto H(s)$ has non zero derivative at $s = \overline{s}$. Assume that:*

(a) *for all $s \in [\overline{s} - \varepsilon, \overline{s}[\bigcup]\overline{s}, \overline{s} + \varepsilon]$, the immersion x_s is* nondegenerate*;*
(b) *the strong Morse indices $\mathrm{i_s}(x_{\overline{s}-\varepsilon})$ and $\mathrm{i_s}(x_{\overline{s}+\varepsilon})$ do not coincide.*

Then, $\overline{s}$ is a bifurcation instant for the family (x_s).

On the other hand, if $(x_s)_{s\in I}$ is a C^1-family in $X_0(c_0, c_1)$ of constant mean curvature immersions, with injective mean curvature function $s \mapsto H(s)$, that are nondegenerate for all *s in the interval I, then there is no bifurcating instant in I for the family (x_s).*

Proof. The result is an application of [21, Theorem 2.1]; Smaller–Wasserman's result is used here in the following setup. The Banach spaces B_0 and B_2 are given respectively by $C^{k-2,\alpha}(\mathcal{C}, \mathbb{R})$ and $C_0^{k,\alpha}(\mathcal{C}, \mathbb{R})$, H is the space $L^2(\mathcal{C})$ (Lebesgue space of square-integrable functions on $\mathcal{C}$ relatively to the measure given by the area form $\mathrm{vol}_{x_{\overline{s}}}$. The

[4] More generally, if $\vec{v}$ is any *Killing* vector field of $\mathbb{R}^3$, then $f = \vec{v} \cdot \vec{n}_x$ is a Jacobi field along x.

interval $[\lambda_1, \lambda_2]$ is given by $[2H_{\min}, 2H_{\max}]$, where $H_{\min}$ and $H_{\max}$ are respectively the minimum and the maximum of the mean curvatures of the immersions x_s; by assumption, there exists a C^1-diffeomorphism $[\lambda_1, \lambda_2] \ni \lambda \mapsto s_\lambda \in [\overline{s} - \varepsilon, \overline{s} + \varepsilon]$ such that, for all $\lambda \in [\lambda_1, \lambda_2]$, the mean curvature of x_{s_λ} is equal to $\frac{\lambda}{2}$. The C^1-path $(u_\lambda)_{\lambda \in [\lambda_1, \lambda_2]}$ in B_2 is defined by $u_\lambda = \Phi_{x_{\overline{a}}}\big([x_\lambda]_{\cong}\big)$; if necessary, ε will be taken small enough to guarantee that $\big[x_\lambda\big]_{\cong}$ belongs to the domain of the chart $\Phi_{x_{\overline{a}}}$. The gradient operator $M_\lambda : C_0^{k,\alpha}(\mathcal{C}, \mathbb{R}) \to C^{k-2,\alpha}(\mathcal{C}, \mathbb{R})$ is given by the standard quasi-linear elliptic constant mean curvature differential operator for value of the mean curvature equal to $\frac{\lambda}{2}$. For all λ, its differential $\mathrm{d}M_\lambda(u_\lambda)$ at the point u_λ is the strongly elliptic linear operator $J_{x_{a_\lambda}}$ defined[5] in (2.3). Strong ellipticity here implies the technical assumption on the finite dimensionality of the eigenspaces of $\mathrm{d}M_\lambda(u_\lambda)$ corresponding to small eigenvalues (recall that the spectrum of a Fredholm operator near zero consists of eigenvalues of finite multiplicity).

The assumption of nondegeneracy of the immersions x_s for $s \neq \overline{s}$ corresponds to assumption (i) in [21, Theorem 2.1].

Finally, the assumption on the jump of the Morse index corresponds to assumption (i) in [21, Theorem 2.1]; note that the dimension of the "eigenspace" of x_λ is equal to the strong Morse index $\mathrm{i_s}(x_\lambda)$ when x_λ is nondegenerate. This concludes the proof of the first statement in the thesis.

The last statement of the thesis follows immediately from the Implicit Function Theorem, applied to the equation $M_\lambda(u) = 0$. □

Assumption (b) in Proposition 2.5 implies that $x_{\overline{s}}$ is a (strongly) degenerate constrained critical point for the area functional; we will see that, in fact, the assumption that the derivative of the mean curvature function be non zero implies that $x_{\overline{s}}$ is also *weakly degenerate* (see Proposition 2.7). Here is an alternative statement of the bifurcation result in terms of weak Morse index jumps.

Corollary 2.6. *The statement of Proposition 2.5 holds if one replaces assumption (b) with:*

$$\left|\mathrm{i_w}(x_{\overline{s}-\varepsilon}) - \mathrm{i_w}(x_{\overline{s}+\varepsilon})\right| \geq 2. \tag{2.6}$$

Proof. Using inequality (2.5), one sees immediately that (2.6) implies assumption (b) in Proposition 2.5. □

2.5. **Eigenvalues and Morse index.** If $x \in X_0(c_0, c_1)$ is a smooth CMC immersion, then the Jacobi operator J_x is self-adjoint and it has compact resolvent. Thus, J_x has spectrum which consists of a strictly increasing and unbounded sequence $\lambda_1 < \lambda_2 \leq \ldots$ of real eigenvalues of finite multiplicity, and the corresponding eigenfunctions form an orthogonal basis of L^2. The number of negative eigenvalues (counted with multiplicity) of J_x is the strong Morse index of x; x is degenerate if some eigenvalue is equal to 0.

The first eigenvalue λ_1, which has a special role in the spectral theory of J, is always *simple*, i.e., of multiplicity 1. The corresponding eigenfunction f_1 can be chosen to be positive in the interior $\mathcal{C} \setminus \partial\mathcal{C}$. In fact, such property characterizes the eigenfunctions of J_x corresponding to the first eigenvalue.

Let us recall the following fact. Let f be a (non zero) eigenfunction corresponding to the eigenvalue λ_k, for some $k \geq 1$. The connected components of the set $\mathcal{C} \setminus f^{-1}(0)$ are called the *nodal domains* of f. Then, the number of nodal domains of f is less than or equal to k; this is known as *Courant's nodal domain theorem*.

If $(x_s)_{s \in I}$ is a smooth variation of x by CMC immersions, then the corresponding eigenvalue functions $s \mapsto \lambda_i(s)$ of the Jacobi operators J_{x_s} are continuous. If $(x_s)_s$ is a variation of x whose dependence on s is real-analytic, then by *Kato Selection Theorem*, see

[5]This is slightly imprecise. In formula (2.3) we give the expression for the second variation of $\mathfrak{f}_\lambda$ at a critical point x, using the chart Φ_x centered at the same point. Here the critical point and the center of the chart are not the same. This is really no big deal; the correct expression for $\mathrm{d}M_\lambda(u_\lambda)$ is a *conjugate* of $J_{x_{s_\lambda}}$.

[11], the eigenvalues are also real-analytic. Assumption (b) in Proposition 2.5 is equivalent to the fact that, for $a = \overline{a}$, some eigenvalue function λ_i changes its sign at $\overline{a}$. We will see below that, in fact, our bifurcation result can only be applied when some eigenvalue λ_i with $i \geq 2$ crosses the value 0.

2.6. **On the assumption of injectivity for the mean curvature function.** Proposition 2.5 uses the assumption that the mean curvature function $s \mapsto H(s)$ has non vanishing derivative at the bifurcation instant $\overline{s}$. Such an assumption is used in the proof in order to parameterize the trivial branch of CMC immersions using the value of the curvature. One can reasonably ask himself whether this assumption is really needed for the result, or not. The following simple 2-dimensional example shows that the answer to this question is *yes*, i.e., bifurcation may not occur otherwise.

Example. Consider the two variable function $f(x, y) = 4y^3 + 6xy^2 - 3xy + 3x^2y$ on the plane. We can look at it as a family of functions of y, parameterized with the parameter x. For each fixed x, we look at the critical points of the function $y \mapsto f(x, y)$, i.e., we look for the zeroes of the partial derivative $\frac{\partial f}{\partial y} = 12y^2 + 12xy - 3x + 3x^2$. Near $(0, 0)$, the points (x, y) solutions of $\frac{\partial f}{\partial y} = 0$ form a smooth curve[6] contained in the half-plane $x \geq 0$, tangent to the vertical axis at $(0, 0)$. Notice that the Implicit Function Theorem cannot be used in this situation, as $\frac{\partial^2 f}{\partial y^2}(0, 0) = 0$. Observe also that the function x is not locally injective on the points of the curve near $(0, 0)$, as for each $x \in]0, 1[$ there are exactly two solutions of $12y^2 + 12xy - 3x + 3x^2 = 0$, one with $y > 0$ and the other with $y < 0$. At all point (x, y) of this curve where $y > 0$, the second derivative $\frac{\partial^2 f}{\partial y^2} = 24y + 12x$ is positive, while it is negative at all points (x, y) of the curve with $y < 0$. Thus, there is a *jump of the Morse* index at the point $(0, 0)$, but there is *no bifurcation*.

Motivated by this observation, let us prove the following result relating the injectivity property of the mean curvature function and the vanishing of the integral of Jacobi fields. We will state the result only for CMC immersions of compact surfaces in $\mathbb{R}^3$, although an analogous result clearly holds in the more general context of CMC immersions of hypersurfaces in arbitrary Riemannian manifolds.

Proposition 2.7. *Let Σ be a compact oriented surface, let $x : \Sigma \to \mathbb{R}^3$ be a smooth CMC immersion, and let J_x be the corresponding Jacobi differential operator. Assume that there exists a smooth 1-parameter variation $x_s : \Sigma \to \mathbb{R}^3$ of x by CMC immersions, $s \in]-\varepsilon, \varepsilon[$, with $x_0 = x$ and $x_s|_{\partial\Sigma} = x|_{\partial\Sigma}$ for all s. Assume also that, denoting by $H(s)$ the mean curvature of x_s, the derivative $H'(0) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} H(s) \neq 0$ (and thus $H(s)$ is injective around $s = 0$). Then, every Jacobi field ϕ along x with $\phi|_{\partial\Sigma} = 0$ satisfies:*

$$\int_\Sigma \phi \,\mathrm{vol}_x = 0.$$

In particular, if the first eigenvalue λ_1 of the Jacobi operator J_x is zero, then there is no such variation x_s of x.

Proof. Let $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} x_s$ be the variational vector field associated to x_s, and set $\psi = V \cdot \vec{n}_x$; then, $\psi|_{\partial\Sigma} = 0$ and (see (2.4)):

$$J_x\psi = 2H'(0) \neq 0.$$

Now, if ϕ is a Jacobi field on Σ with $\phi|_{\partial\Sigma} = 0$, then:

$$0 = \int_\Sigma \psi(J_x\phi) \,\mathrm{vol}_x = \int_\Sigma \phi(J_x\psi) \,\mathrm{vol}_x = 2H'(0) \int_\Sigma \phi \,\mathrm{vol}_x,$$

[6]By explicit calculation, the curve is the graph of the function $x = \frac{1}{2}\left(1 - 4y \pm \sqrt{1 - 8y}\right)$.

i.e.,

$$\int_\Sigma \phi \,\mathrm{vol}_x = 0.$$

The last statement follows from the fact that, when $\lambda_1 = 0$, the non zero eigenfunctions of J_x corresponding to this (simple) eigenvalue do not change sign in the interior of Σ, and thus they have non vanishing integral. □

It is useful to give the following:

Definition 2.8. Let $x : \Sigma \to \mathbb{R}^3$ be a smooth CMC immersion of an oriented compact manifold with boundary $\partial\Sigma$, and let J_x be the corresponding Jacobi differential operator. A simple eigenvalue λ of the Dirichlet problem $J_x f = -\lambda f$, $f|_{\partial\Sigma} = 0$ will be called *regular* if all its eigenfunctions $f \neq 0$ satisfy $\int_\Sigma f \,\mathrm{vol}_x \neq 0$.

For instance, the first eigenvalue λ_1 is always simple and regular. Another interesting example of regular eigenvalues comes from axially symmetric CMC immersions:

2.9. **Example.** Let $x : \Sigma \to \mathbb{R}^3$ be a CMC immersion which is axially symmetric around the x_3-axis, symmetric with respct to the plane $\Pi := \{x_3 = 0\}$, and with $H \neq 0$. Assume that 0 is a simple eigenvalue of J, with corresponding eigenfunction given by an axially symmetric Jacobi field f vanishing on the boundary, which is also symmetric with respect to Π. If the position vector $x = (x_1, x_2, x_3)$ is not tangent to $x(\Sigma)$ at the boundary, then 0 is a regular eigenvalue. Namely, in this case $\int_\Sigma f \,\mathrm{vol}_x \neq 0$; this is proved in Proposition A.3.

It is proved in [12, Theorem 1.2] that, given $x : \Sigma \to \mathbb{R}^3$ a smooth CMC immersion of an oriented compact manifold with boundary $\partial\Sigma$, if some simple regular eigenvalue of J_x vanishes, then there is exactly one 1-parameter variation x_s of x by CMC immersions, with $x_s|\partial\Sigma = x|\partial\Sigma$ for all s. Proposition 2.7 says that the mean curvature of such variation is not monotone. Proposition 2.7 tells us also that our bifurcation result cannot be applied when it is some simple regular eigenvalue of the Jacobi operator that crosses 0. An example of this situation is given when x is the standard round embedding of a half-sphere in $\mathbb{R}^3$, and x_s consists of round spherical caps with the same boundary. The mean curvature has a strict maximum at the half-sphere, see Figure 4. The result of [12, Theorem 1.2] can be restated as follows:

Proposition 2.10. *Given a smooth path of fixed boundary CMC immersions of a compact surface in $\mathbb{R}^3$, then bifurcation of fixed boundary CMC immersions issuing from this family does* not *occur at those degenerate immersions of the family for which the degeneracy is produced by the vanishing of a* simple regular *eigenvalue of the Jacobi operator.*

Remark 2.11. Let Σ be a compact oriented surface, let $x : \Sigma \to \mathbb{R}^3$ be a smooth CMC immersion, and let $x_s : \Sigma \to \mathbb{R}^3$ be a smooth 1-parameter variation of x by CMC immersions, $s \in \left]-\varepsilon, \varepsilon\right[$, with $x_0 = x$ and $x_s|_{\partial\Sigma} = x|_{\partial\Sigma}$ for all s. If the derivative of the mean curvature function $H'(0) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} H(s)$ vanishes, then denoting by $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} x_s$ the variational vector field associated to x_s, the function $\psi = V \cdot \vec{n}_x$ is a Jacobi field along x by (2.4)), which vanishes on $\partial\Sigma$.

2.7. **Eigenvalues of the symmetrized problem.** What we will discuss can be applied to more general variational problem in which Schwarz symmetrization method works.

Denote by Σ the compact manifold

$$\Sigma := [0, \ell] \times \mathbb{S}^{n-1},$$

where $l > 0$ is an interval of $\mathbb{R}$, and let $x : \Sigma \to \mathbb{R}^{n+1}$ be an immersion whose image is invariant by the group $G \cong \mathrm{SO}(n)$ of rotations around the x_{n+1}-axis. The hypersurface x can be regarded as an immersion $x : \Sigma \to \mathbb{R}^{n+1}$ which satisfies

$$x(\sigma, N) = \big(x_1(\sigma)N, x_{n+1}(\sigma)\big), \qquad (\sigma, N) \in \Sigma,$$

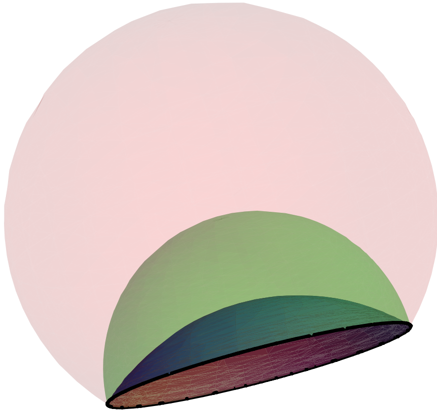

FIGURE 4. Spherical caps with the same boundary. The half-sphere has maximal mean curvature. Bifurcation of fixed boundary CMC surfaces does not occur in this situation.

where $\sigma \mapsto \big(x_1(\sigma), x_{n+1}(\sigma)\big)$ is the generating curve in the (x_1, x_{n+1})-plane of x, parameterized by arclength σ. We denote by $\nu : \Sigma \to \mathbb{S}^n$ the Gauss map of x. Let

$$\tilde{x}_\varepsilon = x + \varepsilon(\xi + \psi\nu) + \mathcal{O}(\varepsilon^2)$$

be an $(n+1)$-dimensional volume-preserving variation of x which fixes the boundary, where ξ is a tangent field to x and ψ is a smooth function. Then, the second variation of the n-dimensional volume of x is given by the quadratic form:

$$I[\psi] := -\int_\Sigma \psi L[\psi] \,\mathrm{vol}_x, \tag{2.7}$$

where vol_x is the volume form of the induced metric and L is the elliptic differential operator

$$L[\psi] := \mathrm{div}(\nabla\psi) + \langle \mathrm{d}\nu, \mathrm{d}\nu\rangle\psi = \Delta\psi + \|\mathrm{d}\nu\|^2\psi. \tag{2.8}$$

The quadratic form L admits a continuous extension to the Sobolev space $H^1_0(\Sigma)$, defined as the completion of C^1_0 with respect to the Hilbert space norm:

$$\|\psi\|^2_{H^1_0} = \int_\Sigma \big[|\psi|^2 + \|\nabla\psi\|^2\big] \,\mathrm{vol}_x.$$

Let us denote by $\|\psi\|_{L^2}$ the L^2-norm of ψ, defined by:

$$\|\psi\|^2_{L^2} = \int_\Sigma |\psi|^2 \,\mathrm{vol}_x.$$

Denote by μ_k the (k-dimensional) volume of the k-dimensional unit sphere $\mathbb{S}^k$. Define functions φ and ζ as

$$\varphi := \frac{1}{\mu_{n-1}} \int_{\mathbb{S}^{n-1}} \psi \,\mathrm{d}\Theta_{n-1}, \quad \text{and} \quad \zeta := \psi - \varphi.$$

Here, Θ_{n-1} denotes the standard volume form of the $(n-1)$-sphere. Then, φ is a function of σ; regarding it as a function on $\Sigma = [0, \ell] \times \mathbb{S}^{n-1}$ that does not depend on the second variable,[7] the immersion $x_\varepsilon^* : \Sigma \to \mathbb{R}^{n+1}$ defined by

$$x_\varepsilon^* = x + \varepsilon\varphi\nu + \mathcal{O}(\varepsilon^2)$$

is the Schwarz symmetrization of the $(n+1)$-dimensional volume-preserving variation

$$x_\varepsilon = x + \varepsilon\psi\nu + \mathcal{O}(\varepsilon^2).$$

Given a function $\varphi : [0, \ell] \to \mathbb{R}$, then $L[\varphi]$ takes the following form:

$$L[\varphi] = \frac{1}{x_1^{n-1}}\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(x_1^{n-1}\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}\Big) + \|\mathrm{d}\nu\|^2\varphi \tag{2.9}$$

$$= \frac{1}{x_1^{n-1}}\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(x_1^{n-1}\frac{\mathrm{d}\varphi}{\mathrm{d}\sigma}\Big) + \big[k_1^2 + (n-1)k_2^2\big]\,\varphi, \tag{2.10}$$

where $k_1, k_2, \cdots, k_n$ ($k_2 = \cdots = k_n$) are principal curvatures of x.

Denote by $\lambda_i := \lambda_i(x)$ the i-th eigenvalue of the following eigenvalue problem:

$$\begin{cases} L[\psi] = -\lambda\psi, & \text{in } \Sigma; \\ \psi = 0, & \text{in } \partial\Sigma. \end{cases} \tag{2.11}$$

The eigenvalues λ_i have a variational characterization, by the following min-max formula:

$$\lambda_{i+1} = \sup_{\substack{V \subset H_0^1(\Sigma) \\ \dim(V)=i}} \inf_{\psi \in V^\perp \setminus 0} \frac{I(\psi)}{\|\psi\|_{L^2}^2}; \tag{2.12}$$

here the supremum is taken over all i-dimensional subspaces V of $H_0^1(\Sigma)$, and the infimum over all $\psi \in H_0^1(\Sigma) \setminus \{0\}$ such that ψ is L^2-orthogonal to V. Similarly, denote by $\hat\lambda_i := \hat\lambda_i(\ell)$ the i-th eigenvalue of the following eigenvalue problem:

$$\begin{cases} L[\varphi] = -\lambda\varphi, & \text{in } [0, \ell]; \\ \varphi(0) = \varphi(\ell) = 0, \end{cases} \tag{2.13}$$

where $\varphi = \varphi(\sigma)$ is a function on $[0, \ell]$. We will call (2.13) the *symmetrized* eigenvalue problem. Min-max formula for these eigenvalues reads:

$$\hat\lambda_{i+1} = \sup_{\substack{\hat V \subset H_0^1([0,\ell]) \\ \dim(\hat V)=i}} \inf_{\varphi \in \hat V^\perp \setminus 0} \frac{I(\varphi)}{\|\varphi\|_{L^2}^2}. \tag{2.14}$$

Let e_1 be the uniquely determined eigenfunction of the eigenvalue λ_1 that satisfies

$$\|e_1\|_{L^2} = 1, \quad \text{and} \quad e_1(p) > 0 \quad \forall\, p \in \Sigma \setminus \partial\Sigma;$$

Similarly, define $\hat e_1$ the (unique) normalized eigenfunction of (2.13) corresponding to $\hat\lambda_1$ which is positive on in the interior of Σ. Sometimes it is easier to estimate $\hat\lambda_i$ than λ_i.

Proposition 2.12. *The following statements hold.*

(a) *Every eigenvalue $\hat\lambda$ of the symmetrized problem* (2.13) *is an eigenvalue also of problem* (2.11).
(b) $\lambda_i \geq \hat\lambda_i$ *for all* $i \geq 1$*;*
(c) $\lambda_1 = \hat\lambda_1$ *and* $e_1 = \hat e_1$.
(d) *If λ_i is a* simple *eigenvalue of* (2.11) *(i.e., the dimension of the corresponding eigenspace is equal to* 1*), then the corresponding eigenfunction is* $\mathrm{SO}(n)$*-invariant, so that λ_i is an eigenvalue also of the symmetrized problem* (2.13).

[7] We will identify implicitly functions on $[0, \ell]$ with $\mathrm{SO}(n)$-invariant functions on Σ.

Proof. Part (a) follows directly from the very definition: eigenvalues of the symmetrized problem are eigenvalues of (2.11) whose eigenfunctions are $\mathrm{SO}(n)$-invariant. Part (b) is proved easily using Rayleigh's formulas (2.12) and (2.14). Observe that that the supremum in (2.14) is taken over a more restricted collections of subspaces of $H^1_0(\Sigma)$; namely, those consisting only of $\mathrm{SO}(n)$-invariant functions. In order to prove (c), let φ be a non trivial eigenfunction of (2.13) corresponding to the eigenvalue $\hat{\lambda}_1$. Then, φ does not change sign in $]0,\ell[$. But φ is also an eigenfunction of (2.11) corresponding to some eigenvalue λ_k, with $k \geq 1$. Since the only eigenfunctions that do not change sign are those corresponding to the first eigenvalue, then $k = 1$ and we are done. For part (d), observe that if $\psi \neq 0$ is any eigenfunction of problem (2.11) and $A \in \mathrm{SO}(n)$, then $\psi \circ A$ is also an eigenfunction of (2.11) with the same eigenvalue as ψ. If such eigenvalue is simple, then $\psi \circ A = g(A)\psi$ for some $g(A) \in \mathbb{R}$. An immediate argument shows that $g(A) = \pm 1$; by continuity, since $g(1) = 1$, one obtains $g(A) = 1$ for all A, i.e., ψ is $\mathrm{SO}(n)$-invariant. □

Notice, however, that in general the equality $\lambda_i = \hat{\lambda}_i$ does not hold for $i \geq 2$.

Remark 2.13. As to the question of bifurcation, it is interesting to observe that the result of Proposition 2.5 admits a formulation for axially symmetric CMC immersions. Assume in the hypotheses of Proposition 2.5 that the path $(x_s)_{s\in[\bar{s}-\varepsilon,\bar{s}+\varepsilon]}$ consists of axially symmetric CMC immersions. Under the same assumptions of Proposition 2.5, and assuming in addition that some eigenvalue of the symmetrized problem crosses the value 0 at $s = \bar{s}$, then there is bifurcating branch issuing at $x_{\bar{s}}$ that consists of axially symmetric CMC immersions with fixed boundary. This is proved by the same argument in the proof of Proposition 2.5, where the abstract bifurcation result of Smoller and Wasserman is applied to the symmetrized variational problem.

2.8. **Equivariant bifurcation.** Let us now discuss another bifurcation result for variational problems invariant by the action of some compact Lie group. For simplicity, we will assume here that the group in question is the special orthogonal group $\mathrm{SO}(n)$, acting by rotation around the x_{n+1}-axis of $\mathbb{R}^{n+1}$. For $g \in \mathrm{SO}(n)$, let $R_g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the corresponding rotation.

In the setup of Subsection 2.7, let us consider the manifold $\Sigma = [0,\ell] \times \mathbb{S}^{n-1}$, with $n \geq 2$. Let $x : \Sigma \to \mathbb{R}^{n+1}$ be a CMC immersion which is invariant by the group $\mathrm{SO}(n)$ of rotations around the x_{n+1}-axis; this means[8] $(R_g \circ x)(\sigma, N) = x(\sigma, gN)$ for all $g \in \mathrm{SO}(n)$ and all $(\sigma, N) \in \Sigma$. There is an action of $\mathrm{SO}(n)$ on the space of (smooth) maps on Σ, defined in the obvious way: to each element $g \in \mathrm{SO}(n)$ one associates the operator T_g, defined by $(T_g\psi)(\sigma, N) = \psi(\sigma, gN)$, for all map $\psi : \Sigma \to \mathbb{R}$ and all $(\sigma, N) \in \Sigma$. Denote by $\mathbb{J}^-_x$ the *negative eigenspace of the Jacobi operator* J_x, i.e., the finite dimensional vector space of smooth maps on Σ spanned by the eigenfunctions of the Jacobi operator J_x corresponding to negative eigenvalues. Using the invariance of x and of the area and the volume functionals, it is easy to see that, given an eigenfunction ψ of the Jacobi operator J_x, then $T_g(\psi)$ is another eigenfunction with the same eigenvalue. In particular, $\mathbb{J}^-_x$ is invariant by T_g for all g, and we obtain a *group representation*:

$$\mathrm{SO}(n) \ni g \longmapsto T_g|_{\mathbb{J}_x} \in \mathrm{GL}\big(\mathbb{J}^-_x\big)$$

of $\mathrm{SO}(n)$ on the finite dimensional space $\mathbb{J}^-_x$, that will be denoted by π_x.

Let us recall that, given a group G and representations $\pi_i : G \to \mathrm{GL}(V_i)$, $i = 1, 2$ on the (finite dimensional) vector spaces V_1 and V_2, π_1 and π_2 are said to be *equivalent* if there exists an isomorphism $L : V_1 \to V_2$ such that $L \circ \pi_1(g) = \pi_2(g) \circ L$ for all $g \in G$.

We have the following refinement of Proposition 2.5:

[8] Up to a choice of a suitable parameterization of x. More generally, x is rotation invariant if the images of the maps x and $R_g \circ x$ coincide for all g.

Proposition 2.14. *Let $\varepsilon > 0$ and let $(x_s)_{s\in[\overline{s}-\varepsilon,\overline{s}+\varepsilon]} : \Sigma \to \mathbb{R}^{n+1}$ be a C^1-path of constant mean curvature* $\mathrm{SO}(n)$*-invariant immersions, such that the mean curvature function $s \mapsto H(s)$ has non zero derivative at $s = \overline{s}$. Assume that:*

(a) *for all $s \in [\overline{s} - \varepsilon, \overline{s}[\bigcup]\overline{s}, \overline{s} + \varepsilon]$, the immersion x_s is* nondegenerate*;*
(b) *the representations $\pi_{x_{\overline{s}-\varepsilon}}$ and $\pi_{x_{\overline{s}+\varepsilon}}$ are not equivalent.*

Then, $\overline{s}$ is a bifurcation instant for the family (x_s).

Proof. This is an application of the equivariant result of Smoller and Wasserman [21, Theorem 3.3]; the assumptions of this theorem are verified as in the proof of Proposition 2.5. □

In the case of $\mathrm{SO}(n)$-invariant CMC immersions, Proposition 2.14 generalizes the result of Proposition 2.5. Namely, the assumption on the jump of the Morse index implies that the spaces $\mathbb{J}^-_{x_{\overline{s}-\varepsilon}}$ and $\mathbb{J}^-_{x_{\overline{s}+\varepsilon}}$ do not have the same dimension, hence the representations $\pi_{x_{\overline{s}-\varepsilon}}$ and $\pi_{x_{\overline{s}+\varepsilon}}$ are not equivalent.

3. BIFURCATION OF FIXED BOUNDARY CMC NODOIDS

3.1. **Description of the nodoids: the family Σ_{t_0}.** We want to describe by explicit equations the family of nodoids passing through two given coaxial circles C_0 and C_1 of the same radius $r_* > 0$, lying on the parallel planes Π_0 and Π_1 whose distance is $h_* > 0$, and that are symmetric with respect to the reflection around the plane Π that lies half-way between Π_0 and Π_1. Let (x_1, x_2, x_3) be the canonical coordinates of $\mathbb{R}^3$. Up to a rigid motion of $\mathbb{R}^3$, we can assume that the symmetry axis of the nodoids is the x_3-axis, and that planes Π_0 and Π_1 are given respectively by $x_3 = -h_*/2$ and $x_3 = h_*/2$, while Π is given by $x_3 = 0$.

There is a two-parameter family of Delaunay's surfaces with given symmetry axis. This family can be indexed using two parameters: H is the value of the mean curvature, and c, whose interpretation is given below. We may choose the unit normal to the surface so that H is negative. In the case of nodoids, c is also negative.

An explicit parameterization for the generatrix in the x_1x_3-plane of these surfaces of revolution, the nodary curve (Figure 1), is given by (cf. [13, §5]):

$$\text{(3.1)} \quad x_1 = q_1(t) := \frac{\cos t + \sqrt{\cos^2 t + a}}{2|H|},$$
$$x_3 = q_3(t) := \frac{1}{2|H|}\int_0^t \frac{\cos\tau + \sqrt{\cos^2\tau + a}}{\sqrt{\cos^2\tau + a}} \cos\tau \,\mathrm{d}\tau,$$

where we have set[9]

$$a = 2cH > 0. \tag{3.2}$$

Here t is a parameter varying in some interval $[a, b]$ to be determined by the boundary conditions. Note that the unit normal $\nu(t)$ to the nodary (3.1) at the instant t is given by:

$$\nu(t) = \big(\cos t, \sin t\big),$$

independently on the value of the parameters a and H.

There are two 1-parameter families of nodoids satisfying the given boundary conditions and the symmetry condition. The first family contains nodoids that have an *even* number of bulges in the slab $-\frac{h_0}{2} \le z_3 \le \frac{h_0}{2}$, and the second family contains nodoids that have an *odd* number of bulges in the slab, see Figure 8. As to the bifurcation problem, the situation

[9]The constant c is given by the conservation law (Noether's theorem):

$$2x_1(t)\cos t + 2Hx_1(t)^2 \equiv c.$$

This is derived from Noether's Theorem and the fact that the functional is invariant with respect to vertical translation. It can also be obtained from Codazzi equation, see (A.3).

is totally analogous for the two families, and we will only consider the first family. This will be denoted by Σ_{t_0}, with $t_0 > 0$, and it can be described as follows.

Given $a > 0$, $H < 0$ and $t_0 > 0$, let us denote by Σ_{a,H,t_0} the immersed surface parameterized by the equations:

$$x_1 = q_1(t)\,\cos\theta, \quad x_2 = q_1(t)\,\sin\theta, \quad x_3 = q_3(t), \tag{3.3}$$

where[10] $(t,\theta) \in [-t_0, t_0] \times [0, 2\pi]$. The boundary of the nodoid Σ_{a,H,t_0} consists of two circles of radius $x(t_0)$ on the planes $x_3 = q_3(t_0)$, and thus one has to impose

$$-\,2Hr_* = \cos t_0 + \sqrt{\cos^2 t_0 + a}, \tag{3.4}$$

and

$$-\,Hh_* = \int_0^{t_0} \frac{\cos\tau + \sqrt{\cos^2\tau + a}}{\sqrt{\cos^2\tau + a}}\cos\tau\;\mathrm{d}\tau. \tag{3.5}$$

Observe that the nodoid Σ_{a,H,t_0} has vertical normal on the boundary precisely when $q_3'(t_0) = 0$, i.e., when t_0 is of the form $\frac{\pi}{2} + k\pi$, for $k \in \mathbb{N}$.

From (3.4) and (3.5), we get the following:

$$\frac{h_*}{2r_*}\big(\cos t_0 + \sqrt{\cos^2 t_0 + a}\big) - \left[\sin t_0 + \int_0^{t_0} \frac{\cos^2\tau}{\sqrt{\cos^2\tau + a}}\,\mathrm{d}\tau\right] = 0. \tag{3.6}$$

Proposition 3.1. *For every value of the constant $\frac{h_*}{2r_*}$, equation* (3.6) *defines implicitly a real analytic function $a = a(t_0)$, taking values in $\mathbb{R}^+$, and whose domain consists of the union of open intervals and an open half-line $]t_*, +\infty[$ and it contains all instants $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{N}$ (see Figure 5). This function satisfies:*

$$\lim_{t_0\to+\infty} a(t_0) = +\infty, \tag{3.7}$$

and consequently, the function $H(t_0) = -\frac{1}{2r_*}\big[\cos t_0 + \sqrt{\cos^2 t_0 + a(t_0)}\big]$ satisfies:

$$\lim_{t_0\to+\infty} H(t_0) = -\infty. \tag{3.8}$$

Proof. Denote by $\mathcal{H}(t_0, a)$ the left-hand side of (3.6). For all $t_0 \in [0, +\infty[$:

$$\lim_{a\to+\infty} \mathcal{H}(t_0, a) = +\infty;$$

moreover, for t_0 sufficiently large or[11] for t_0 near a point in $\frac{\pi}{2} + \pi\mathbb{N}$:

$$\mathcal{H}(t_0, 0) = \frac{h_*}{2r_*}\big(\cos t_0 + |\cos t_0|\big) - \sin t_0 - \int_0^{t_0} |\cos\tau|\,\mathrm{d}\tau < 0.$$

Finally, the partial derivative:

$$\frac{\partial\mathcal{H}}{\partial a} = \frac{h_*}{4r_*\sqrt{\cos^2 t_0 + a}} + \frac{1}{2}\int_0^{t_0} \frac{\cos^2\tau}{(\cos^2\tau + a)^{\frac{3}{2}}}\,\mathrm{d}\tau$$

is strictly positive in $[0, +\infty[\, \times\,]0, +\infty[$. The domain $\mathcal{D}$ of the desired function a consists of all points t_0 where $\mathcal{H}(t_0, 0) < 0$, and it depends on the value of the ratio $\frac{h_*}{2r_*}$; the smoothness of a follows from the Implicit Function Theorem. In fact, since all the functions involved are real-analytic, then a is a real-analytic function of t_0.

Let us prove (3.7). By contradiction, assume that there exists an unbounded sequence $t_n \to \infty$ and a positive real number M such that $a(t_n) \le M$ for all n. Then:

$$\liminf_{n\to\infty} \int_0^{t_n} \frac{\cos^2\tau}{\sqrt{\cos^2\tau + a(t_n)}}\,\mathrm{d}\tau \ge \lim_{n\to\infty} \int_0^{t_n} \frac{\cos^2\tau}{\sqrt{\cos^2\tau + M}}\,\mathrm{d}\tau = +\infty.$$

---

[10] Nodoids having an odd number of bulges in the slab $-\frac{h_0}{2} \le z_3 \le \frac{h_0}{2}$ have the same parametric equations, with parameter t varying in $[-t_0, t_0 + 2\pi]$.

[11] Namely, $\mathcal{H}(\frac{\pi}{2} + k\pi, 0) = (-1)^{k+1} - 1 - 2k < 0$ for all $k \in \mathbb{N}$; moreover, if t_0 is large enough to guarantee that $\int_0^{t_0} |\cos\tau|\,\mathrm{d}\tau > 1 + \frac{h_*}{r_*}$, then $\mathcal{H}(t_0, 0) < -1 - \frac{h_*}{r_*} + \frac{h_*}{2r_*}\big(\cos t_0 + |\cos t_0|\big) - \sin t_0 < 0$.

Then, using (3.6) one would have:

$$\lim_{n\to\infty} \frac{h_*}{2r_*}\big(\cos t_n + \sqrt{\cos^2 t_n + a(t_n)}\big) - \sin t_n = +\infty,$$

which can only occur if $\lim\limits_{n\to\infty} a(t_n) = +\infty$. This proves (3.7).

Equality (3.8) follows readily from (3.4) and (3.7). $\square$

Definition 3.2. The real-analytic family $\Sigma_{a(t_0),H(t_0),t_0}$ will be denoted by Σ_{t_0}; it consists of all nodoids satisfying the boundary conditions, that are symmetric with respect to the (x_1x_2)-plane.

The unit normal vector field to the surface is given by:

$$\nu(t,\theta) = \big(\cos t\cos\theta, \cos t\cos\theta, \sin t\big).$$

When $t_0 \in \frac{\pi}{2} + \pi\mathbb{N}$, the nodoid Σ_{t_0} is tangent to the planes $x_3 = \pm h_*/2$ at its boundary, while when $t_0 \in \pi\mathbb{N}$, Σ_{t_0} is perpendicular to these planes at its boundary. For these values of the parameter t_0, the CMC surface Σ_{t_0} is a degenerate constrained critical point of the area functional subject to $\mathrm{volume} =$ constant.

3.2. **On the mean curvature function.** Let us now consider the mean curvature function $t_0 \mapsto H(t_0) = H\big(t_0, a(t_0)\big)$ of Σ_{t_0}; let us show that, for $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{N}$, the derivative $\frac{\mathrm{d}}{\mathrm{d}t_0}H(t_0)$ is non zero, so that $H(t_0)$ is strictly monotone near these instants. First, observe that, from (3.6), one computes easily:

$$\frac{\partial\mathcal{H}}{\partial t_0}(\pi/2 + k\pi, a) = -\frac{h_*}{2r_*}\sin(\pi/2 + k\pi) = (-1)^{k+1}\frac{h_*}{2r_*} \neq 0,$$

and so:

$$a'(\pi/2+k\pi) = -\frac{\partial\mathcal{H}}{\partial t_0}\big(\pi/2+k\pi, a(\pi/2+k\pi)\big)\cdot\left[\frac{\partial\mathcal{H}}{\partial a}\big(\pi/2 + k\pi, a(\pi/2 + k\pi)\big)\right]^{-1} \neq 0.$$

From (3.5), we get:

$$\frac{\partial H}{\partial t_0}\Big|_{t_0=\pi/2+k\pi} = 0,$$

and

$$\frac{\partial H}{\partial a}(\pi/2 + k\pi, a) = \frac{1}{2h_*}\int_0^{\pi/2+k\pi} \frac{\cos^2\tau}{(\cos^2\tau + a)^{3/2}}\,\mathrm{d}\tau > 0,$$

hence:

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=\pi/2+k\pi} H(t_0) = \frac{\partial H}{\partial a}\big(\pi/2 + k\pi, a(\pi/2 + k\pi)\big)\cdot a'(\pi/2 + k\pi) \tag{3.9}$$

$$= (-1)^k\frac{1}{4r_*}\int_0^{\pi/2+k\pi}\frac{\cos^2\tau}{(\cos^2\tau + a)^{3/2}}\,\mathrm{d}\tau\left(\frac{h_*}{4r_*\sqrt{a}}+\frac{1}{2}\int_0^{\pi/2+k\pi}\frac{\cos^2\tau}{(\cos^2\tau + a)^{3/2}}\,\mathrm{d}\tau\right)^{-1} \neq 0.$$

A straightforward analysis of the sign in the above inequalities shows that the function $|H(t_0)|$ is increasing for t_0 near $\frac{\pi}{2}+2k\pi$ and decreasing for t_0 near $\frac{\pi}{2}+(2k+1)\pi$, $k \in \mathbb{N}$.

We will need the following:

Lemma 3.3. *For all $k \geq 0$, there exists (a unique) $s_k \in \left]k\pi, k\pi + \frac{\pi}{2}\right[$ such that*[12]

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=s_k} H(t_0) = 0.$$

[12]Actually, the proof works only if the entire interval $[k\pi, k\pi + \frac{\pi}{2}]$ is contained in the domain of the function $a = a(t_0)$ defined in Subsection 3.1.

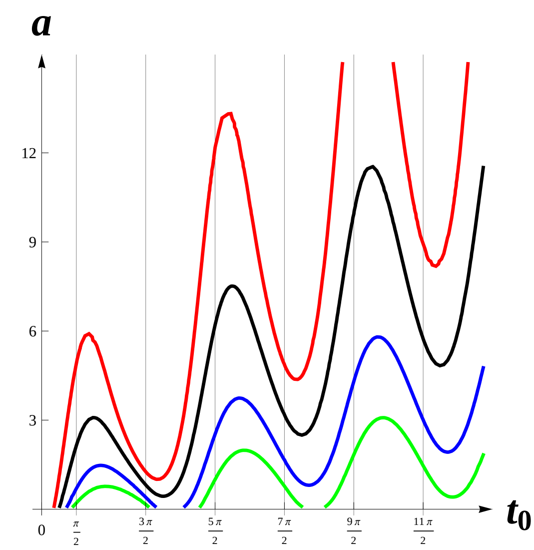

FIGURE 5. Graphs of the implicit function $a = a(t_0)$ defined by the equation $\mathcal{H}(t_0, a) =$ for different values of the constant $\frac{h_*}{2r_*}$. The instants $t_0 = \frac{\pi}{2} + k\pi$ correspond to the degenerate nodoids studied in Subsections 3.2 and 3.3; they belong to the domain of the function $a = a(t_0)$ for all values of the constant $\frac{h_*}{2r_*}$.

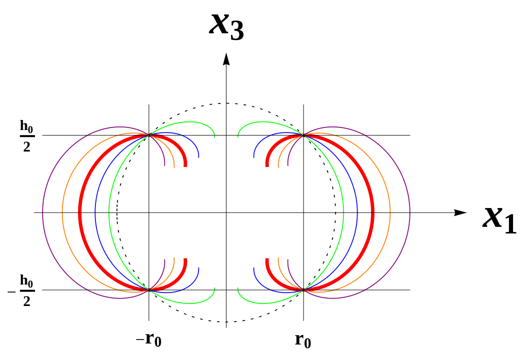

FIGURE 6. Nodary curves that generate nodoids which pass through 2 circles. The bifurcation point is in the middle (thicker/red), it has horizontal tangent at the point of intersection with the circles. The inner circle is a limit of the family when $a \to 0$.

Proof. Let us check that $\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=k\pi} H(t_0)$ and $\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=k\pi+\frac{\pi}{2}} H(t_0)$ have opposite sign.

A straightforward calculation gives:

$$\frac{\partial \mathcal{H}}{\partial t_0}(k\pi, a) = (-1)^{k+1} - \frac{1}{\sqrt{1+a}};$$

and from (3.4):

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=k\pi} H(t_0) = -\frac{1}{4r_*}\frac{a'(k\pi)}{\sqrt{1+a(k\pi)}} = \frac{\frac{\partial \mathcal{H}}{\partial t_0}(k\pi, a)}{4r_* \frac{\partial \mathcal{H}}{\partial a}(k\pi, a)\sqrt{1+a(k\pi)}}. \tag{3.10}$$

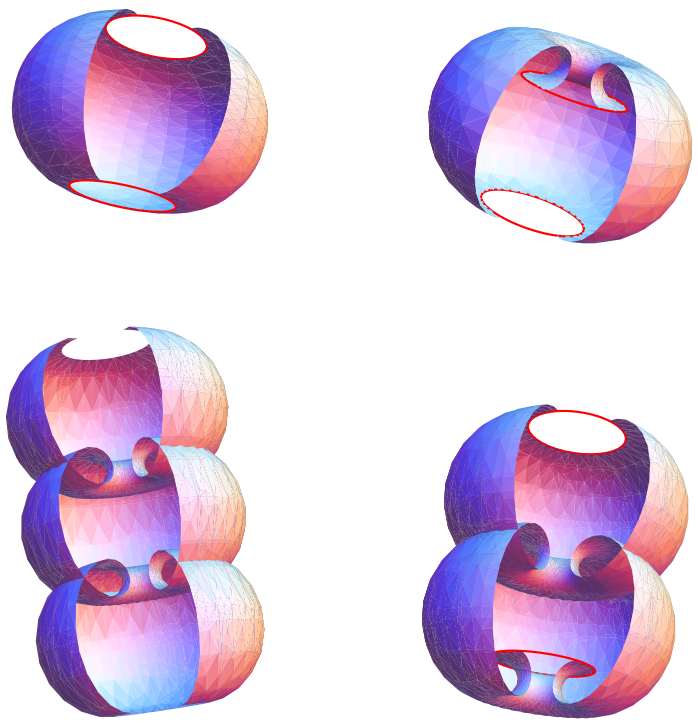

FIGURE 7. Degenerate nodoids are tangent to the planes containing their boundary. On the left, nodoids from the family Σ, on the right nodoids that are not symmetric with respect to the reflection around the plane Π.

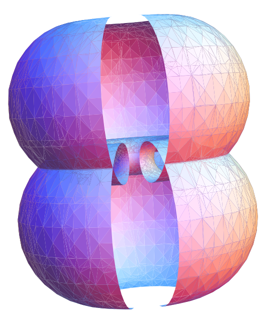

FIGURE 8. Degenerate nodoid with two bulges.

Thus, $\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=k\pi} H(t_0)$ has the same sign as $\frac{\partial \mathcal{H}}{\partial t_0}(k\pi, a)$: negative if k is even, and positive if k is odd. On the other hand, from (3.9) we get that $\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=k\pi+\frac{\pi}{2}} H(t_0)$ is positive if k is even, and negative if k is odd.

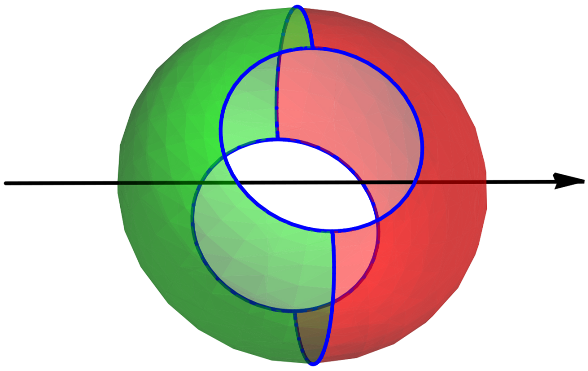

FIGURE 9. The two nodal domains of the Jacobi field ν_1 on the first degenerate nodoid $\Sigma_{\frac{\pi}{2}}$. The arrow gives the direction of the x_1-axis.

This shows that there exists $s_k \in \left]k\pi, k\pi + \frac{\pi}{2}\right[$ such that $\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=s_k} H(t_0) = 0$. Uniqueness will be established later using the Jacobi equation of the symmetrized problem, see Proposition 3.8, part (a). □

3.3. **Bifurcation at the instants $t_0 = \frac{\pi}{2} + k\pi$.** We will now apply Proposition 2.5 to show the existence of a bifurcating branch of fixed boundary CMC immersions issuing from the family Σ at the instants $t_0 = \frac{\pi}{2} + k\pi$, $k \geq 0$. Let us prove some preliminary results.

Lemma 3.4. *Let $\lambda_k(t_0)$ denote the k-th eigenvalue of the CMC immersion Σ_{t_0}, $k \geq 1$. Then, the function $t_0 \mapsto \lambda_k(t_0)$ is real-analytic. Moreover:*

(a) $\lambda_2\big(\frac{\pi}{2}\big) = \lambda_3\big(\frac{\pi}{2}\big) = 0$;

(b) $\lambda_m\big(\frac{\pi}{2} + k\pi\big) = \lambda_{m+1}\big(\frac{\pi}{2} + k\pi\big) = 0$ *for some* $m \geq 2 + 4k$.

Proof. The family Σ_{t_0} has real-analytic dependence on t_0, and so does each λ_k, as it has been observed in Subsection 2.3.

It is known (see [14, Theorem 9.1]) that the first degenerate nodoid $\Sigma_{\frac{\pi}{2}}$ is *stable*, in the sense that the second variation of the area is nonnegative for all volume preserving variations that fix the boundary, and this implies $\lambda_2(\frac{\pi}{2}) \geq 0$. Consider the functions $\nu_i = \vec{n} \cdot \vec{e}_i$, $i = 1, 2, 3$, where $\vec{n}$ is the unit normal outward point vector to $\Sigma_{\frac{\pi}{2}}$ and $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ is the canonical basis of $\mathbb{R}^3$. As we have observed, ν_i is a Jacobi field for $\Sigma_{\frac{\pi}{2}}$, and clearly $\nu_1|_{\partial\Sigma_{\frac{\pi}{2}}} = \nu_2|_{\partial\Sigma_{\frac{\pi}{2}}} = 0$. Thus, ν_1 and ν_2 are eigenfunctions of the Jacobi operator. These two functions have exactly two nodal domains in $\Sigma_{\frac{\pi}{2}}$, see Figure 9, and by Courant's nodal domain theorem, there exists $k \geq 2$ such that $\lambda_k(\Sigma_{\frac{\pi}{2}}) = 0$. Thus, $0 \leq \lambda_2(\Sigma_{\frac{\pi}{2}}) \leq \lambda_k(\Sigma_{\frac{\pi}{2}}) = 0$, and so, $\lambda_2(\Sigma_{\frac{\pi}{2}}) = 0$. Since the first eigenvalue has multiplicity 1, $\lambda_1(\Sigma_{\frac{\pi}{2}}) < \lambda_2(\Sigma_{\frac{\pi}{2}}) = 0$ holds.

Finally, observe that the eigenvalue 0 has multiplicity greater than or equal to 2, which implies $\lambda_2(\Sigma_{\frac{\pi}{2}}) = \lambda_3(\Sigma_{\frac{\pi}{2}}) = 0$, proving (a).

For part (b), observe that the functions ν_1 and ν_2 are Jacobi fields vanishing on the boundary of $\Sigma_{\frac{\pi}{2}+k\pi}$ for all $k \in \mathbb{N}$, and they have exactly $2 + 4k$ nodal domains. The proof of (b) follows directly from Courant's nodal domain theorem. □

A more accurate description of the set of degeneracy instants and of the eigenvalues of the family Σ will be given below, see Subsection 3.8.

Proposition 3.5. *The instants $t_0 = \frac{\pi}{2} + k\pi$, $k \geq 0$, are isolated degeneracy instant for the nodoid family Σ, and there is jump of the Morse index at each of these instants.*

Proof. Fix $k \geq 0$ and set $t_* = \frac{\pi}{2} + k\pi$; for (s, t_0) near (t_*, t_*), let us denote by $\mathrm{i}(s, t_0)$ the strong Morse index of the nodoid $\Sigma_{a(s),H(s),t_0}$ as a constrained critical point of the fixed

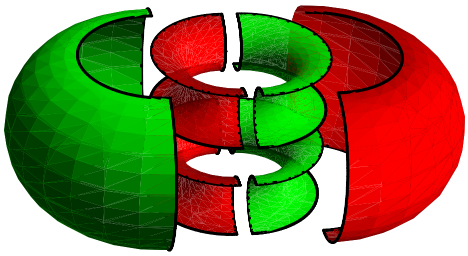

FIGURE 10. The nodoid $\Sigma_{\frac{3}{2}\pi}$ decomposed into six nodal domains of the function ν_1.

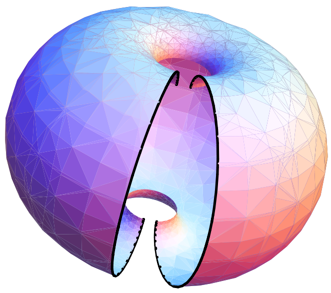

FIGURE 11. The nodoid $\overline{\Sigma}$.

boundary CMC variational problem. Note that $\Sigma_{a(s),H(s),t_0} \subsetneq \Sigma_{a(s),H(s),t_1}$ for $t_0 < t_1$, and for all s. We need to show that $\mathfrak{i}(t_0, t_0)$ has a jump at $t_0 = t_*$, and towards this goal we make the following observations.

(1) For all s, the instant t_* is an isolated degeneracy instant for the family $t_0 \mapsto \Sigma_{a(s),H(s),t_0}$. This follows easily from the strict monotonicity of the eigenvalues of the Jacobi operator with respect to inclusions.
(2) If J_{s,t_0} denotes the Jacobi operator of $\Sigma_{a(s),H(s),t_0}$, then the map $(s, t_0) \mapsto J_{s,t_0}$ is continuous with respect to the operator topology in the space of bounded linear operators from $C^{2,\alpha}(\mathcal{C}, \mathbb{R})$ to $C^{0,\alpha}(\mathcal{C}, \mathbb{R})$. It is real-analytic in both variables s and t_0, and it takes values in the open subset of (essentially positive) Fredholm operators.
(3) For (s, t_0) near (t_*, t_*), the degeneracy of Σ_{s,t_0} is determined by the vanishing of a finite number of non constant real-analytic functions, namely, some eigenvalue $\lambda_k(s, t_0)$ of the Jacobi operator J_{s,t_0}. For every fixed s, the function $t_0 \mapsto \lambda_k(s, t_0)$ has a zero of finite order, hence isolated, at $t_0 = t_*$; by continuity, one can find $\varepsilon, \eta > 0$ sufficiently small, such that the unique zero of $[t_* - \eta, t_* + \eta] \ni t_0 \mapsto \lambda_k(s, t_0)$, for $s \in [t_* - \varepsilon, t_* + \varepsilon]$, is $t_0 = t_*$.
(4) $\mathfrak{i}(s, t_* + \eta)$ is constant for $s \in [t_*, t_* + \varepsilon]$. This follows from the fact that $\Sigma_{s,t_*+\eta}$ is nondegenerate for all $s \in [t_*, t_* + \varepsilon]$. Recall that the index of a continuous path of nondegenerate path of essentially positive Fredholm operators is constant. Similarly, $\mathfrak{i}(s, t_* - \eta)$ is constant for $s \in [t_* - \varepsilon, t_*]$.

(5) $\mathfrak{i}(t_*, t_* + \varepsilon) - \mathfrak{i}(t_*, t_* - \varepsilon) = 2$. This follows from the strict monotonicity of the eigenvalues, and from the fact that the nullity of the degenerate nodoid $\Sigma_{a(t_*),H(t_*),t_*}$ is equal to 2.

If $\varepsilon' = \min\{\varepsilon, \eta\}$, then based on the above observation we compute:

$$\begin{aligned}\mathfrak{i}(t_* + \varepsilon', t_* + \varepsilon') &- \mathfrak{i}(t_* - \varepsilon', t_* - \varepsilon') \\ &= \mathfrak{i}(t_* + \varepsilon', t_* + \varepsilon') - \mathfrak{i}(t_*, t_* + \varepsilon') + \mathfrak{i}(t_*, t_* + \varepsilon') - \mathfrak{i}(t_*, t_* - \varepsilon') \\ &\qquad + \mathfrak{i}(t_*, t_* - \varepsilon') - \mathfrak{i}(t_* - \varepsilon', t_* - \varepsilon') = 2.\end{aligned}$$

This concludes the proof. □

We can now finalize the proof of the statement on non rotational bifurcation at the instants $t_0 = \frac{\pi}{2} + k\pi$.

Proof of statement (1) *of Theorem.* For all $k \in \mathbb{N}$, denote by $x^k : \mathcal{C} \to \mathbb{R}^3$ any smooth parameterization of $\Sigma^k := \Sigma_{\frac{\pi}{2}+k\pi}$. By (a) and (b) of Lemma 3.4, x^k is degenerate in the sense of Definition 2.3. Observe that[13] $\int_{\mathcal{C}} \nu_1 \operatorname{vol}_x = \int_{\mathcal{C}} \nu_2 \operatorname{vol}_{x^k} = 0$. Namely, consider the isometry Ψ of $\mathbb{R}^3$ defined by $\Psi(x_1, x_2, x_3) = (x_1, x_2, -x_3)$; then, Ψ preserves Σ^k, i.e., $\Psi(\Sigma^k) = \Sigma^k$, and $\nu_i \circ \Psi = -\nu_i$, $i = 1, 2$.

Using Proposition 2.5, the proof of existence of bifurcation at Σ^k for the family Σ_{t_0} follows by observing that:

- for t_0 near $\frac{\pi}{2} + k\pi$ and $t_0 \neq \frac{\pi}{2} + k\pi$, the nodoid Σ_{t_0} is nondegenerate, as proved in Proposition 3.5;
- there is a jump of the strong Morse index of Σ_{t_0} at $t_0 = \frac{\pi}{2} + k\pi$, by Proposition 3.5;
- the function $t_0 \mapsto H(t_0) < 0$ of the mean curvature of Σ_{t_0} has non zero derivative at $t_0 = \frac{\pi}{2} + k\pi$, as proved in Subsection 3.2.

As to the break of symmetry of the bifurcating branch, one simply observes that axially symmetric CMC surfaces are determined completely by the values of the parameters H and c; the dependence on such parameters is continuous. So, if $\Sigma = \Sigma(H, c)$ is a nodoid ($H < 0$ and $c < 0$), and $\Sigma' = \Sigma(H', c')$ is an axially symmetric CMC surface with the same boundary of Σ and very close to Σ, then, H' is close to H. Hence c' is close to c, and therefore $H' < 0$ and $c' < 0$ hold. Hence, Σ' is also a nodoid. Since the CMC immersions of the bifurcating branch are not isometrically congruent to any nodoid, it follows that they can't be axially symmetric.

This concludes the proof. □

3.4. **Bifurcation at the instants $k\pi$, $k \geq 1$.** Let us now study the degenerate nodoids corresponding to the instants $t_0 = k\pi$, with $k \geq 1$. At these instants, the function ν_3 is the Jacobi field that vanishes on the boundary. Notice that this function is axially symmetric, and thus it is also a Jacobi field for the symmetrized problem, recall Subsection 2.7. Let us denote by $I_3 : \mathbb{R}^3 \to \mathbb{R}^3$ the map $I_3(x_1, x, x_3) = (x_1, x_2, -x_3)$; note that $\nu_3 \circ I_3 = -\nu_3$.

Proposition 3.6. *Let $\hat{\lambda}_k(t_0)$ denote the k-th symmetrized eigenvalue of the CMC immersion Σ_{t_0}, $k \geq 1$. Then, the function $t_0 \mapsto \hat{\lambda}_k(t_0)$ is real-analytic, and strictly decreasing. Moreover:*

(a) $\hat{\lambda}_{2k}(k\pi) = 0$ *for all* $k \geq 1$*;*
(b) *the (unique) zero of* $\hat{\lambda}_{2k+1}$ *is in* $]k\pi, (k+1)\pi[$*;*
(c) *if f is an eigenfunction of $\hat{\lambda}_k(t_0)$ for some $k \geq 1$, then $f \circ I_3 = (-1)^{k+1} f$.*

Proof. Part (a) follows from Courant Nodal Domain Theorem, in the form of Sturm Oscillation Theorem for the symmetrized 1-dimensional problem. Namely, the function ν_3 has exactly $2k$ nodal domains in $\Sigma_{k\pi}$, thus the corresponding eigenvalue is $\hat{\lambda}_{2k}(k\pi)$. For

[13] i.e., x^k is also *weakly degenerate*

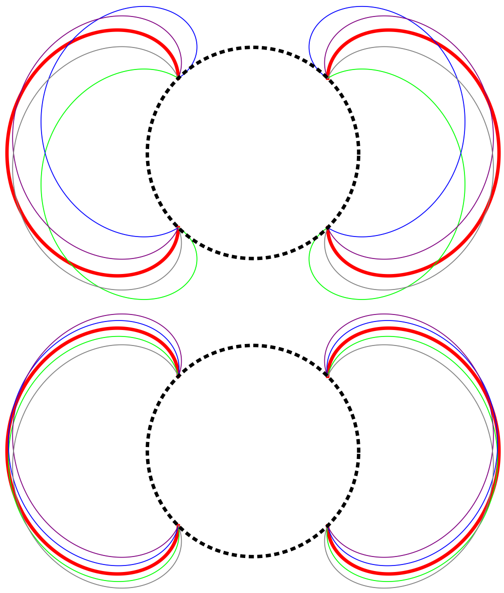

FIGURE 12. Nodaries of bifurcating branch of nodoids at the instant $t_0 = \pi$.

part (b), observe that by the strict domain monotonicity of the eigenvalues, each eigenvalue has at most one zero. Moreover, the eigenvalues of the symmetrized problem are simple, thus $\hat\lambda_{2k}(t_0) < \hat\lambda_{2k+1}(t_0) < \hat\lambda_{2k+2}(t_0)$ for all t_0, thus $\lambda_{2k+1}(k\pi) > 0$ and $\lambda_{2k+1}\big((k+1)\pi\big) < 0$. Part (c) follows from a general fact on CMC surfaces of revolution. Eigenfunctions of the k-th symmetrized eigenvalues are of the form $f_k(t)\cos\theta$, $f_k(t)\sin\theta$, where f_k is even if $k+1$ is even and odd if $k+1$ is odd. This is discussed in detail in Appendix A. □

We will show later that, in fact, the unique zero of the function $\hat\lambda_{2k+1}$ coincides with the instant $s_k \in \big]k\pi, k\pi + \frac{\pi}{2}\big[$ introduced in Lemma 3.3, see Proposition 3.8, part (a).

We can now conclude the proof of the statement on bifurcation at the instants $t_0 = k\pi$.

Proof of statement (2) *of Theorem.* It suffices to apply the abstract bifurcation result of Smoller and Wasserman to the symmetrized variational problem. By Proposition 3.6, there is jump of the Morse index of the symmetrized problem at the instants $t_0 = k\pi$, $k \ge 1$. Moreover, we have shown in the proof of Lemma 3.3, see formula (3.10), that the derivative of the mean curvature function $H(t_0)$ at $t_0 = k\pi$ is non zero. By Proposition 2.5, there is a bifurcating branch of CMC immersions of the cylinder, issuing from $\Sigma_{k\pi}$ for all $k \ge 1$, and since the Jacobi field ν_3 is axially symmetric, then the bifurcating branch consists of axially symmetric immersions (see Remark 2.13). Arguing as in the proof of statement (1) of the Theorem, one proves that the bifurcating branch consists of nodoids with fixed boundary. Since they do not belong to the family Σ, they fail to be symmetric with resect to reflections around the plane Π. □

We will now give an explicit description of these bifurcating branches.

3.5. **The bifurcating branches of non symmetric nodoids.**

Each nodoid[14] $\Sigma_{k\pi}$, $k \ge 1$, belongs to a 1-parameter family of fixed boundary nodoids that are not symmetric with

[14] Again, we assume here that $k\pi$ belongs to the domain of the function $a = a(t_0)$ introduced in Subsection 3.1.

respect to reflections around the plane Π ($x_3 = 0$). Let $k \geq 1$ be fixed; for $\varepsilon > 0$ small enough, there exists a real-analytic function $]-\varepsilon, \varepsilon[\ \ni s \mapsto a^k(s) > 0$ such that, setting:

$$H^k(s) = -\frac{1}{2r_*}\Big[\cos(k\pi - s) + \sqrt{\cos^2(k\pi - s) + a^k(s)}\Big],$$

the following equality holds

$$\int_{-k\pi-s}^{k\pi-s} \frac{\cos\tau + \sqrt{\cos^2\tau + a^k(s)}}{\sqrt{\cos^2\tau + a^k(s)}}\cos\tau\,\mathrm{d}\tau = -2H^k(s)h_*$$

for all $s \in\ ]-\varepsilon, \varepsilon[$. The proof of the existence of such function a^k is obtained from the Implicit Function Theorem, in total analogy with the discussion in Subsection 3.1; observe that for $s = 0$ we recover the data of the symmetric nodoid $\Sigma_{k\pi}$, i.e., $a^k(0) = a(k\pi)$ and $H^k(0) = H(k\pi)$.

For all $s \in\ ]-\varepsilon, \varepsilon[$, define $\mathcal{N}_{k,s}$ the surface in $\mathbb{R}^3$ defined by parametric equations:

$$x_1 = \frac{\cos t + \sqrt{\cos^2 t + a^k(s)}}{-2H^k(s)}\cos\theta, \quad x_2 = \frac{\cos t + \sqrt{\cos^2 t + a^k(s)}}{-2H^k(s)}\sin\theta$$
$$x_3 = -\frac{h_*}{2} - \frac{1}{2H^k(s)}\int_{-k\pi-s}^{t}\frac{\cos\tau + \sqrt{\cos^2\tau + a^k(s)}}{\sqrt{\cos^2\tau + a^k(s)}}\cos\tau\,\mathrm{d}\tau,$$

where $(t, \theta) \in [-k\pi - s, k\pi - s] \times [0, 2\pi]$.

This is a real-analytic family of nodoids satisfying the same boundary conditions as the family Σ, and that are not symmetric with respect to the plane $x_3 = 0$ except when $s = 0$, see Figure 12. For all k, $\mathcal{N}_{k,0} = \Sigma_{k\pi}$, and $s \mapsto \mathcal{N}_{k,s}$ is the bifurcating branch of the family Σ issuing at $\Sigma_{k\pi}$.

3.6. **Degeneracy instants where bifurcation does not occur.** We will study the instants s_k introduced in Lemma 3.3, showing that they are degenerate instants, but that there is no bifurcation at these instants. We prove first a result on the vanishing of the derivative of the mean curvature function; let us consider the functions $H(t_0)$ and $a(t_0)$ defined in Subsection 3.1.

Lemma 3.7. *If $H'(t_*) = 0$, then $a'(t_*) \neq 0$.*

Proof. Assume $H'(t_*) = 0$; differentiating (3.4) we get:

$$a'(t_*) = 2\sin t_*\Big[\cos t_* + \sqrt{\cos^2 t_* + a(t_*)}\Big].$$

The right-hand side of this equality does not vanish for all $t_* \neq k\pi$, $k \in \mathbb{N}$; on the other hand, t_* is not an integer multiple of π, since $H'(k\pi) \neq 0$ for all $k \in \mathbb{N}$ (see (3.10)). Thus $a'(t_*) \neq 0$. □

Proposition 3.8. *The following statements hold.*

(a) *The mean curvature function H has vanishing derivative* only *at the instants s_k introduced in Lemma 3.3.*
(b) *For all $k \geq 0$, s_k is a degenerate instant for the family Σ which corresponds to the vanishing of an eigenvalue of the symmetrized problem.*
(c) *No bifurcation for the family Σ occurs at s_k, unless s_k coincides with one of the other degeneracy instants described in the paper.*

Proof. Let t_* be such that $H'(t_*) = 0$, and denote by $\vec{n}$ the normal to Σ_{t_*}. Consider the map $f = V \cdot \vec{n}$, where V is the variational vector field of the variation $t_0 \mapsto \Sigma_{t_0}$ at $t_0 = t_*$. Clearly, f vanishes on $\partial\Sigma_{t_*}$, and it is an axially symmetric Jacobi field along Σ_{t_*} by (2.4). Let us show that f is not identically zero. Since $\vec{n}(0, 0)$ is the first vector of the canonical

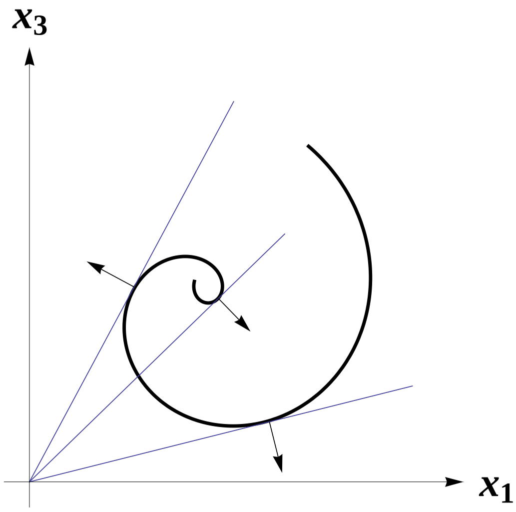

FIGURE 13. If the position vector $x = (x_1, x_3)$ is tangent to the curve γ in the first quadrant, then the normal $\vec{n} = (n_1, n_3)$ to γ at the point of tangency satisfies $n_1 \cdot n_3 \leq 0$, i.e., $\vec{n}$ gives the direction of a line with nonpositive angular coefficient. In the case of the generatrix of the nodoid Σ_{s_k}, the normal at the boundary is $(\cos s_k, \sin s_k)$, where $s_k \in \left]k\pi, k\pi + \frac{\pi}{2}\right[$, and so $\cos s_k \sin s_k > 0$. This implies that the position vector is not tangent to Σ_{s_k} at its boundary.

basis of $\mathbb{R}^3$, the value of f at $(t, \theta) = (0, 0)$ is the first component of the variational vector field $V(0, 0)$. This is given by the derivative of the map $t_0 \mapsto \frac{1+\sqrt{1+a(t_0)}}{-2H(t_0)}$ at $t_0 = t_*$:

$$f(0,0) = \frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=t_*} \frac{1+\sqrt{1+a(t_0)}}{-2H(t_0)} = -\frac{a'(t_*)}{4H(t_*)\sqrt{1+a(t_*)}} \neq 0,$$

by Lemma 3.7. This shows that f is not identically zero, hence every instant t_* at which $H'(t_*) = 0$ is a degeneracy instant for the symmetrized problem, proving (b). By Proposition 3.6, since $k\pi < s_k < (k+1)\pi$, $\hat{\lambda}_{2k}(s_k) < 0$ and $\hat{\lambda}_{2k+2}(s_k) > 0$, thus $\hat{\lambda}_{2k+1}(s_k) = 0$. So, the only degeneracy instants for the symmetrized problem are $k\pi$ and s_k, and therefore there are no other instants where H' vanishes. This proves (a), and the uniqueness claim in Lemma 3.3.

As to part (c), we observe that the degeneracy at s_k is produced by the vanishing of an eigenvalue of the axially symmetric problem, with corresponding Jacobi field which is even, see Appendix A. Let us prove that 0 is a regular eigenvalue at s_k, in the sense of Definition 2.8; to this aim, it suffices to show that the position vector $x = (x_1, x_2, x_3)$ is not tangent to Σ_{s_k} at $\partial\Sigma_{s_k}$, see Example 2.9. This follows form an elementary geometrical argument (see Figure 13), using the fact that $s_k \in \left]k\pi, k\pi + \frac{\pi}{2}\right[$ (see Lemma 3.7). Hence, 0 is a regular eigenvalue at s_k, and by Proposition 2.10, no bifurcation occurs at s_k. $\square$

3.7. **Large degeneracy instants.** The method of separation of variables provides all possible Jacobi fields and eigenfunctions of the Jacobi operator, see Appendix A. We will use Sturm–Liouville theory to show the existence of arbitrarily large degeneracy instants for the nodoid family Σ, where bifurcation occurs with break of both type of symmetry.

Proposition 3.9. *There are arbitrarily large degeneracy instants for the family Σ that do not belong to the set $\{k\pi/2,\ s_{k-1} : k \geq 1\}$. The multiplicity of these degeneracy instants* *is greater than or equal to* 2*. If* t_0 *is one of these instants, denoting by* f_1 *and* f_2 *two linearly independent Jacobi fields vanishing on* $\partial\Sigma_{t_0}$*, then, for* $i = 1, 2$*:*

(a) $f_i \circ I_3 = \pm f_i$*, (both possibilities occur);*
(b) f_i *is not axially symmetric, but it is invariant by rotations of an angle* $\frac{2\pi}{n}$ *for some* $n \geq 2$*.*

Moreover, bifurcation occurs at t_0 *by a branch of non axially symmetric CMC immersions of the cylinder* $\mathcal{C}$ *with fixed boundary. The bifurcating branch consists of CMC surfaces that are not symmetric with respect to reflections around the plane* Π *exactly when* $f_i \circ I_3 = -f_i$*, and that are symmetric with respect to a finite (non trivial) group of rotations around the axis.*

Proof. We use the method of separation of variables in the Jacobi equation, which is discussed in Appendix A. In the notations of the appendix, we will consider an arc-length parameterization of the nodary curve $\gamma_{t_0} = (x_1^{t_0}, x_3^{t_0})$ in the (x_1x_3)-plane which is the generatrix of Σ_{t_0}. Non axially symmetric Jacobi fields along Σ_{t_0} can be obtained of the form $f_1 = S\cos(n\theta)$, $f_2 = S\sin(n\theta)$, where $n \geq 1$ and $S : \mathbb{R} \to \mathbb{R}$ is a solution of the Sturm–Liouville equation:

$$(x_1^{t_0} S')' + \Big(x_1^{t_0}\|\mathrm{d}\nu_{t_0}\|^2 - \tfrac{n^2}{x_1^{t_0}}\Big)S = 0, \tag{3.11}$$

$\mathrm{d}\nu_{t_0}$ is the second fundamental form of Σ_{t_0}. Such Jacobi fields are invariant by the rotation of $\frac{2\pi}{n}$. Moreover, $f_i \circ I_3 = \pm f_i$ according to whether S is an even or an odd function; note that the coefficients of the ODE (3.11) are even functions, and this implies that such equation admits a pair of linearly independent solutions consisting of an even and an odd function, corresponding the the initial values $S(0) = 1$, $S'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$ respectively. Notice that in the case $n = 1$, the even solution (3.11) is given by the function $\dot{x}_3^{t_0}$ (see Appendix A), and this gives rise to the Jacobi fields $\nu_1 = \dot{x}_3^{t_0}\cos\theta$ and $\nu_2 = \dot{x}_3^{t_0}\sin\theta$.

A nodal domain inside Σ_{t_0} is obtained when either the even or the odd solution of (3.11) has a zero in the interval $]0, L_{t_0}/2]$, where L_{t_0} is the length of γ_{t_0}. However, a non trivial Jacobi field vanishing on the boundary of Σ_{t_0} is obtained from a solution of (3.11) only when this has a zero *precisely* at $L_{t_0}/2$. We claim that statements (a) and (b) will be proved once we show that, given any $n \geq 2$, then for t_0 large enough the odd solution of (3.11) has a zero in $]0, L_{t_0}/2]$ (this will imply that also the even solution does, by Sturm oscillation theorem). Namely, denote by S_{n,t_0} the odd solution of (3.11), and assume that given $n \geq 2$ there exists t_0 large such that the first positive zero of S_{n,t_0} lies in $]0, L_{t_0}/2]$. Clearly, such t_0 is bounded away from zero, i.e., there exists $t_* > 0$ such that S_{n,t_0} does not vanish in $]0, L_{t_0}/2]$ for all $t_0 \leq t_*$. Thus, by continuity, there must exist $t_0 > t_*$ such that $S_{n,t_0}(L_{t_0}/2) = 0$

Let us prove that, for any given $n \geq 2$, one can find t_0 sufficiently large such that every solution of (3.11) has two consecutive zeroes at distance less than or equal to L_{t_0}. Denote by $k_1^{t_0}$ and $k_2^{t_0}$ the principal curvatures of Σ_{t_0}, so that $\|\mathrm{d}\nu_{t_0}\|^2 = (k_1^{t_0})^2 + (k_2^{t_0})^2$; one computes:

$$k_1^{t_0} = H(t_0) + \dot{x}_3^{t_0}/x_1^{t_0}, \quad k_2^{t_0} = -\dot{x}_3^{t_0}/x_1^{t_0},$$

see Appendix A. The basic estimates needed for our argument are as follows:

(1) $\lim\limits_{t_0\to+\infty} a(t_0) = +\infty$, see (3.7);
(2) $-H(t_0) \cong \frac{1}{2r_*}a(t_0)^{\frac{1}{2}}$ as $t_0 \to \infty$, see (3.4);
(3) $x_1^{t_0}$ tends to the constant r_* uniformly as $t_0 \to +\infty$, this is proved using (1) and formula (3.1);
(4) $|\dot{x}_3^{t_0}| \leq 1$, thus $|k_2^{t_0}|$ is uniformly bounded as $t_0 \to \infty$;
(5) $\|\mathrm{d}\nu_{t_0}\|^2 \cong \frac{a(t_0)}{4r_*^2}$ as $t_0 \to +\infty$;

(6) $\big(x_1^{t_0}\|\mathrm{d}\nu_{t_0}\|^2 - \frac{n^2}{x_1^{t_0}}\big) \cong \dfrac{a(t_0)}{4r_*}$ as $t_0 \to +\infty$;

(7) $L_{t_0} = \dfrac{1}{-H(t_0)}\displaystyle\int_0^{t_0}\Big[1+\frac{\cos\tau}{\sqrt{\cos^2\tau + a(t_0)}}\Big]\mathrm{d}\tau \geq \frac{t_0}{-2H(t_0)} \cong \frac{4r_* t_0}{a(t_0)^{\frac12}}$ as $t_0 \to +\infty$;

Using (3) and (6), Sturm comparison theorem tells us that the distance between two consecutive zeroes of a solution of (3.11) is comparable with the distance between two consecutive zeros of the linear ODE with constant coefficient:

$$S'' + \Big(\frac{a(t_0)}{4r_*^2} - \frac{n^2}{r_*^3}\Big)S = 0.$$

For such equation, the distance between two consecutive zeros of any solution is of the order of $a(t_0)^{-\frac12}$. By (7), for t_0 large enough, $L_{t_0}/2$ is larger than this value, which implies that the odd solution of (3.11) must have a zero in $]0, L_{t_0}/2]$. This concludes the proof of (a) and (b).

As to the statement on bifurcation at these degenerate instants, this is obtained as an application of the equivariant result of Proposition 2.14. First we observe that the assumption on the non vanishing of the derivative H' is proved in Proposition 3.7. The passage through one of the degenerate instants in question produces eigenfunctions of the Jacobi operator corresponding to negative eigenvalues, whose stabilizer relatively to the $\mathbb{S}^1$-action is a non trivial cyclic group, say of order n. Thus, passing through such degenerate instant gives a jump in the dimension of the space spanned by eigenfunctions of the Jacobi operator with negative eigenvalue and whose $\mathbb{S}^1$-stabilizer has finite order $n > 1$. This implies a jump of the isomorphism class of the $\mathbb{S}^1$-representation on the negative eigenspace of the Jacobi operator, as discussed in Subsection 2.8. □

3.8. **Conclusions.** In conclusion, we have found the following sequence of degenerate instants for the nodoid family Σ:

$$s_0 < \tfrac{\pi}{2} < \pi < s_1 < \tfrac32\pi < 2\pi < s_2 < \tfrac52\pi < \\ \ldots < k\pi < s_k < k\pi + \tfrac12\pi < (k+1)\pi < \ldots$$

The eigenvalues $\frac{\pi}{2} + k\pi$, $k \geq 0$, have multiplicities equal to 2, and the corresponding eigenfunctions are not axially symmetric. All the other eigenvalues are simple, and they correspond to axially symmetric eigenfunctions. For all $k \geq 1$:

$$\lambda_{m_k}(k\pi) = \hat\lambda_{2k}(k\pi) = 0$$

for some $m_k \geq k$, and at these instants there is bifurcation by axially symmetric nodoids that are not symmetric with respect to the horizontal plane. For all $k \geq 0$:

$$\lambda_{l_k}\big(k\pi + \tfrac{\pi}{2}\big) = \lambda_{l_k+1}\big(k\pi + \tfrac{\pi}{2}\big) = 0$$

for some $l_k \geq 4k+2$, and at these instants there is bifurcation by a branch of non axially symmetric CMC immersions. Note that the Jacobi fields corresponding to λ_{4k+2} and λ_{4k+3} at the instants $k\pi + \frac{\pi}{2}$ have exactly $4k+2$ nodal domains.

A third type of degeneracy instants are the s_k's, where the mean curvature function has vanishing derivative. They correspond to zeroes of the even axially symmetric Jacobi field. For all $k \geq 0$:

$$\lambda_{n_k}(s_k) = \hat\lambda_{2k+1}(s_k) = 0$$

for some $n_k \geq 4k+1$, and there is no bifurcation at these degenerate instants.

Although a complete analysis of all the degeneracy instants seems quite involved, the method of separation of variables and Sturm–Liouville theory allow to prove the existence of a further class of degeneracy instants. The corresponding Jacobi fields vanishing of the boundary are of the form $S(t)\sin(n\theta)$ and $S(t)\cos(n\theta)$, with n arbitrarily large, and S is either an even or an odd function.

4. Final remarks and conjectures

Bifurcation at the embedded nodoid $\Sigma_{\frac{\pi}{2}}$ is of particular interest. We conjecture that the non-nodoidal branch at $\Sigma_{\frac{\pi}{2}}$ contains surface which minimize area for volume preserving fixed boundary variations. A proof of this fact requires an analysis of the stability for the bifurcating branch, which is the subject of an ongoing research project, see [15]. It is generally believed that absolute minimizers are "as symmetric as possible". In the case of one circular boundary component and genus zero surfaces, the only stable equilibria are axially symmetric ([1]). This implies that, with one circular boundary component and surfaces of higher genus, no minimizer exists. So, it is quite interesting that there might be a non axially symmetric minimizer for 2 circular boundaries.

We also conjecture that, for large volumes, the solutions of the isoperimetric problem for surfaces bounded by our contour are not axially symmetric.

5. Extensions to anisotropic mean curvature

The bifurcation theory of the nodoid discussed above fits into the far more general theory of anisotropic nodoids which arise as critical points for axially symmetric anisotropic surface energies. The generalization is not only mathematical; the physical problem of determining the equilibrium shape of a liquid-liquid interface which is tethered to two parallel planes along two coaxial circles is a special case of an anisotropic-liquid interface having the same boundary conditions.

5.1. **Constant anisotropic mean curvature surfaces.** Let $\gamma = \gamma(\nu_3)$ be a positive, C^2 function on $\mathbb{S}^2$. We assume the following *convexity condition*. We define the functions

$$1/\mu_2 := \gamma - \nu_3\gamma'(\nu_3)\,, \quad 1/\mu_2 := (1-\nu_3^2)\gamma''(\nu_3) + 1/\mu_2\,.$$

We will also assume here that γ is an even function, although most of what is said here holds when this is not the case. These functions are assumed to be positive. They are the principal curvatures of a convex surface of revolution W, called the *Wulff shape*, which is parameterized over $\mathbb{S}^2$ as follows. The profile curve of W is given in terms of the vertical coordinate on $\mathbb{S}^2$ by

$$u = u(\nu_3) := \frac{\sqrt{1-\nu_3^2}}{\mu_2}\,, \qquad v = v(\nu_3) := \frac{\nu_3}{\mu_2} + \gamma'(\nu_3)\,.$$

Then W can be parameterized,

$$\chi(\nu_3,\theta) = \big(ue^{i\theta}, v\big)\,. \tag{5.1}$$

The positivity of $1/\mu_i$, $i = 1,2$ is equivalent to the fact that this surface W is convex, the μ_i's are the principal curvatures of W with respect to the inward pointing normal.

To a smooth, compact surface $X : \Sigma \to \mathbb{R}^3$ with normal ν, we assign the axially symmetric energy

$$\mathcal{F}[X] := \int_\Sigma \gamma(\nu_3)\,\mathrm{d}\Sigma\,.$$

For a smooth variation of the surface $X_\varepsilon = X + \varepsilon(\delta X) + \mathcal{O}(\varepsilon^2)$, we have the first variation formula

$$\delta\mathcal{F} =: -\int_\Sigma \Lambda\delta X\cdot\nu\,\mathrm{d}\Sigma + \oint_{\partial\Sigma} X_L\times\chi\cdot\delta X\,\mathrm{d}L\,, \tag{5.2}$$

which defines the *anisotropic mean curvature* Λ. Above we have used χ to denote the composition of the map given in 5.1 with the Gauss map ν. For the isotropic case $\gamma \equiv 1$, we have $\Lambda = 2H$. The equation $\Lambda \equiv$ constant characterizes critical points of the functional $\mathcal{F}$ subject to the constraint that the enclosed three dimensional volume is fixed. The convexity condition imposed above insures that the equation $\Lambda \equiv$ constant is elliptic. Specifically, the linearized operator J defined by the equation,

$$\delta\Lambda =: J[\psi] + \nabla\Lambda \cdot \xi \ , \tag{5.3}$$

defines a linear, strongly elliptic self-adjoint differential operator.

The surface W occurs among the critical points of the volume constrained energy functional. In fact, it is the absolute minimizer of $\mathcal{F}$ among all surfaces enclosing the same volume. This is known as Wulff's Theorem.

The standard examples that occur in applications are the so called Rapini–Papoular functionals $\gamma = 1 + e\nu_3^2$, where e is a constant. The convexity condition is satisfied when $|e| < 1$.

Assuming that a given surface Σ is a critical point of a volume constrained anisotropic energy functional, i.e., that $\Lambda \equiv$ const., the second variation of the fixed boundary problem is given by

$$\delta^2\mathcal{F} = -\int_\Sigma \psi J[\psi] \, \mathrm{d}\Sigma \ ,$$

where J is given by (5.3) and $\psi := \delta X \cdot \nu$. We note that if δX is taken to be the infinitesimal generator of a family of translations, i.e., $\delta X = E \in \mathbf{R}^3$, then if $\Lambda =$ constant, we obtain from 5.3, $J[\nu \cdot E] = 0$. The same argument shows that if δX is taken to be the infinitesimal generator of a family of rotations around the x_3-axis, then $J[\delta X \cdot \nu] = 0$.

5.2. **Anisotropic nodoids.** The axially symmetric surfaces with constant anisotropic mean curvature are called *anisotropic Delaunay surfaces* ([13]). They are exactly the axially symmetric critical points of the functional $\mathcal{F}$, subject to a volume constraint. To find these surfaces explicitly, note that vertical translation is a symmetry of both the functional $\mathcal{F}$ and the volume functional. Assuming a surface of revolution is a critical point and deforming it by a one parameter family of vertical translations, $\delta X = E_3 = E_3^T + \nu_3\nu$, we obtain that the first variation, as given by (5.2), vanishes. Writing this out explicitly for the given variations yields

$$0 = -\Lambda \int_\Sigma \Lambda\nu_2 \, \mathrm{d}\Sigma + \oint_{\partial\Sigma} u \frac{E_3^T}{|E_3^T|} \cdot \eta \, \mathrm{d}L \ ,$$

where η denotes the outward pointing unit conormal to $\partial\Sigma$.

By applying the divergence theorem to the surface integral and noticing that the integrand is constant in the line integral, we obtain the conservation law

$$2ux + \Lambda x^2 \equiv \text{constant} =: c \ . \tag{5.4}$$

As in the isotropic case, the anisotropic Delaunay surfaces fall into six classes. In all cases, we can assume, through a choice of orientation, that $\Lambda \leq 0$ holds. The anisotropic nodoids occur when the scale invariant quantity $a := \Lambda c$ is positive.

In order to parameterize the anisotropic nodoids, we first extend the function u to a periodic function by defining:

$$U(\cos t) := \frac{\cos t}{\mu_2(\cos t)} \ .$$

In this case, an explicit parameterization is given by first defining

$$q_1(t) := \frac{U(\cos t) + \sqrt{U^2(\cos t) + a}}{|\Lambda|},$$

$$q_3(t) := \frac{1}{|\Lambda|} \int_0^t \frac{U(\cos\tau) + \sqrt{U^2(\cos\tau) + a}}{\sqrt{U^2(\cos\tau) + a}} \mathrm{d}v(\cos\tau) \ .$$

Then the surface can be parameterized $X := (q_1(t)e^{i\theta}, q_3(t))$. The outward pointing normal is given by $\nu = [(q_1'(t))^2 + (q_3'(t))^2]^{-1/2}(q_3'(t)e^{i\theta}, -q_1'(t))$.

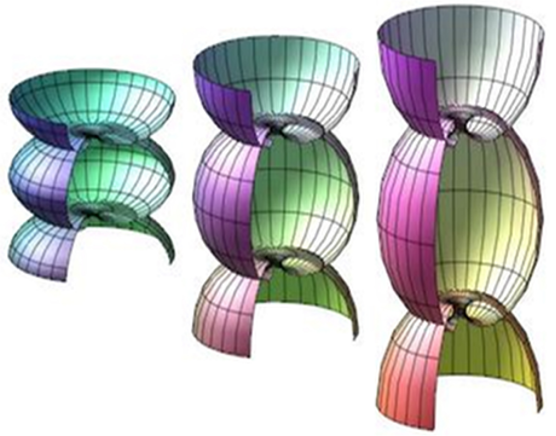

FIGURE 14. Anisotropic nodoids for the functionals $\gamma = 1 + e\nu_3^2$, with $e = -0.4$ (left), $e = 0$ (center) and $e = 0.4$ (right). In all cases $\Lambda = -1$ and $a = 0.3$.

The boundary of the anisotropic nodoid Σ_{a,Λ,t_0} consists of two circles of radius $q_1(t_0)$ on the planes $x_3 = q_3(t_0)$, and thus one has to impose

$$-\Lambda r_* = U(\cos t_0) + \sqrt{U^2(\cos t_0) + a}, \tag{5.5}$$

and

$$-\frac{\Lambda h_*}{2} = \int_0^{t_0} \frac{U(\cos\tau) + \sqrt{U^2(\cos\tau) + a}}{\sqrt{U^2(\cos\tau) + a}}\, \mathrm{d}v(\cos\tau). \tag{5.6}$$

Note that the nodoid Σ_{a,Λ,t_0} has vertical normal on the boundary precisely when $q_3'(t_0) = 0$, i.e., when t_0 is of the form $\frac{\pi}{2} + k\pi$, for $k \in \mathbb{N}$.

From (5.5) and (5.6), we get the following:

$$\frac{h_*}{2r_*}\big(U(\cos t_0) + \sqrt{U^2(\cos t_0) + a}\big) - \left[v(\cos(t_0)) + \int_0^{t_0} \frac{U(\cos\tau)}{\sqrt{U^2(\cos\tau) + a}}\, \mathrm{d}v(\cos\tau)\right] = 0. \tag{5.7}$$

We claim that for every value of the constant $\frac{h_*}{2r_*}$, equation (5.7) defines implicitly a smooth function $a = a(t_0)$, taking values in $\mathbb{R}^+$, and whose domain consists of the union of open intervals and an open half-line $]t_*, +\infty[$ and it contains all instants $t_0 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{N}$. In order to prove the claim, denote by $\mathcal{H}(t_0, a)$ the left-hand side of (5.7). For all $t_0 \in [0, +\infty[$:

$$\lim_{a \to +\infty} \mathcal{H}(t_0, a) = +\infty;$$

moreover, for t_0 sufficiently large or for t_0 near a point in $\frac{\pi}{2} + \pi\mathbb{N}$:

$$\mathcal{H}(t_0, 0) = \frac{h_*}{2r_*}\big(U(\cos t_0) + |U(\cos t_0)|\big) - \left(v(\cos(t_0)) + \int_0^{t_0} \operatorname{sign}(U(\cos\tau))\, \mathrm{d}v(\cos\tau)\right) < 0.$$

(Note that u is negative exactly on the part of W where v is decreasing). Finally, the partial derivative:

$$\frac{\partial\mathcal{H}}{\partial a} = \frac{h_*}{4r_*\sqrt{U^2(\cos t_0) + a}} + \frac{1}{2}\int_0^{t_0} \frac{U(\cos\tau)}{(U^2(\cos\tau) + a)^{\frac{3}{2}}}\, \mathrm{d}v(\cos\tau)$$

is strictly positive in $[0,+\infty[\times]0,+\infty[$. The domain $\mathcal{D}$ of the desired function a consists of all points t_0 where $\mathcal{H}(t_0,0)<0$, and it depends on the value of the ratio $\frac{h_*}{r_*}$; the smoothness of a follows from the Implicit Function Theorem. For example, if W is real-analytic, then all the functions involved are real-analytic, and so a is a real-analytic function of t_0.

Definition 5.1. The real-analytic family $\Sigma_{a(t_0),\Lambda_{t_0},t_0}$ will be denoted by Σ_{t_0}; it consists of all nodoids satisfying the boundary conditions, that are symmetric with respect to the (x_1,x_2)-plane.

For $t_0\in\frac{\pi}{2}+\pi\mathbb{N}$, Σ_{t_0} is degenerate. The proof of existence of a bifurcating branch of fixed boundary constant anisotropic curvature issuing from each of these degenerate anisotropic nodoids is totally analogous to the isotropic case, and it will be omitted. We will only show that, also for the anisotropic degenerate nodoids, the anisotropic mean curvature function has non zero derivative at the instants $t_0\in\frac{\pi}{2}+\pi\mathbb{N}$.

5.3. **On the anisotropic mean curvature function.** Let us now consider the anisotropic mean curvature function $t_0\mapsto\Lambda_{t_0}=\Lambda\big(t_0,a(t_0)\big)$ of Σ_{t_0}; let us show that, for $t_0=\frac{\pi}{2}+k\pi$, $k\in\mathbb{N}$, the derivative $\frac{\mathrm{d}}{\mathrm{d}t_0}\Lambda_{t_0}$ is non zero, so that $\Lambda(t_0)$ is strictly monotone near the degenerate instants. One can observe from the Wulff shape that

$$U(\nu_3=\pm1)=0=v_{\nu_3}(\nu_3=\pm1)\,,\quad U_{\nu_3}(\nu_3=\pm1)\neq0\neq v(\nu_3=\pm1)\,. \tag{5.8}$$

Then, observe that, from (5.7), one computes easily:

$$\frac{\partial\mathcal{H}}{\partial t_0}(\pi/2{+}k\pi,a)=-\frac{h_*}{2r_*}U_{\nu_3}(\nu_3=\pm1)\sin(\pi/2{+}k\pi)=(-1)^{k+1}\frac{h_*}{2r_*}U_{\nu_3}(\nu_3=\pm1)\neq0,$$

and so:

$$a'(\pi/2{+}k\pi)=-\frac{\partial\mathcal{H}}{\partial t_0}\big(\pi/2{+}k\pi,a(\pi/2{+}k\pi)\big)\cdot\left[\frac{\partial\mathcal{H}}{\partial a}\big(\pi/2+k\pi,a(\pi/2+k\pi)\big)\right]^{-1}\neq0.$$

From (5.6) and (5.8), we get:

$$\frac{d\Lambda}{dt_0}\Big|_{t_0=\pi/2+k\pi}=0,$$

and

$$h_*\frac{\partial\Lambda}{\partial a}(\pi/2+k\pi,a)=\int_0^{\pi/2+k\pi}\frac{U(\cos\tau)}{(U^2(\cos\tau)+a)^{3/2}}\,\mathrm{d}v(\cos(\tau))>0,$$

hence:

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\Big|_{t_0=\pi/2+k\pi}\Lambda_{t_0}=\frac{\partial\Lambda}{\partial a}\big(\pi/2+k\pi,a(\pi/2+k\pi)\big)\cdot a'(\pi/2+k\pi)\neq0.$$

A straightforward analysis of the sign in the above inequalities shows that the function $|\Lambda_{t_0}|$ is increasing for t_0 near $\frac{\pi}{2}+2k\pi$ and decreasing for t_0 near $\frac{\pi}{2}+(2k+1)\pi$, $k\in\mathbb{N}$.

Appendix A. Separation of variables in the Jacobi equation of axially symmetric CMC surfaces

Let us consider a smooth regular (not necessarily injective) curve $\gamma=(x,z)$:

$$(x,z):\big[-\tfrac{L}{2},\tfrac{L}{2}\big]\longrightarrow\mathbb{R}^2$$

parameterized by arclength, and the corresponding surface of revolution Σ in $\mathbb{R}^3$ given by parametric equations:

$$X=\begin{cases}x_1=x(\sigma)\cos\theta,\\ x_2=x(\sigma)\sin\theta,\\ x_3=z(\sigma),\end{cases}\qquad\sigma\in\big[-\tfrac{L}{2},\tfrac{L}{2}\big],\ \theta\in[0,2\pi].$$

This is an axially symmetric surfaces in $\mathbb{R}^3$, with symmetry axis given by the x_3-axis. Let us assume that:

(a) $x > 0$ is an even function;
(b) z is an odd function;
(c) x and $\dot{z}$ are periodic functions of period $T_0 > 0$.

Assumptions (a) and (b) mean that Σ is symmetric with respect to the plane $x_3 = 0$. Assumption (c) says that Σ is a portion of a periodic infinite surface, and its boundary consists of two coaxial circles of the same radius lying on the planes $x_3 = \pm z(L/2)$. By (a), $\dot{x}(0) = 0$, thus $\dot{z}(0) = \pm 1$; we can assume $\dot{z}(0) = 1$ (if not, replace z with $-z$).

An orthonormal basis of the pull-back metric on the cylinder $\mathcal{C} = [-L/2, L/2] \times [0, 2\pi]$ is given by:

$$w_1 = \frac{\partial}{\partial \sigma}, \qquad w_2 = \frac{1}{x}\frac{\partial}{\partial \theta},$$

and the area form is given by $x \mathrm{d}\sigma\,\mathrm{d}\theta$. The Gauss map $\nu : \mathcal{C} \to \mathbb{S}^2$ is given by $\nu(\sigma, \theta) = \pm\big(\dot{z}(\sigma)\cos\theta, \dot{z}(\sigma)\sin\theta, -\dot{x}(\sigma)\big)$; we choose the orientation of ν *outward pointing* at the point $X(0,0) = \big(x(0), 0, 0\big)$, i.e., $\nu(0,0) = e_1$, thus

$$\nu(\sigma, \theta) = \big(\dot{z}(\sigma)\cos\theta, \dot{z}(\sigma)\sin\theta, -\dot{x}(\sigma)\big).$$

The differential $\mathrm{d}\nu$ (the second fundamental form of X) is written relatively to the basis $\{w_1, w_2\}$ and $\{\frac{\partial X}{\partial \sigma}, \frac{1}{x}\frac{\partial X}{\partial \theta}\}$ as:

$$\mathrm{d}\nu(\sigma, \theta) \cong \begin{pmatrix} -\|\ddot{\gamma}(\sigma)\| & 0 \\ 0 & -\frac{\dot{z}(\sigma)}{x(\sigma)} \end{pmatrix}.$$

The principal curvatures of Σ are given by:

$$k_1 = \ddot{x}\dot{z} - \dot{x}\ddot{z}, \quad k_2 = -\frac{\dot{z}}{x}. \tag{A.1}$$

Note that k_1 is the curvature of the plane curve γ, thus:

$$\ddot{x} = k_1 \dot{z}, \quad \text{and} \quad \ddot{z} = -k_1 \dot{x}. \tag{A.2}$$

Let us assume that Σ has constant mean curvature H, i.e.:

$$2H = k_1 + k_2 = \ddot{x}\dot{z} - \dot{x}\ddot{z} - \frac{\dot{z}}{x} = \text{constant};$$

Codazzi equation then gives:

$$x^2(k_1 - k_2) = c \quad \text{(constant)},$$

from which we get the following conservation law:

$$2x\dot{z} + 2Hx^2 = c; \tag{A.3}$$

recall from (3.2) the identity $2cH = a$. This can also be obtained as an application of Noether's theorem, using the fact that the area functional is invariant by vertical translations. From (A.3), a straightforward calculation yields:

$$k_1 = H + \frac{c}{2x^2}, \quad k_2 = H - \frac{c}{2x^2}, \tag{A.4}$$

and differentiating we get:

$$(xk_1)' = k_2 \dot{x}, \quad (xk_2)' = k_1 \dot{x}, \tag{A.5}$$

which are the Codazzi equations.

The Hilbert–Schmidt norm $\|\mathrm{d}\nu\|^2$ is computed easily:

$$\|\mathrm{d}\nu\|^2 = \|\mathrm{d}\nu(w_1)\|^2 + \|\mathrm{d}\nu(w_2)\|^2 = k_1^2 + k_2^2 = \|\ddot{\gamma}\|^2 + \frac{\dot{z}^2}{x^2};$$

observe that k_1 and k_2 are even periodic functions, and so is $\|\mathrm{d}\nu\|^2$. Using (A.3), we obtain another expression for $\|\mathrm{d}\nu\|^2$, as follows:

$$\|\mathrm{d}\nu\|^2 = k_1^2 + k_2^2 = \left(2H + \frac{\dot z}{x}\right)^2 + \left(\frac{\dot z}{x}\right)^2 = \frac{1}{x^2}\big[2\dot z^2 + 2Hc\big] = \frac{1}{x^2}\big[2\dot z^2 + a\big]. \tag{A.6}$$

The Laplacian Δ of the pull-back metric is computed easily as:

$$\Delta = \frac{1}{x}\frac{\partial}{\partial\sigma}\left(x\frac{\partial}{\partial\sigma}\right) + \frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}.$$

Let J denote the Jacobi operator of Σ, given by:

$$J = \Delta + \|\mathrm{d}\nu\|^2 = \frac{1}{x}\frac{\partial}{\partial\sigma}\left(x\frac{\partial}{\partial\sigma}\right) + \frac{1}{x^2}\frac{\partial^2}{\partial\theta^2} + (\ddot x\dot z - \dot x\ddot z)^2 + \frac{\dot z^2}{x^2}.$$

The corresponding eigenvalue problem is:

$$JF = -\lambda F, \tag{A.7}$$

with boundary conditions

$$F(-L/2,\theta) = F(L/2,\theta) = 0, \qquad \forall\,\theta; \tag{A.8}$$

we use the method of separation of variables for this linear equation, i.e., we look for solutions $F(\sigma,\theta)$ that are product $F = S(\sigma)T(\theta)$. This leads to the following pairs of boundary value problems for ODE's:

$$T'' + \kappa T = 0, \quad T(0) = T(2\pi), \quad T'(0) = T'(2\pi), \tag{A.9}$$

and

$$-(xS')' + \left(\frac{\kappa}{x} - x\|\mathrm{d}\nu\|^2\right)S = \lambda x S, \quad S\big(-\tfrac{L}{2}\big) = S\big(\tfrac{L}{2}\big) = 0. \tag{A.10}$$

Problem (A.9) has non trivial solutions when $\kappa = n^2$, $n \in \mathbb{N}$, with corresponding eigenfunctions $\cos n\theta$ and $\sin n\theta$; substituting $\kappa = n^2$ in (A.10) we get:

$$\begin{cases} -(xS')' + \left(\dfrac{n^2}{x} - x\|\mathrm{d}\nu\|^2\right)S = \lambda x S, \\ S\big(-\tfrac{L}{2}\big) = S\big(\tfrac{L}{2}\big) = 0. \end{cases} \tag{A.11}$$

Solutions of (A.11) with $n = 0$ are precisely the axially symmetric solutions of (A.7); in particular, the axially symmetric Jacobi fields are the solutions of (A.11) with $n = 0$ and $\lambda = 0$. Observe that the differential equation in (A.11) is a *Sturm–Liouville equation*; an immediate application of Sturm theory gives the following.

Proposition A.1. *The following statements hold:*

(a) *For all $n \ge 0$, there exists a diverging sequence $\lambda_1^{(n)} < \lambda_2^{(n)} < \dots$ of eigenvalues of the boundary value problem* (A.11). *The corresponding eigensolutions $S_{n,1}$, $S_{n,2}$, …, can be chosen to form an orthonormal basis of the Lebesgue space $L^2\big([-L/2, L/2]\big)$ relatively to the measure* $\mathrm{d}\mu = x\mathrm{d}\sigma$. *For all $j \ge 1$, $S_{n,j}$ has exactly $j-1$ zeroes in* $]-L/2, L/2[$.

(b) *For all eigenvalue λ of problem* (A.7) *with boundary conditions $F(-L/2,\theta) = F(L/2,\theta) = 0$, there exists a finite sequence $n_1(\lambda)$,… $n_{m_\lambda}(\lambda)$ of nonnegative integers such that, for $n = n_j(\lambda)$, problem* (A.11) *has a non trivial solution $S^{\lambda,j}$. Every eigensolution of* (A.7) *is a linear combination of the functions*

$$S^{\lambda,j}(\sigma)\cos\big(n_j(\lambda)\theta\big) \quad \textit{and} \quad S^{\lambda,j}(\sigma)\sin\big(n_j(\lambda)\theta\big),$$

$j = 1, \dots, m_\lambda$.

Proof. Part (a) is precisely Sturm–Liouville theory of the boundary value problem (A.11).

For part (b), observe that if the function:

$$\frac{n^2}{x} - x\|\mathrm{d}\nu\|^2 - \lambda x =: \varphi \tag{A.12}$$

is strictly positive on $[-L/2, L/2]$, then no nontrivial solution of the ODE in (A.11) can have two distinct zeroes in $[-L/2, L/2]$. This is proved with an elementary argument on the sign of the second derivative at a critical point of a solution of the ODE. Now, since $x > 0$, given any $\lambda \in \mathbb{R}$ then for n large enough the quantity (A.12) is positive. This implies that for each given λ, there is only a finite sequence of n's for which (A.11) has non trivial solutions. Note that, by our periodicity assumptions, an upper bound for the values of n for which (A.11) has non trivial solutions can be found[15] independent of the constant L.

For each $n \in \mathbb{N} \cup \{0\}$, let $S_{n,k}$ be a k-th eigenfunction of (A.11). $\{S_{n,k}\ ;\ k \in \mathbb{N}\}$ can be chosen so that it forms an orthogonal basis of $L^2\big([0, 2\pi], \mathrm{d}\theta\big)$ and for all n, the functions $S_{n,k}$, $k \geq 1$, form an orthonormal basis of $L^2\big([-L/2, L/2], x\,\mathrm{d}\sigma\big)$, then the doubly indexed family

$$\mathcal{F} = \big\{S_{n,k}(\sigma)\cos(n\theta), S_{n,k}(\sigma)\sin(n\theta)\big\}_{n\geq 0, k\geq 1}$$

forms an orthogonal basis of $L^2(\mathcal{C}, x\,\mathrm{d}\sigma\mathrm{d}\theta)$, and it consists of eigenfunctions of (A.7).

Since eigenfunctions of (A.7) corresponding to distinct eigenvalues are orthogonal in $L^2(\mathcal{C}, x\,\mathrm{d}\sigma\mathrm{d}\theta)$, it follows that every eigenfunction of (A.7) with eigenvalue λ is a linear combination of those $S_{n,k}\cos(n\theta)$ and $S_{n,k}\sin(n\theta)$ for which $\lambda_k^{(n)} = \lambda$. □

Corollary A.2. *Let $L > 0$, $\lambda \in \mathbb{R}$ be fixed, and assume that every non zero solution F of* (A.7) *and* (A.8) *is not axially symmetric. Then, $\int_\Sigma F\,\mathrm{d}\Sigma = 0$.*

Proof. By part (b) of Proposition A.1, every such F is a linear combination of functions of the form $S\cdot\cos(n\theta)$ and $S\cdot\sin(n\theta)$, with $n > 0$, and $\int_0^{2\pi}\cos(n\theta)\,\mathrm{d}\theta = \int_0^{2\pi}\sin(n\theta)\,\mathrm{d}\theta = 0$ for all $n > 0$. □

A.1. **Axially symmetric Jacobi fields.** Let us now consider the ODE:

$$(xS')' + x\|\mathrm{d}\nu\|^2 S = 0; \tag{A.13}$$

this corresponds to the equation in (A.11) with $n = \lambda = 0$. We can extend by periodicity its coefficients, and we study its solutions on the entire real line. Such solutions correspond to axially symmetric Jacobi fields along Σ. Denote by $S_\mathrm{o}, S_\mathrm{e} : \mathbb{R} \to \mathbb{R}$ the solutions of (A.13) satisfying:

$$S_\mathrm{o}(0) = 0, \quad S_\mathrm{o}'(0) = 1, \quad \text{and} \quad S_\mathrm{e}(0) = 1, \quad S_\mathrm{e}'(0) = 0.$$

The function S_o is a constant multiple of $\dot{x}$:

$$S_\mathrm{o}(\sigma) = \ddot{x}(0)^{-1}\dot{x}(\sigma).$$

Namely, one checks easily that the function $\dot{x}$ is a solution of (A.13); using (A.1), (A.2) and (A.5) we compute:

$$(x\ddot{x})' = (xk_1\dot{z})' = (xk_1)'\dot{z} + xk_1\ddot{z} = \dot{x}k_2\dot{z} - k_1^2 x\dot{x} = -x(k_1^2 + k_2^2)\dot{x}.$$

This solution is the axially symmetric Jacobi field ν_3; it is an odd periodic function. Now, using the fact that x and $\|\mathrm{d}\nu\|^2$ are even functions, it is easily checked that the function S_e is also even.

Zeroes of the functions S_o and S_e correspond to values of L such that the CMC surface Σ is degenerate as a constrained critical point of the area functional. Since $x\|\mathrm{d}\nu\|^2 > 0$, both S_o and S_e have an infinite sequence of zeroes, occurring at points symmetric with

[15] For instance, $n^2 \leq \max_{[0,t_0]} x^2\big[\lambda + \|\mathrm{d}\nu\|^2\big]$.

respect to the origin. It is easily checked that if a linear combination $\alpha S_{\rm o}+\beta S_{\rm e}$ has zeros at two points symmetric with respect to the origin, then either α or β must be zero. In other words, the only degeneracy instants of Σ by axially symmetric Jacobi fields occur at the zeroes of $S_{\rm o}$ and of $S_{\rm e}$.

If $r_{\rm o}$ is a zero of $S_{\rm o}$, then clearly $\int_{-r_{\rm o}}^{r_{\rm o}} xS_{\rm o}\,{\rm d}\sigma=0$. If $r_{\rm e}$ is the first positive zero of $S_{\rm e}$, then:

$$\int_{-r_{\rm e}}^{r_{\rm e}} xS_{\rm e}\,{\rm d}\sigma\neq 0.$$

Namely, $S_{\rm e}$ is positive in $\left]-r_{\rm e},r_{\rm e}\right[$. Let us give a geometrical condition equivalent to the non vanishing of the integral of $S_{\rm e}$ between two symmetric zeroes.

Proposition A.3. *Assume $H\neq 0$, and let $r_{\rm e}>0$ be a zero of $S_{\rm e}$. Then, the integral $\int_{-r_{\rm e}}^{r_{\rm e}} xS_{\rm e}\,{\rm d}\sigma$ is zero if and only if the position vector $\big(x_1(r_{\rm e}),x_3(r_{\rm e})\big)$ is tangent to the curve γ at $\gamma(r_{\rm e})$.*

Proof. Consider the support function $q:[-L/2,L/2]\to\mathbb{R}$ defined by $q=X\cdot\nu$. Then, $Jq=-2H$. Namely, q can be written as $\frac{\rm d}{{\rm d}t}\vert_{t=1}X_t\cdot\nu$, where $X_t=tX$ is a CMC variation of X with mean curvature $H(X_t)=\frac{1}{t}H$. Using (2.4), we have:

$$Jq=2\frac{\rm d}{{\rm d}t}\Big\vert_{t=1}\big(\tfrac{1}{t}H\big)=-2H\neq 0.$$

Partial integration gives:

$$\begin{aligned}-2H\int_{-r_{\rm e}}^{r_{\rm e}} xS_{\rm e}\,{\rm d}\sigma&=\int_{-r_{\rm e}}^{r_{\rm e}} S_{\rm e}(Jq)x\,{\rm d}\sigma\\&=\int_{-r_{\rm e}}^{r_{\rm e}}(JS_{\rm e})qx\,{\rm d}\sigma-2S_{\rm e}'(r_e)q(r_{\rm e})x(r_{\rm e})=-2S_{\rm e}'(r_e)q(r_{\rm e})x(r_{\rm e}).\end{aligned}$$

Since $S_{\rm e}'(r_e)x(r_{\rm e})\neq 0$, it follows that $\int_{-r_{\rm e}}^{r_{\rm e}} xS_{\rm e}\,{\rm d}\sigma=0$ if and only if $q(r_{\rm e})=0$, which is the thesis. $\square$

A.2. **Non axially symmetric Jacobi fields.** Let us now consider the equation:

$$(xS')'+x\|{\rm d}\nu\|^2S-\frac{n^2}{x}S=0,\quad n\geq 1.$$

Each solution of this equation provides two linearly independent non axially symmetric Jacobi fields along Σ; if S is a non zero solution of this equation, the corresponding Jacobi fields $J_1=S\cdot\sin(n\theta)$ and $J_2=S\cos(n\theta)$ are invariant by the rotation of $\frac{2\pi}{n}$. Moreover, if S is not even, then J_1 and J_2 are not symmetric with respect to reflections around the plane $x_3=0$.

As above, one proves that for all $n\geq 1$ this equation admits two linearly independent solutions, one even and one odd, corresponding to the initial values $S(0)=1$, $S'(0)=0$ and $S(0)=0$, $S'(0)=1$, respectively. However, for n large, such solutions may not have nodal domains, i.e., may not have two distinct zeroes. This is the case, for instance, when $n>\max x\|{\rm d}\nu\|^2$.

When $n=1$, the function $\dot z$ is the even solution of the equation. This is checked using again the identities (A.1), (A.2) and (A.5); setting $S=\dot z$:

$$\begin{aligned}&(xS')'=(x\ddot z)'=-(k_1\dot x x)'=-(k_1x)'\dot x-k_1x\ddot x=-k_2\dot x^2-k_1^2x\dot z\\&=-k_2(1-\dot z^2)-k_1^2x\dot z=-k_2+k_2\dot z^2-k_1^2x\dot z=-k_2-(k_1^2+k_2^2)x\dot z=\big[\tfrac{1}{x}-(k_1^2+k_2^2)x\big]S.\end{aligned}$$

The functions $\dot z\cos\theta$ and $\dot z\sin\theta$ are the Jacobi fields ν_1 and ν_2.

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