Measurement error model with a general class of error distribution for the surrogate variable

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Motivating example

Sleep disordered breathing

One interest in epidemiological studies is to analyze the relation between:

blood pressure (Y) and **sleep disordered breathing** (x)

and other variable such as gender (w_1), age (w_2) and body mass index (w_3).

Problem:

The sleep disordered breathing cannot be observed directly.

The surrogate variable

The **apnea-hypopnea index** (X) is observed in the place of the sleep disordered breathing:

> *It is the number of occurrences of apnea and hypopnea per sleep hour.*

- *A*pnea occurs when there is no breathing during 10 seconds;
- *H*ypopnea occurs when there is a breathing reduction detected by airway obstruction noises.

The AHI (X) and SDB (x) are assumed to be connected by (Li, Palta and Shao, 2004)

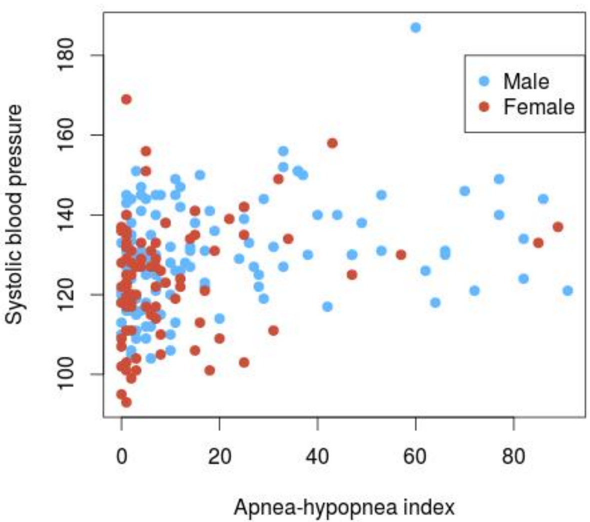
$$X|x \sim Poisson(x).$$

Wisconsin sleep cohort study data

Ind	SBP	AHI	AGE	BMI	Gender
1	130	3	51	20.08	M
2	121	7	56	23.12	M
3	125	5	58	30.93	M
4	110	0	35	27.47	M
⋮	⋮	⋮	⋮	⋮	⋮
210	113	16	50	44.38	F
211	151	5	55	21.63	F
212	131	6	50	37.19	F
213	119	1	61	29.37	F

SBP: systolic blood pressure
AHI: apnea-hypopnea index
BMI: body mass index
sample size: 130 males and 83 females

Wisconsin sleep cohort study data



On the simple measurement error model

Typically, the measurement error model is composed by two equations.

The **regression equation**:

$$Y_i = \beta + \gamma x_i + e_i$$

The **measurement equation**:

$$X_i = x_i + u_i \quad \text{or} \quad X_i = u_i x_i$$

where u_i,e_i, $i = 1, \ldots, n$, are independent random variables (usually normally distributed).

Structural model: x_i, $i = 1, \ldots, n$, are random variables.

Functional model: x_i, $i = 1, \ldots, n$, are incidental parameters.

The measurement error equation

The measurement equations could simply be replaced by:

$$X_i|x_i \stackrel{ind}{\sim} F_{X_i|x_i}$$

which contains all the probabilistic information of the measurement equation.

In this presentation, I consider the above expression in the place of the measurement error equation.

The proposed Model

The proposed model

Let $(Y_i, \boldsymbol{W}_i^\top, \boldsymbol{X}_i^\top)^\top$, $i \geq 1$, be vectors related by the following equations

$$\begin{array}{rcl} Y_i & = & \boldsymbol{\beta}^\top \boldsymbol{W}_i + \boldsymbol{\gamma}^\top \boldsymbol{x}_i + e_i, \\ \boldsymbol{X}_i|\boldsymbol{x}_i & \stackrel{ind}{\sim} & F_{\boldsymbol{X}_i|\boldsymbol{x}_i} \in \mathcal{C}(\boldsymbol{x}_i, g_1, g_2), \end{array} \tag{1}$$

- Y_i is the dependent random variable,
- $\boldsymbol{W}_i \in \mathbb{R}^q$ is a vector of covariate measured without error,
- $\boldsymbol{x}_i \in \mathbb{R}^p$ is a vector of unobservable covariates
- $\boldsymbol{X}_i \in \mathbb{R}^p$ is the surrogate of $\boldsymbol{x}_i$,
- $e_i \stackrel{iid}{\sim} N(0, \sigma^2)$ is the model error (it could be a skewed distribution).

Also, $F_{\boldsymbol{X}_i|\boldsymbol{x}_i}$ is the unknown distribution of $\boldsymbol{X}_i$ given $\boldsymbol{x}_i$ which lies in the class of distributions $\mathcal{C}(\boldsymbol{x}_i, g_1, g_2)$, where the functions $g_1(.)$ and $g_2(.)$ are known and must satisfy the following conditions

$$E[g_1(\boldsymbol{X}_i)|\boldsymbol{x}_i] = \boldsymbol{x}_i \quad \text{and} \quad E[g_2(\boldsymbol{X}_i)|\boldsymbol{x}_i] = \boldsymbol{x}_i\boldsymbol{x}_i^\top \tag{2}$$

Remarks

- We use the corrected score method proposed by Nakamura (1990) to conduct inferences about the parameters $\boldsymbol{\beta}$, γ and σ^2.
- It is not necessary to know the shape of $F_{\boldsymbol{X}_i|\boldsymbol{x_i}}$,
- It is only required to know the shape of g_1 and g_2 to employ this methodology.

Next we present same examples of g_1 and g_2.

Particular cases

Normal distribution and $p = 1$

Assume that $X_i|x_i \sim \mathsf{N}(x_i, \phi)$, where $\phi > 0$ is known. Then,

- $E(X_i|x_i) = x_i$ and $E(X_i^2|x_i) = \phi + x_i^2$
- $g_1(X_i) = X_i$ and $g_2(X_i) = X_i^2 - \phi$.

It is the additive model: $X_i = x_i + u_i$, where $u_i \sim N(0, \phi)$.

Notice that any distribution $F_{X_i|x_i}$ that yields the same g_1 and g_2 as above is such that $F_{X_i|x_i} \in \mathcal{C}(x_i, g_1, g_2)$.

Poisson distribution and $p = 1$

Assume that $X_i|x_i \sim \text{Poisson}(x_i)$. Then,

- $E(X_i|x_i) = x_i$ and $E(X_i^2 - X_i|x_i) = x_i^2$
- $g_1(X_i) = X_i$ and $g_2(X_i) = X_i^2 - X_i$.

Notice that any distribution $F_{X_i|x_i}$ such that

$$E(X_i|x_i) = Var(X_i|x_i) = x_i$$

produces the same g_1 and g_2 as above is such that $F_{X_i|x_i} \in \mathcal{C}(x_i, g_1, g_2)$.

Multiplicative normal model or Gamma distribution and $p = 1$

Assume $X_i|x_i \sim \mathcal{N}(x_i, x_i^2\phi)$, with $\phi > 0$ known, then

- $E(X_i|x_i) = x_i$ and $E(X_i^2|x_i) = (\phi + 1)x_i^2$
- $g_1(X_i) = X_i$ and $g_2(X_i) = X_i^2/(\phi + 1)$.

It is the multiplicative model: $X_i = x_i u_i$, where $u_i \sim N(1, \phi)$.

Notice that, $X_i|x_i \sim \text{Gamma}(x_i, \phi)$, where $E(X_i|x_i) = x_i$ and $Var(X_i|x_i) = x_i^2\phi$ also yields the same functions above. This gamma distribution is a reparameterization of the usual version.

That is, $\mathcal{N}(x_i, x_i^2\phi), \text{Gamma}(x_i, \phi) \in \mathcal{C}(x_i, g_1, g_2)$

Examples of g_1 and g_2 for the multivariate normal distribution

Assume $\boldsymbol{X}_i|\boldsymbol{x}_i \sim \mathsf{N}_p(\boldsymbol{x}_i, \boldsymbol{\Sigma}_i)$, where $\boldsymbol{\Sigma}_i$ is known for each $i = 1, \ldots, n$. Then,

- $E(\boldsymbol{X}_i|\boldsymbol{x}_i) = \boldsymbol{x_i}$ and $E(\boldsymbol{X}_i\boldsymbol{X}_i^\top|\boldsymbol{x}_i) = \boldsymbol{\Sigma}_i + \boldsymbol{x}_i\boldsymbol{x}_i^\top$
- $g_1(\boldsymbol{X}_i) = \boldsymbol{X}_i$ and $g_2(\boldsymbol{X}_i) = \boldsymbol{X}_i\boldsymbol{X}_i^\top - \boldsymbol{\Sigma}_i$.

Examples of g_1 and g_2 for a 'mixed' multivariate distribution

Assume $\boldsymbol{X}_i = (X_{1i}, X_{2i})^\top$ such that $X_{1i} \sim \mathsf{N}(x_{1i}, \phi_i)$, $X_{2i} \sim \mathsf{Poisson}(x_{2i})$ and $\mathsf{Cov}(X_{1i}, X_{2i}) = a_i$ known for each $i = 1, \ldots, n$. Then,

- $E(\boldsymbol{X}_i|\boldsymbol{x}_i) = \boldsymbol{x_i}$ and $E(\boldsymbol{X}_i\boldsymbol{X}_i^\top|\boldsymbol{x}_i) = \begin{bmatrix} x_{1i}^2 + \phi_i & x_{1i}x_{2i} + a_i \\ x_{1i}x_{2i} + a_i & x_{2i}^2 + x_{2i} \end{bmatrix}$
- $g_1(\boldsymbol{X}_i) = \boldsymbol{X}_i, \qquad g_2(\boldsymbol{X}_i) = \begin{bmatrix} X_{1i}^2 - \phi_i & X_{1i}X_{2i} - a_i \\ X_{1i}X_{2i} - a_i & X_{2i}^2 - X_{2i} \end{bmatrix}.$

Estimation procedure

Estimation procedure

We use the corrected score methodology proposed by Nakamura (1990).

We need to find a pseudo-log-likelihood function ℓ^+ which depends only on the observed data $(\boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X})$ such that

$$E\left[\ell^+(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X}) \middle| \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x}\right] = \ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x})$$

where

$\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\gamma}^\top, \sigma^2)^\top$

$\ell^+(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X})$ is the corrected log-likelihood function

$\ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x})$ is the “true” log-likelihood function as if $\boldsymbol{x}$ were observed.

How to find ℓ^+

In order to find ℓ^+, we use the likelihood function attained by means of the model

$$Y_i = \boldsymbol{\beta}^\top \boldsymbol{W}_i + \boldsymbol{\gamma}^\top \boldsymbol{x}_i + e_i. \tag{3}$$

Let $\ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x})$ be the log-likelihood function related with (3), then

$$\ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x}) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)^2 + \\ - 2(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)\boldsymbol{x}_i^\top \boldsymbol{\gamma} + \boldsymbol{\gamma}^\top \boldsymbol{x}_i \boldsymbol{x}_i^\top \boldsymbol{\gamma}\}$$

How to find ℓ^+

Replacing $\boldsymbol{x}_i$ with $\boldsymbol{X}_i$, we obtain the "naïve" log-likelihood function $\ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X})$:

$$\ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X}) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)^2 + \\ - 2(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)\boldsymbol{X}_i^\top \boldsymbol{\gamma} + \boldsymbol{\gamma}^\top \boldsymbol{X}_i \boldsymbol{X}_i^\top \boldsymbol{\gamma}\}$$

and from the naïve log-likelihood function we obtain the corrected log-likelihood function

$$\ell^+(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X}) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)^2 + \\ - 2(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)g_1(\boldsymbol{X}_i)^\top \boldsymbol{\gamma} + \boldsymbol{\gamma}^\top g_2(\boldsymbol{X}_i)\boldsymbol{\gamma}\}$$

which satisfies

$$E\left[\ell^+(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{X}) \middle| \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x}\right] = \ell(\boldsymbol{\theta}, \boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{x})$$

Estimators

Maximizing ℓ^+ with respect to the parameters, we obtain

$$\hat{\boldsymbol{\beta}}_n = \left(\sum_{i=1}^{n} \boldsymbol{W}_i \boldsymbol{W}_i^\top\right)^{-1} \sum_{i=1}^{n} \boldsymbol{W}_i \left[Y_i - \boldsymbol{\gamma}^\top g_1(\boldsymbol{X}_i)\right],$$

$$\hat{\gamma}_n = \boldsymbol{H}_n^{-1} \left[\sum_{i=1}^{n} g_1(\boldsymbol{X}_i) Y_i - \sum_{i=1}^{n} g_1(\boldsymbol{X}_i) \boldsymbol{W}_i^\top \left(\sum_{i=1}^{n} \boldsymbol{W}_i \boldsymbol{W}_i^\top\right)^{-1} \sum_{i=1}^{n} \boldsymbol{W}_i Y_i\right]$$

and

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \left\{(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i)^2 - 2(Y_i - \boldsymbol{\beta}^\top \boldsymbol{W}_i) g_1(\boldsymbol{X}_i)^\top \boldsymbol{\gamma} + \boldsymbol{\gamma}^\top g_2(\boldsymbol{X}_i) \boldsymbol{\gamma}\right\},$$

where
$\boldsymbol{H}_n = \sum_i g_2(\boldsymbol{X}_i)^\top - \sum_i g_1(\boldsymbol{X}_i) \boldsymbol{W}_i^\top \left(\sum_i \boldsymbol{W}_i \boldsymbol{W}_i^\top\right)^{-1} \sum_i \boldsymbol{W}_i g_1(\boldsymbol{X}_i)^\top$.

Asymptotic distribution

Under certain regular conditions (Gimenez and Bolfarine, 1997), we have that

$$\sqrt{n}\boldsymbol{L}_n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow{\mathcal{D}} \mathcal{N}_s(\boldsymbol{0}, \boldsymbol{I})$$

where $s = p + q + 1$,

$$\boldsymbol{L}_n^{1/2} = \bar{\boldsymbol{\Gamma}}_n(\hat{\boldsymbol{\theta}}_n)^{-1/2}\bar{\boldsymbol{\Lambda}}_n(\hat{\boldsymbol{\theta}}_n),$$

$$\bar{\boldsymbol{\Gamma}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \ell_i^+(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\frac{\partial \ell_i^+(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^\top}, \qquad \bar{\boldsymbol{\Lambda}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \ell_i^+(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^\top}$$

and

$$\ell^+(\boldsymbol{\theta}) = \sum_{i=1}^{n}\ell_i^+(\boldsymbol{\theta})$$

Testing a general linear hypothesis

Wald statistics can be used to test the general null hypothesis $H_0 : \boldsymbol{C\theta} = \boldsymbol{d}$ against $H_1 : \boldsymbol{C\theta} \neq \boldsymbol{d}$,

$$\mathcal{W}_{H_0} = n\big(\boldsymbol{C}\hat{\boldsymbol{\theta}}_n - \boldsymbol{d}\big)^\top \big[\boldsymbol{C}\boldsymbol{L}_n^{-1}\boldsymbol{C}^\top\big]^{-1}\big(\boldsymbol{C}\hat{\boldsymbol{\theta}}_n - \boldsymbol{d}\big) \xrightarrow{\mathcal{D}} \chi^2_k$$

where $k = \mathrm{rank}(\boldsymbol{C})$.

The asymptotic covariance matrix for $\hat{\boldsymbol{\theta}}_n$ can be estimated by

$$\mathrm{Cov}_a(\hat{\boldsymbol{\theta}}_n) = \frac{1}{n}\boldsymbol{L}_n^{-1} = \frac{1}{n}\boldsymbol{L}_n^{-1/2}\boldsymbol{L}_n^{-1/2\top}$$

Examples

Normal case

In the next slides, I consider the model: $Y_i = \beta + \gamma x_i + e_i$, where $e_i \overset{iid}{\sim} N(0, \sigma^2)$.

1. Normal: $X_i | x_i \sim N(x_i, \phi_i^2)$ with ϕ_i known for each $i = 1, \ldots, n$.

$$\hat{\beta}_n = \bar{Y} - \hat{\gamma}_n \bar{X}, \qquad \hat{\gamma}_n = \frac{\sum_{i=1}^n Y_i X_i - n\,\bar{Y}\,\bar{X}}{\sum_{i=1}^n X_i^2 - n\bar{X}^2 - \sum_{i=1}^n \phi_i^2}$$

and $$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \left\{ (Y_i - \hat{\beta}_n - \hat{\gamma}_n X_i)^2 - \phi_i^2 \hat{\gamma}_n^2 \right\}.$$

Gamma distribution

2. Gamma: $X_i|x_i \sim \text{Gamma}(x_i, \phi)$ with ϕ known.

$$\hat{\beta}_n = \bar{Y} - \hat{\gamma}_n \bar{X}, \qquad \hat{\gamma}_n = \frac{\sum_{i=1}^n Y_i X_i - n \bar{Y} \bar{X}}{(1+\phi)^{-1} \sum_{i=1}^n X_i^2 - n \bar{X}^2}$$

and

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \left\{ (Y_i - \hat{\beta}_n - \hat{\gamma}_n X_i)^2 - \frac{\phi}{1+\phi} \hat{\gamma}_n^2 X_i^2 \right\}.$$

Poisson distribution

Poisson: $X_i | x_i \sim \text{Poisson}(x_i)$.

$$\hat{\beta}_n = \bar{Y} - \hat{\gamma}_n \bar{X}, \qquad \hat{\gamma}_n = \frac{\sum_{i=1}^n Y_i X_i - n\,\bar{Y}\,\bar{X}}{\sum_{i=1}^n X_i^2 - n\bar{X}(1+\bar{X})}$$

and

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \left\{ (Y_i - \hat{\beta}_n - \hat{\gamma}_n X_i)^2 - \hat{\gamma}_n^2 X_i \right\}.$$

This case was studied by Li, Palta and Shao (2004).

Simulations

Monte Carlo simulations

Consider the model $Y_i = \beta_1 + \beta_2 T_i + \gamma x_i + e_i$, where T_i represents the treatment indicator. The parameter values are: $\beta_1 = 1$, $\beta_2 = 2$, $\gamma = 1$ and $\text{Var}(e_i) = 5$.

Consider also that $x_i \sim Uniform\{1, \dots, 10\}$, $i = 1, \dots, n$. We generate $25\ 000$ Monte Carlo simulations under two scenarios:

i. $X_i | x_i \sim \text{Gamma}(x_i, 0.01)$
ii. $X_i | x_i \sim \text{Poisson}(x_i)$,

The following sample sizes are considered $n = 20$, $n = 50$, $n = 100$ and $n = 200$.

The Bias and the Mean Square Error (MSE) are presented.

Gamma distribution

	Sample size	Negative variance	Gamma			
			β_1	β_2	γ	σ^2
MSE	n=50	0.58%	0.6749	0.3428	0.0188	1.6996
Bias			−0.2056	0.0450	0.0360	−0.7091
MSE	n=100	0%	0.2593	0.1470	0.0103	0.7130
Bias			−0.1023	−0.0004	0.0207	−0.3618
MSE	n=200	0%	0.1576	0.0842	0.0045	0.4280
Bias			−0.0570	0.0009	0.0099	−0.1954

Poisson distribution

	Sample size	Negative variance	Poisson β_1	β_2	γ	σ^2
MSE	n=50	3.67%	0.7792	0.4069	0.0217	2.2102
Bias			−0.1877	0.0310	0.0313	−0.7669
MSE	n=100	0.32%	0.4037	0.1912	0.0123	1.1485
Bias			−0.1351	0.0095	0.0245	−0.4648
MSE	n=200	0.01%	0.2221	0.0970	0.0066	0.5713
Bias			−0.0716	−0.0075	0.0128	−0.2118

Testing $H_0 : \gamma = \gamma_0$: significance level 5%

		Gamma		Poisson	
		Proposed Model	Naïve Model	Proposed Model	Naïve Model
$n = 50$					
	-3	5.70	<0.01	6.00	2.43
	-2	5.08	0.01	6.02	1.56
γ_0	-1	4.26	0.01	4.83	0.29
	1	4.50	0.04	4.74	0.24
	2	5.16	<0.01	5.46	1.36
	3	5.31	0.01	5.69	2.24

Testing $H_0 : \gamma = \gamma_0$: significance level 5%

		Gamma		Poisson	
		Proposed Model	Naïve Model	Proposed Model	Naïve Model
$n = 100$					
	-3	5.44	$<$0.01	5.83	1.32
	-2	5.03	$<$0.01	5.62	0.63
γ_0	-1	4.64	0.02	4.77	0.01
	1	4.86	0.02	4.80	0.03
	2	4.96	$<$0.01	5.50	0.60
	3	4.98	$<$0.01	6.06	1.16

Testing $H_0 : \gamma = \gamma_0$: significance level 5%

		Gamma		Poisson	
		Proposed Model	Naïve Model	Proposed Model	Naïve Model
$n = 200$					
	-3	5.16	$<$0.01	5.88	0.51
	-2	5.03	$<$0.01	5.30	0.10
γ_0	-1	4.71	0.01	4.73	$<$0.01
	1	4.91	$<$0.01	4.74	$<$0.01
	2	4.75	$<$0.01	5.22	0.14
	3	5.44	$<$0.01	5.95	0.48

Application

Wisconsin sleep cohort study data

The model for the Wisconsin sleep cohort study data is

$$Y_i = \beta_0 + \beta_1 W_{1i} + \beta_2 W_{2i} + \beta_3 W_{3i} + \gamma x_i + e_i,$$

where W_{1i} : is the age; W_{2i} : is the body mass index; W_{3i} : is the gender (1=male; 0 = Female); x_i : represents the sleep disordered breathing; and X_i is the observed apnea-hypopnea index (Poisson distribution).

	Naïve model Estimative (SD)	Proposed model Estimative (SD)
β_0	78 (7.53)	78 (7.60)
β_1	0.41 (0.11)	0.41 (0.11)
β_2	0.80 (0.16)	0.79 (0.16)
β_3	5.86 (1.82)	5.81 (1.84)
γ	0.59 (0.30)	0.62 (0.30)
σ^2	150	145

Final remarks

Final remarks

- This work generalizes the results proposed in Li, Palta and Shao (2004). The authors considered only a Poisson distribution to the surrogate covariate.

- We are still working on a polynomial regression model:

$$\begin{aligned} Y_i &= \boldsymbol{\beta}^\top \boldsymbol{W}_i + \gamma_1 x_i + \gamma_2 x_i^2 + \ldots + \gamma_p x_i^p + e_i \\ X_i | x_i &\sim \mathcal{G} \in \mathcal{C}(x_i, g_1, g_2, \ldots, g_{2p}), \end{aligned}$$

- The distribution of the error e_i could be in the class of elliptical or skew distributions.

References

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Thank you