In addition to serving ex officio on the Board of Governors and various Association committees (the Executive and Finance Committees, the Committees on Publications and on Membership, and the Joint Committee on Places of Meetings), he has served by appointment on nominating committees, on the planning committee for the fiftieth anniversary celebration, and as chairman of the Committee on the Structure of the Government of the Association. He has represented the Association in the Division of Mathematics of the National Research Council, on the Policy Committee for Mathematics and then the Conference Board of the Mathematical Sciences, where he was elected to an additional term as member-at-large. He was Chairman in 1951–52 of the mathematics section of the American Society for Engineering Education and participated three summers (1959–61) in writing projects of the School Mathematics Study Group.

Over and beyond this handsome record of accomplishment shine the amiable warmth and sterling spirit of the man himself—and his wife, Marian, who has shared in his mission to mathematics. It has been service with the extra zeal and zest of a man who enjoys working with other people and accepts their foibles with a chuckle. For all this we proudly and gratefully hail Harry Merrill Gehman.

A. W. TUCKER

AVOIDING THE JORDAN CANONICAL FORM IN THE DISCUSSION OF LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

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Consider the differential equation

$$\dot{x} = Ax; \quad x(0) = x_0; \quad 0 \leqq t < \infty, \tag{1}$$

where x and x_0 are n-vectors and A is an $n \times n$ matrix of constants. In this paper we present two methods, believed to be new, for explicitly writing down the solution of (1) without making any preliminary transformations. This is particularly useful, both for teaching and applied work, when the matrix A cannot be diagonalized, since the necessity of discussing or finding the Jordan Canonical Form (J.C.F.) of A is completely by-passed.

If e^{At} is defined as usual by a power series it is well known (see [1]) that the solution of (1) is

$$x = e^{At}x_0,$$

so the problem is to calculate the function e^{At}. In [2], this is done via the J.C.F. of A. In [1], it is shown how the J.C.F. can be by-passed by a transformation which reduces A to a triangular form in which the off diagonal elements are arbitrarily small. While this approach permits a theoretical discussion of the form of exp $\{At\}$ and its behavior as t becomes infinite (Theorem 7 of [1]), it is not intended as a practical method for calculating the function. The following two theorems suggest an alternate approach which can be used both for calculation and for expository discussion. It may be noted that the formula of Theorem 2 is simpler than that of Theorem 1 since the r_i are easier to calculate than the q_i.

In order to state Theorem 1 simply, it will be convenient to introduce some notation.

Let A be an $n \times n$ matrix of constants, and let

$$f(\lambda) \equiv |\lambda I - A| \equiv \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

be the characteristic polynomial of A. Construct the scalar function $z(t)$ which is the solution of the differential equation

$$z^{(n)} + c_{n-1}z^{(n-1)} + \cdots + c_1\dot{z} + c_0 z = 0 \tag{2}$$

with initial conditions

$$z(0) = \dot{z}(0) = \cdots = z^{(n-2)}(0) = 0; \qquad z^{(n-1)}(0) = 1. \tag{3}$$

We observe at this point that regardless of the multiplicities of the roots of $f(\lambda) = 0$, once these are obtained it is trivial to write down the general solution of (2). Then one solves a single set of linear algebraic equations to satisfy the initial conditions (3). Since the right member of each of these equations is zero except for the last, solving them entails only finding the cofactors of the elements of the *last row* of the associated matrix. It is not necessary to invert the matrix itself. For teaching purposes, the point is that the form of the general solution of (2) can be obtained quickly and easily by elementary methods.

Now define

$$Z(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ \cdot \\ \cdot \\ \cdot \\ z^{(n-1)}(t) \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} c_1 & c_2 \cdots c_{n-1} & 1 \\ c_2 & c_3 \cdots 1 & \\ \cdot & \cdot & \\ \cdot & \cdot & 0 \\ c_{n-1} & 1 & \\ 1 & & \end{bmatrix}$$

We then have the following

THEOREM 1.

$$e^{At} = \sum_{j=0}^{n-1} q_j(t) A^j, \tag{4}$$

where $q_0(t), \cdots, q_{n-1}(t)$ *are the elements of the column vector*

$$\mathfrak{q}(t) = CZ(t). \tag{5}$$

Before we prove this, a remark is in order as to what happens when $f(\lambda)$ has multiple roots but the minimal polynomial of A has distinct factors so that A can in fact be diagonalized. It appears at first glance that our formula will contain powers of t, yet we know this cannot be the case. What occurs, of course, is that the powers of t in (4) just cancel each other out. This is the nice feature of formula (4); it is true for *all* matrices A, and we never have to concern ourselves about the nature of the minimal polynomial of A, or its J.C.F., and no preliminary transformations of any kind need be made.

Proof. We will show that if

$$\Phi(t) = \sum_{j=0}^{n-1} q_j(t) A^j \tag{6}$$

then $d\Phi/dt = A\Phi$ and $\Phi(0) = I$, so $\Phi(t) = e^{At}$. Since only $q_0(t)$ involves $z^{(n-1)}(t)$, $q_j(0) = 0$ for $j \geqq 1$. Clearly, $q_0(0) = 1$. Thus, $\Phi(0) = I$.

Now consider $(d\Phi/dt) - A\Phi$. Differentiating (6), and applying the Hamilton-Cayley Theorem

$$A^n + \sum_{j=0}^{n-1} c_j A^j = 0,$$

we obtain

$$\frac{d\Phi}{dt} - A\Phi = (\dot{q}_0 + c_0 q_{n-1}) + \sum_{j=0}^{n-1} (\dot{q}_j - q_{j-1} + c_j q_{n-1}) A^j.$$

It will suffice, therefore, to show that

$$\dot{q}_0(t) \equiv -c_0 q_{n-1}(t),$$
$$\dot{q}_j(t) \equiv q_{j-1}(t) - c_j q_{n-1}(t) \qquad j = 1, \cdots, (n-1).$$

From the definition (5),

$$q_j(t) \equiv \sum_{k=1}^{n-j-1} c_{k+j} z^{(k-1)} + z^{(n-j-1)}. \tag{7}$$

Therefore $\dot{q}_j(t) \equiv \sum_{k=1}^{n-j-1} c_{k+j} z^{(k)} + z^{(n-j)}$.

But $q_{n-1} \equiv z$, so we have

$$\dot{q}_j + c_j q_{n-j} \equiv \sum_{k=0}^{n-j-1} c_{k+j} z^{(k)} + z^{(n-j)} \quad \text{for } j = 0, 1, \cdots, n-1. \tag{8}$$

If $j = 0$, this yields

$$\dot{q}_0 + c_0 q_{n-1} \equiv \sum_{k=0}^{n-1} c_k z^{(k)} + z^{(n)}$$

which is zero because of (2).

If $j \geqq 1$, replace j by $j-1$ in (7) and change the summation index from k to $k+1$ to get

$$q_{j-1}(t) \equiv \sum_{k=0}^{n-j-1} c_{k+j} z^{(k)} + z^{(n-j)}. \tag{9}$$

Comparing (9) and (8) we have $\dot{q}_j + c_j q_{n-1}(t) \equiv q_{j-1}(t)$ for $j=1, 2, \cdots, n-1$.

Students will want to see the formula (4) *derived*. One merely presents the proof backward, beginning with the observation that because of the Hamilton-Cayley theorem, exp $\{At\}$ should be expressible in the form (6). Regarding the q_j as unknowns, and applying the differential equation which exp $\{At\}$ satisfies, leads directly to (4).

A second explicit formula for exp $\{At\}$, which also holds for all matrices A, is the following: Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A in some arbitrary but specified order. These are not necessarily distinct. Then

THEOREM 2. $e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j$, *where*

$$P_0 = I; \qquad P_j = \prod_{k=1}^{j} (A - \lambda_k I), \qquad (j = 1, \cdots, n),$$

and $r_1(t), \cdots, r_n(t)$ *is the solution of the triangular system*

$$\begin{cases} \dot{r}_1 = \lambda_1 r_1 & \\ \dot{r}_j = r_{j-1} + \lambda_j r_j & (j = 2, \cdots n) \\ r_1(0) = 1; \quad r_j(0) = 0 & (j = 2, \cdots n). \end{cases}$$

Proof. Let

$$\Phi(t) \equiv \sum_{j=0}^{n-1} r_{j+1}(t) P_j \tag{10}$$

and define $r_0(t) \equiv 0$. Then from (10) and the equations satisfied by the $r_j(t)$ we have, after collecting terms in r_j,

$$\dot{\Phi} - \lambda_n \Phi = \sum_{0}^{n-2} [P_{j+1} + (\lambda_{j+1} - \lambda_n) P_j] r_{j+1}.$$

Using $P_{j+1} \equiv (A - \lambda_{j+1} I) P_j$ in this gives

$$\begin{aligned} \dot{\Phi} - \lambda_n \Phi &= (A - \lambda_n I)(\Phi - r_n(t) P_{n-1}) \\ &= (A - \lambda_n I)\Phi - r_n(t) P_n. \end{aligned}$$

But $P_n \equiv 0$ from the Hamilton-Cayley Theorem, so $\dot{\Phi} = A\Phi$. Then since $\Phi(0) = I$ it follows that $\Phi(t) = e^{At}$.

Example. If one desires a numerical example for class presentation an appropriate matrix A can be prepared in advance by arbitrarily choosing a set of eigenvalues, a Jordan Canonical Form J and a nonsingular matrix S and calculating

$$A = SJS^{-1}.$$

Then beginning with A, one simply calculates the set $\{q_i(t)\}$ and/or $\{r_i(t)\}$. Consider the case of a 3×3 matrix having eigenvalues (λ, λ, μ). There are two subcases; the one in which the normal form of A is diagonal, and the one in which it is not. These two subcases are taken care of automatically by the given formula for exp $\{At\}$, and do not enter at all into the calculation of the $\{q_i\}$ or $\{r_i\}$.

As an example we will explicitly find the sets $\{q_i\}$ and $\{r_i\}$ for the case of a 3×3 matrix with eigenvalues $(\lambda, \lambda, \lambda)$. We note that aside from the trivial case in which the normal form (and hence A itself) is diagonal, there are two distinct nondiagonal normal forms that A may have. As above, these do not have to be treated separately.

From Theorem 1,

$$f(x) \equiv (x-\lambda)^3 = x^3 - 3\lambda x^2 + 3\lambda^2 x - \lambda^3$$

so $c_1 = 3\lambda^2$, $c_2 = -3\lambda$. Obviously $z(t) = (a_1 + a_2 t + a_3 t^2)e^{\lambda t}$. Applying the initial conditions to find the a_i yields $z = \frac{1}{2}t^2 e^{\lambda t}$, so

$$Z(t) = \tfrac{1}{2}e^{\lambda t}\begin{bmatrix} t^2 \\ \lambda t^2 + 2t \\ \lambda^2 t^2 + 4\lambda t + 2 \end{bmatrix}.$$

Then since

$$C = \begin{bmatrix} 3\lambda^2 & -3\lambda & 1 \\ -3\lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathrm{q} = CZ(t) = \tfrac{1}{2}e^{\lambda t}\begin{bmatrix} \lambda^2 t^2 - 2\lambda t + 2 \\ -2\lambda t^2 + 2t \\ t^2 \end{bmatrix}.$$

Thus

$$e^{At} = \tfrac{1}{2}e^{\lambda t}\{(\lambda^2 t^2 - 2\lambda t + 2)I + (-2\lambda t^2 + 2t)A + t^2 A^2\} \tag{11}$$

for every 3×3 matrix A having all three eigenvalues equal to λ. The corresponding formula from Theorem 2 is obtained by solving the system

$$\begin{cases} \dot r_1 = \lambda r_1 \\ \dot r_2 = r_1 + \lambda r_2 \\ \dot r_3 = r_2 + \lambda r_3 \end{cases}$$

with the specified initial values. This immediately gives

$$r_1 = e^{\lambda t}; \qquad r_2 = te^{\lambda t}; \qquad r_3 = \frac{t^2}{2} e^{\lambda t}$$

so

$$e^{At} = \tfrac{1}{2}e^{\lambda t}\{2I + 2t(A - \lambda I) + t^2(A - \lambda I)^2\}. \tag{12}$$

Of course, if we collect powers of A in (12) we will obtain (11).

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THE IMPORTANCE OF ASYMPTOTIC ANALYSIS IN APPLIED MATHEMATICS

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Friedrichs began his 1954 Gibbs lecture on asymptotic phenomena in mathematical physics [6] by saying "the problems I intend to speak about belong to the somewhat undefined and disputed region at the border between mathematics and physics." In the ten years since his lecture there has been a growing tendency to define this region and some others as the domain of applied mathematics. A number of explicit university applied mathematics programs have been started, the Society for Industrial and Applied Mathematics has been founded in the United States and the Institute of Mathematics and its Applications in England.

It ought to be helpful to have on record various opinions on what this new, or renewed, activity is all about. The general nature of applied mathematics has been ably discussed [7], [8] and the time now appears ripe for more detailed remarks on the elements of an applied mathematical outlook. The purpose of the following supplement to Friedrichs' remarks and references is to stress the importance of asymptotic analysis in applied mathematics. Although they will not be mentioned here, there are of course several other important aspects of the subject. One of these is the use of computing machines. Murray [13] concentrates so exclusively on this one aspect that he gives the impression that applied mathematics is a problem solving service rather than the independent science envisioned here and elsewhere. An indication of the central position of asymptotics is that in concentrating on it we do not present an unbalanced picture of applied mathematics.