A CHANGE OF VARIABLE FOR THE RIEMANN INTEGRAL ON THE REAL LINE

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SOME COMMENTS

We show a *Change of Variable for the Riemann Integral on the Real Line* that has the following attributes.

(1) It is not a particular case of the influential *general version* of H. Kestelman.

(2) It does not require the integrability or the continuity of the derivative of the substitution map.

(3) It expands the scope of the Change of Variable Theorem. Including those usually found in textbooks.

SOME VERSIONS OF THE CHANGE OF VARIABLE FORMULA

(Spivak, Calculus) *If* f *and* g' **are continuous**, *then*

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x))g'(x)\,dx\,.$$

(Apostol, Análisis Matemático, translated) *If* g *has* **continuous derivative** *on* $[c,d]$, *and* f *is* **continuous** *on* $g\big([c,d]\big)$, *then*

$$\int_{g(c)}^{g(d)} f(x)\,dx = \int_c^d f[g(t)]g'(t)\,dt\,.$$

(Lang, Undergraduate Analysis) *Let* J_1, J_2 *be intervals each having more than one point, and let* $f : J_1 \to J_2$ *and* $g : J_2 \to \mathbb{R}$ **be continuous**. *Assume that* f *is differentiable, and that* f' **is continuous**. *Then for any* $a, b \in J_1$ *we have*

$$\int_a^b g(f(x))f'(x)\,dx = \int_{f(a)}^{f(b)} g(u)\,du\,.$$

(Knapp, Basic Real Analysis) *Let* f **be integrable** *on* $[a, b]$*. Let* φ *be a continuous strictly increasing function from an interval* $[A, B]$ *onto* $[a, b]$*, suppose that the inverse function* $\varphi^{-1} : [a, b] \to [A, B]$ *is continuous, and suppose finally that* φ *is differentiable on* (A, B) *with* φ' *uniformly continuous. Then the product* $(f \circ \varphi)\varphi'$ *is integrable on* $[A, B]$*, and*

$$\int_a^b f(x)\,dx = \int_A^B f[\varphi(y)]\varphi'(y)\,dy\,.$$

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Remark 1. The hypotheses "$\varphi : [A, B] \to [a, b]$ is continuous, strictly increasing, and onto" imply φ^{-1} continuous. Thus, φ is bicontinuous.

Remark 2. The hypothesis "φ' uniformly continuous on (a, b)" implies that the derivative φ' is bounded and integrable over $[a, b]$.

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(Rudin, Principles of Mathematical Analysis) *Suppose* f **is integrable** *on* $[a, b]$*. Suppose* φ *is a strictly increasing continuous function that maps an interval* $[A, B]$ *onto* $[a, b]$*. Assume* φ' *is integrable on* $[A, B]$*. Then*

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\varphi'(y)\,dy\,.$$

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Remark 1. The hypotheses "$\varphi : [A, B] \to [a, b]$ is continuous, strictly increasing, and onto" imply φ^{-1} continuous. Thus, φ is bicontinuous.

Remark 2. Rudin proves the Change of Variable Formula for the Riemann-Stieltjes Integral. Then, he cites the correspondent formula for the Riemann Integral as a particular case.

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("General Version") (H. Kestelman, 1961) *Let* $h : [c, d] \to \mathbb{R}$ *be integrable. Let us fix an arbitrary point* $a \in [c, d]$*. Given* $x \in [c, d]$*, we write*

$$g(x) = \int_a^x h(t)\, dt.$$

Let $f : g\big([c, d]\big) \to \mathbb{R}$ *be* **integrable**. *Then,* $(f \circ g)h$ *is integrable on* $[c, d]$ *and*

$$\int_{g(c)}^{g(d)} f(x)\, dx = \int_c^d f[g(t)]h(t)\, dt.$$

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Remark 1. Kestelman's proof uses the concept *measure zero*.

Remark 2. Davies simplifies Kestelman's and avoids *measure zero*.

Remark 3. Some articles on the theme are: Sarkhel and Výborný (RAEX, 1996 – 97), Bagby (RAEX, 2001 – 2002), and Torchinsky (RAEX, 2020).

Remark 4. Torchinsky's new book *A Modern View of the Riemann Integral* (2022) discuss much of the substitution formulas.

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A TABLE OF HYPOTHESIS FOR THE FORMULA

	main function	the change of variable function
Apostol	continuous	continuous derivative
Lang	continuous	continuous derivative
Spivak	continuous	continuous derivative
Knapp	integrable	bicontinuous **uniformly cont. derivative on the interior**
Rudin	integrable	bicontinuous **integrable derivative**
Kestelman	integrable	**an integral function**

Notations and Definitions

We adopt the following type of partitions, for the Riemann integral,

$$\mathcal{X} = \{a = x_0 \leq \cdots \leq x_n = b\}.$$

The *norm* of $\mathcal{X}$ is written as $|\mathcal{X}|$.

Given a bounded $f : [a, b] \to \mathbb{R}$, we indicate the inferior and the superior Riemann sums of f with respect to the partition $\mathcal{X}$ by, respectively,

$$s(f, \mathcal{X}) \text{ and } S(f, \mathcal{X}).$$

Given a real map $\varphi : [\alpha, \beta] \to \mathbb{R}$, we say that φ is *monotone* if it is increasing (not necessarily strictly), decreasing (not necessarily strictly), or constant.
We say that φ is *piecewise monotone* if there exists a finite sequence $\{\alpha = t_0 < \cdots < t_N = \beta\}$ such that φ is monotone in each open sub-interval (t_j, t_{j+1}) for every $j = 0, \ldots, N - 1$.

THE THEOREM

Theorem (A Generalized Change of Variable Theorem). *Let us consider*

$$f : [a, b] \longrightarrow \mathbb{R} \textit{ integrable } \textit{ and } \varphi : [\alpha, \beta] \longrightarrow [a, b]$$

surjective, increasing (not necessarily strictly increasing) and continuous. Suppose φ differentiable on the open interval (α, β). The following are true.

- *If φ' is integrable on $[\alpha, \beta]$, then the map $(f \circ \varphi)\varphi'$ is integrable on $[\alpha, \beta]$.*
- *If the product $(f \circ \varphi)\varphi'$ is integrable on $[\alpha, \beta]$, then we have the formula*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

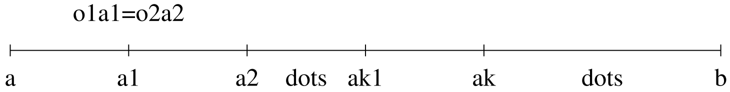
Proof. We split the proof into eight small steps.

1. The hypothesis that φ is continuous is superfluous.
   The proof is a Calculus 101 exercise.
2. The function φ is uniformly continuous.
   It is trivial.
3. We have $\varphi' \geq 0$ on the open interval (α, β).
   It is true, since φ is increasing.
4. To integrate φ', we may define $\varphi'(\alpha)$ and $\varphi'(\beta)$ arbitrarily.
   No comment!

5. Given $\mathcal{T} = \{\alpha = t_0 \leq \cdots \leq t_n = \beta\}$ a partition of $[\alpha, \beta]$, it follows that $\varphi(\mathcal{T}) = \mathcal{X} = \{a = x_0 \leq \cdots \leq x_n = b\}$ is a partition of $[a, b]$.

   That is, we have $x_i = \varphi(t_i)$ for each $i = 0, \ldots, n$.
   The mean-value theorem yields a point $\overline{t_i} \in [t_{i-1}, t_i]$ satisfying

$$\Delta x_i = \varphi(t_i) - \varphi(t_{i-1}) = \varphi'(\overline{t_i})\Delta t_i.$$



6. If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$.
   It follows from the uniform continuity of φ.

7. If φ' is integrable, then $(f \circ \varphi)\varphi'$ is integrable.

Let τ_i be arbitrary in $[t_{i-1}, t_i]$. Since φ is increasing, $\varphi(\tau_i) \in [x_{i-1}, x_i]$.
In what follows, for simplicity, we omit the summation index.
Let us investigate the Riemann sum $\left[\text{remember } \Delta x_i = \varphi'(\bar{t}_i)\Delta t_i\right]$

$$\sum f(\varphi(\tau_i))\varphi'(\tau_i)\Delta t_i = \sum f(\varphi(\tau_i))\Delta x_i + \sum f(\varphi(\tau_i))\left[\varphi'(\tau_i)) - \varphi'(\overline{t_i})\right]\Delta t_i.$$

If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the first sum on the right goes to $\int_a^b f dx$.

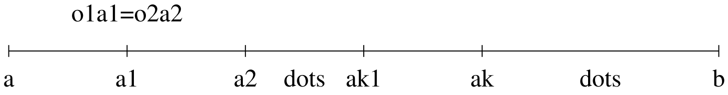
Let M be a constant such that $|f| \leq M$ (obviously, f is bounded). Then,

$$\left|\sum f(\varphi(\tau_i))\left[\varphi'(\tau_i)) - \varphi'(\overline{t_i})\right]\Delta t_i\right| \leq M\left[S(\varphi', \mathcal{T}) - s(\varphi', \mathcal{T})\right] \xrightarrow{|\mathcal{T}| \to 0} 0.$$

Thus, $(f \circ \varphi)\varphi'$ is integrable.
The value of its integral equals the one of f.

8. If $(f \circ \varphi)\varphi'$ is integrable, the value of its integral equals the one of f.

With the above notation, we choose $\tau_i = \overline{t_i}$ and write $\overline{x_i} = \varphi(\overline{t_i})$.



Hence, we have

$$\sum f(\varphi(\overline{t_i}))\varphi'(\overline{t_i})\Delta t_i = \sum f(\overline{x_i})\Delta x_i.$$

If $|\mathcal{T}| \to 0$, the left hand side goes to the integral of $(f \circ \varphi)\varphi'$.
If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the right side goes to the integral of f.

The proof of the theorem is complete $\square$

A COROLLARY

Corollary. *Keeping all the other hypotheses of the theorem, suppose that the continuous and onto* $\varphi : [\alpha, \beta] \to [a, b]$ *satisfies one of the following conditions.*

(a) φ *is monotone.*

(b) φ *is piecewise monotone.*

(c) φ *is piecewise monotone in* $[\alpha + \epsilon, \beta]$*, for each* $0 < \epsilon < \beta - \alpha$*.*

(d) φ' *has a finite number of zeros.*

(e) φ' *has a finite number of zeros in* $[\alpha + \epsilon, \beta)$*, for each* $0 < \epsilon < \beta - \alpha$*.*

Then, the following two claims are true.

- *If* φ' *is integrable, then* $(f \circ \varphi)\varphi'$ *also does.*
- *If* $(f \circ \varphi)\varphi'$ *is integrable, then*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)dt.$$

TWO EXAMPLES

Example 1. Consider

$$f(x) = x \text{ where } x \in [0,1], \quad \text{and} \quad \varphi(t) = \sqrt{t} \text{ where } t \in [0,1].$$

Evidently, f is integrable. Moreover, $\varphi : [0,1] \to [0,1]$ is surjective, increasing, and continuous. The derivative φ' is defined on the open interval $(0,1)$ and

$$\varphi'(t) = \frac{1}{2\sqrt{t}}.$$

So, φ' is not bounded and thus not integrable on $(0,1)$. However,

$$f(\varphi(t))\varphi'(t) = \frac{\sqrt{t}}{2\sqrt{t}} = \frac{1}{2}$$

is integrable. From the above theorem we find

$$\int_0^1 x\,dx = \int_0^1 \frac{1}{2}\,dt. \quad \square$$

Since φ' is not integrable, Kestelman's version does not apply to Example 1.

Example 2 [an example for the Corollary, items (c) and (e)]. Consider

$$f(x) = x^3, \text{ if } x \in \left[0, \frac{2}{\pi}\right], \qquad \text{and} \qquad \varphi(t) = \begin{cases} 0, & \text{if } t = 0, \\ t \sin \frac{1}{t}, & \text{if } t \in \left(0, \frac{2}{\pi}\right]. \end{cases}$$

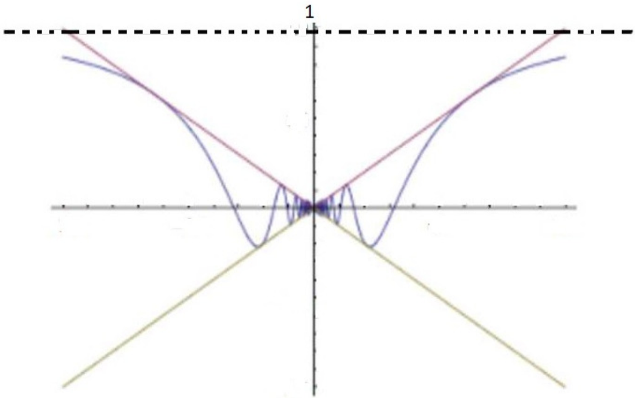


Figure 1: The graph of φ.

Clearly, f is integrable while φ is continuous and oscillates near zero. We have

$$\varphi'(t) = \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}.$$

Hence, φ' is unbounded and not integrable on $[0, 2/\pi]$.
Now, take $\epsilon > 0$. Thus, φ' has infinite zeros on $[0, \epsilon]$. Conversely, φ' has a finite number of zeros on $[\epsilon, 2/\pi]$, and φ is piecewise monotone on $[\epsilon, 2/\pi]$.

Near zero (thus, on $[0, 2/\pi]$), we have the integrability of

$$(f \circ \varphi)(t)\varphi'(t) = t^3\left(\sin^3\frac{1}{t}\right)\left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right).$$

By the Corollary, either item (c) or item (e), we find

$$\int_0^{\frac{\pi}{2}} x^3\, dx = \int_0^{\frac{2}{\pi}} [\varphi(t)]^3 \varphi'(t)\, dt.$$

From which follows

$$\int_0^{\frac{\pi}{2}} x^3\, dx = \left.\frac{\varphi^4(t)}{4}\right|_0^{\frac{2}{\pi}} = \frac{\pi^4}{64}. \quad \square$$

Since φ' is not integrable, Kestelman's version does not apply to Example 2.

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