# Manifold submetries, and polynomial algebras

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Modern Trends in Differential Geometry

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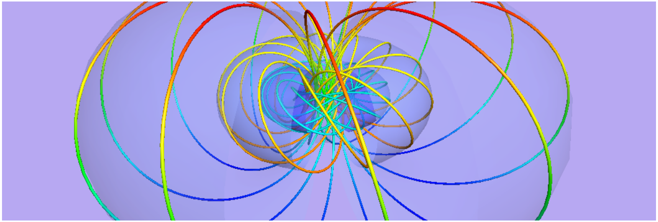

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$M$ Riemannian manifold, $X$ metric space.

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Continuous map $\pi : M \to X$ such that:

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Want to look at local structure of manifolds submetries.

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- Related to collapse with lower curvature bounds (Cheeger, Yamaguchi, Shioya-Yamaguchi, Wilking, ...).

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Classify all SMS's.

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**Question 2**

Find constructions and structure of SMS's.

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### Theorem (R. '14)

$\mathcal{C} = \{P_0, \dots P_m\} \subset \mathrm{Sym}^2(n)$ *Clifford system*

$$\begin{aligned} \pi_{\mathcal{C}} : \mathbb{S}^{n-1} &\longrightarrow \mathbb{D}^{m+1} \subset \mathbb{R}^{m+1} \\ v &\longmapsto (\langle P_0 v, v\rangle, \dots \langle P_m v, v\rangle) \end{aligned}$$

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All known SMS's are obtained from Clifford and homogeneous examples, together with two operations between them (spherical join, composition).

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(Gorodski-Lytchak): Study of orthogonal representations, from the point of view of $\pi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}/G$.

**Algebraicity Theorem (Lytchak, R., '15)**

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**Key point:** Averaging operator.

# Two maps

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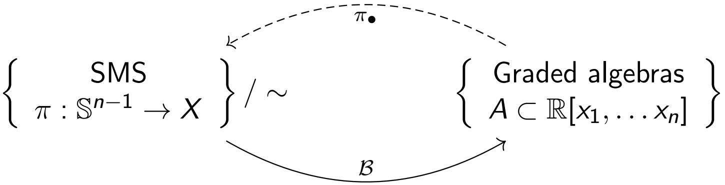


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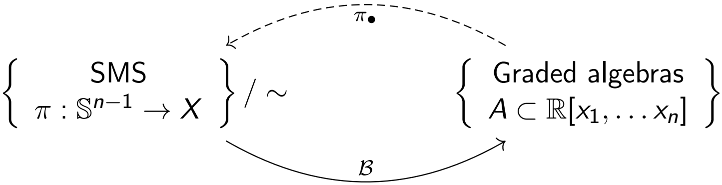


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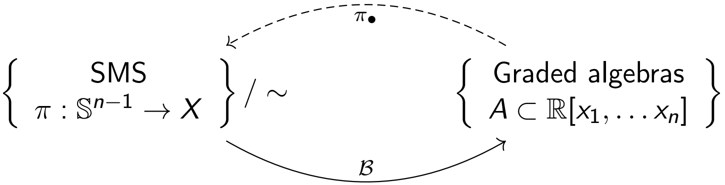


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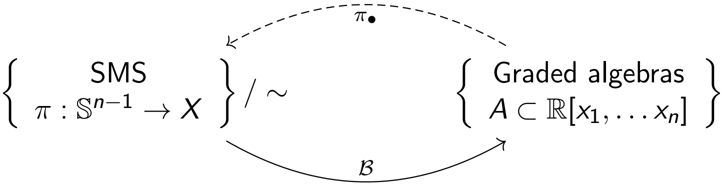


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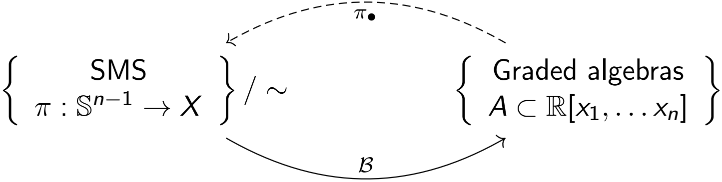


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$\pi_\bullet$ is not a SMS in general, but $\pi_\bullet(\mathcal{B}(\pi)) \sim \pi$.

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**Theorem (Alexandrino, R.)**

*For any SMS $\pi : \mathbb{S}^{n-1} \to X$, $\mathcal{B}(\pi)$ is maximal and Laplacian.*

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**Corollary**

*There is a bijection*

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# From Laplacian algebras to SMS

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**Step 3:** $\rho_{reg}$ **extends to a manifold submetry** $\hat{\pi} : \mathbb{S}^{n-1} \to \hat{X}_A$.

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**Theorem (First fundamental theorem of $O(n)$, Weyl)**

*The algebra of $O(n)$-invariant polynomials is generated by the $P_{ij}$.*

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**Theorem (Mendes, R., '16)**

*YES, in the following situations:*

- *$A$ generated by 2 polynomials.*
- *$A$ is generated by quadratic polynomials.*

Obtained via generalization of Weyl's First Fundamental Theorem, in the non homogeneous setting.

# An application to (inverse) Invariant theory

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**Corollary (Mendes, R.)**

*Suppose* $A \subset \mathbb{R}[x_1, \dots x_n]$ *is a maximal Laplacian algebra, with* $trdeg.K(A) = n$ ($K(A) =$ *field of fractions of* $A$). *Then* $A = \mathbb{R}[x_1, \dots x_n]^\Gamma$ *for some finite group* $\Gamma \subset O(n)$.

# Thank you!