A toric geometry road from Kähler metrics to contact topology

I Kähler metrics

- symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, $\omega_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$
- Kähler metric $g(\cdot,\cdot) = \omega_0(\cdot, J\cdot)$,
  J = compatible complex structure, i.e.

$$J = \begin{bmatrix} S^{-1}R & -S^{-1} \\ RS^{-1}R + S & -RS^{-1} \end{bmatrix}$$ with R, S real $(n\times n)$ symmetric, $\boxed{S>0}$

- If $\boxed{R=0}$ then $J = \begin{bmatrix} 0 & -S^{-1} \\ S & 0 \end{bmatrix}$ and

$$g = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 & -S^{-1} \\ S & 0 \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix}$$

- (Almost) Kähler metric on Darboux chart
  $\left(U \subset \mathbb{R}^{2n},\ \omega = dx\wedge dy = \sum_{j=1}^{n} dx_j \wedge dy_j \right)$

$$g = \begin{bmatrix} S(x,y) & 0 \\ 0 & S^{-1}(x,y) \end{bmatrix}$$

- If $\boxed{\text{independent of } y}$ then g is Kähler, i.e

  J is integrable iff $S(x) = \mathrm{Hess}_x(s(x))$,

  <u>symplectic potential</u> $\boxed{s : U \longrightarrow \mathbb{R}}$

Holomorphic coordinates : $z(x,y) = \frac{\partial s}{\partial x}(x) + iy$

Scalar curvature : $\boxed{Sc = -\sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \, \partial x_k} \equiv -(s^{jk})_{jk}}$

where $(s^{jk})_{j,k=1}^{n} = \mathrm{Hess}_x(s)^{-1}$

- This local description of a very particular class of Kähler metrics becomes a

  global description for all toric Kähler metrics

  (Guillemin 1994, A. 1998, ...)

(II) Toric Kähler metrics

- Toric symplectic manifold: (M^{2n}, ω) with

  $$\tau : \mathbb{T}^n \cong \mathbb{R}^{2n}/2\pi\mathbb{Z}^n \hookrightarrow \mathrm{Ham}(M, \omega)$$

- Moment map $\mu : M \to \mathbb{R}^n$, $\mu = (h_1, \ldots, h_n)$

  $P := \mu(M) \equiv$ finite intersection of closed half-spaces

  $$\boxed{P = \bigcap_{j=1}^{d} \left\{ x \in \mathbb{R}^n : \ \ell_j(x) := \langle x, \nu_j \rangle + \lambda_j \geq 0 \right\}}$$

  $\nu_j \in \mathbb{R}^n$, $\lambda_j \in \mathbb{R}$

$\mu^{-1}(\mathring{P}) \subset M$ is open dense

$\cong$

$\mathring{P} \times \mathbb{T}^n$ such that $\omega = dx \wedge dy$

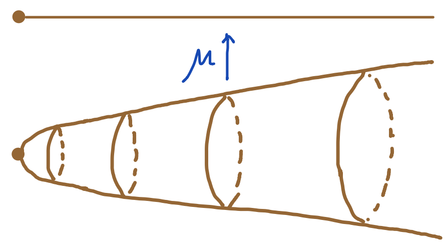
$(x, y) \leftarrow$ **action - angle coordinates**

- Any toric Kähler metric on M can be written as before when restricted to $\mu^{-1}(\mathring{P}) \subset M$, with symplectic potential $s : \mathring{P} \longrightarrow \mathbb{R}$ of the form

$$\boxed{s(x) = \frac{1}{2}\left(\sum_{j=1}^{d} \ell_j(x)\log \ell_j(x) + h(x)\right), \quad h \in C^\infty(P)}$$

- Examples

1) $P = \{x \in \mathbb{R}^n : x \geq 0\}$

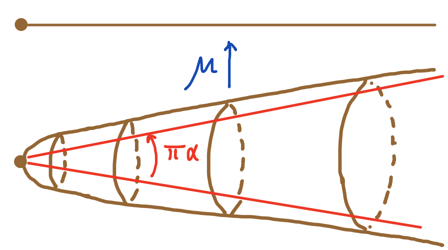


$s(x) = \frac{1}{2} x \log x$

$s''(x) = \frac{1}{2x}$

$g_{(x,y)} = \begin{bmatrix} \frac{1}{2x} & 0 \\ 0 & 2x \end{bmatrix} \equiv$ flat metric in action-angle coordinates

2) $P = \{x \in \mathbb{R}^n : \frac{x}{\alpha} \geq 0\}$



$s(x) = \frac{1}{2} \frac{x \log x}{\alpha}$

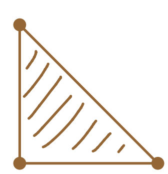
$s''(x) = \frac{1}{2x\alpha}$

$g_{(x,y)} = \begin{bmatrix} \frac{1}{2x\alpha} & 0 \\ 0 & \alpha 2x \end{bmatrix} \equiv$ conical flat metric in action-angle coordinates

3) $\mathbb{CP}^n$

$$s(x) = \frac{1}{2}\left(\sum_{j=1}^{n} x_j \log x_j + \left(a - \sum_j x_j\right)\log\left(a - \sum_j x_j\right)\right), \quad a > 0$$

Fubini - Study metric

$\mathbb{CP}^2 \xrightarrow{\mu}$ 

4) [R. Bryant, A. (2001)]

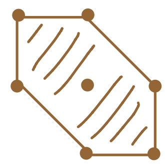
$$s(x) = \frac{1}{2}\left(\sum_{j=1}^{n} \frac{x_j \log x_j}{\alpha_j} + \frac{\left(a - \sum_j x_j\right) \log \left(a - \sum_j x_j\right)}{\alpha_{n+1}} - \ell_\Sigma \log \ell_\Sigma \right)$$

with $\ell_\Sigma(x) = \sum_{j=1}^{n} \frac{x_j}{\alpha_j} + \frac{\left(a - \sum_{j=1}^{n} x_j\right)}{\alpha_{n+1}}$

$$Sc(x) = \frac{2n}{a}\left(\sum_{j=1}^{n+1} \alpha_j\right) + \frac{2(n+2)}{a} \sum_{j=1}^{n} (\alpha_{n+1} - \alpha_j)\, x_j$$

Conical extremal Kähler metrics on $\mathbb{CP}^n$

(include extremal Kähler metrics on weighted projective spaces).

5) $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2} \xrightarrow{\mu}$ 

Siu 1988 $\Rightarrow$ KE metric

$\boxed{s(x) = ?}$

- Donaldson's work on equation $\boxed{-\left(s^{jk}\right)_{jk} = \text{const.}}$

Theorem (2009, dim 4)

A K-stable polarized toric surface admits a constant scalar curvature Kähler metric.

Note: the converse is due to Zhou-Zhu (2008).

- Scalar flat solutions in dim = 4 via Donaldson's generalized Joyce construction (2009)

For any Kähler metric on 4-manifold

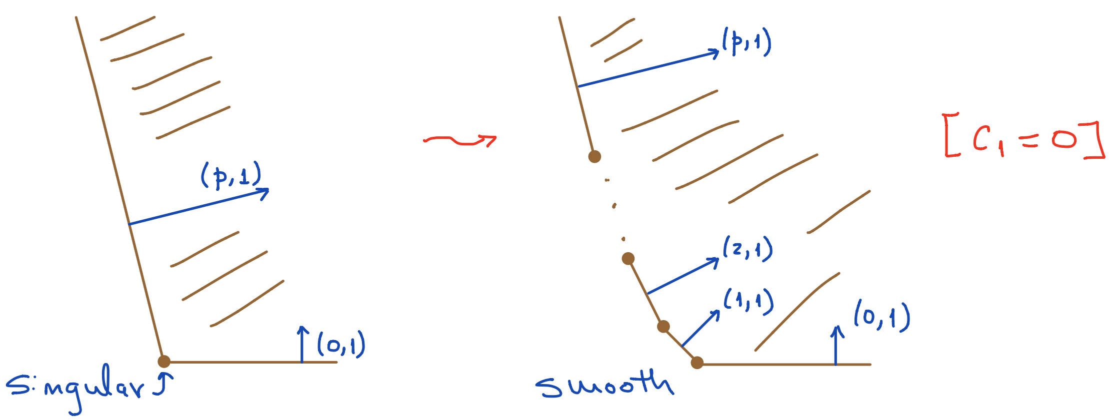
(a) scalar flat $\Leftrightarrow$ anti-self-dual

(b) Ricci flat $\Leftrightarrow$ hyperKähler

$\rightsquigarrow$ Gibbons - Hawking, Hitchin, Kronheimer, LeBrun, Joyce, Calderbank - Singer, Donaldson $\rightsquigarrow$ generalized Taub-NUT in $\mathbb{R}^4$ and D. Wright unpublished PhD thesis

Theorem [A.- Sena Dias (2012)]
Any strictly unbounded symplectic toric 4-manifold [$c_1 = 0$] admits an ALE scalar flat [Ricci flat] Kähler metric and a 2-parameter [1-parameter] family of complete scalar flat [Ricci flat] Kähler metrics asymptotic to generalized Taub-NUT.

Example: toric resolution of A_p-singularity



- Calabi's (1982) cohomogeneity one solutions via ODE's

$$s(x) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \log x_i + h(r)\right), \quad r = x_1 + \cdots + x_n$$

Extremal Kähler metric, i.e. Sc $\equiv$ affine function of r, iff

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br - Cr^{n+1} - Dr^{n+2}}$$

$$\boxed{Sc(r) = 2(n+1)\big((2+n)Dr + nC\big)}$$

$$\Updownarrow$$

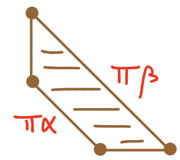
Kähler-Einstein iff $B = D = 0$

Example $(n=2)$:

$0 < \alpha < 2$

$0 < \beta < 1$

$$M^4(\alpha,\beta) \xrightarrow{\mu}$$



$$(2-\alpha)^2(1+\alpha) = (2+\beta)^2(1-\beta)$$

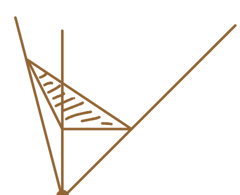
Kähler - Einstein metric on nontrivial S^2-bundle over S^2 with conical singularities along 0-section and ∞-section.

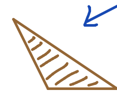
III Toric Kähler-Sasaki geometry

([GMSW] Gauntlett-Martelli-Sparks-Waldram (2004)
[MSY] Martelli-Sparks-Yau (2006))

Basic example:

$$\begin{tikzcd}
S^{2n+1} \arrow[r, "\text{symplectization}"'] \arrow[d, "\text{quotient}"] & \mathbb{R}^{2(n+1)} \setminus \{0\} \arrow[dl, "\text{symplectic reduction}"] \\
\mathbb{C}P^n = S^{2n+1} / \text{diagonal } S^1\text{-action} &
\end{tikzcd}$$





General case:

$$\begin{tikzcd}
\text{toric contact Sasaki Einstein} \quad N^{2n+1} \arrow[r, "\text{symplectization}"] \arrow[d, "\text{quotient}"] & W^{2(n+1)} \cong N \times \mathbb{R} \quad \text{toric symplectic cone Kähler Ricci flat} \arrow[dl, "\text{symplectic reduction}"] \\
\text{toric symplectic Kähler Einstein} \quad M^{2n} = N/K , \; K = \text{Reeb vector field} &
\end{tikzcd}$$

Note: can have N^{2n+1} smooth with M^{2n} singular.

- [A. (2010)]

[GMSW] Sasaki-Einstein 5-manifolds $Y^{p,q}$ $(\cong S^2 \times S^3)$
$(0 < q < p, \; \gcd(q,p) = 1)$

$$\Big\downarrow \text{quotient}$$

Calabi's Kähler-Einstein 4-manifolds $M^4(\alpha(p,q), \beta(p,q))$

[MSY] $\mu(Y^{p,q} \times \mathbb{R}) \subset \mathbb{R}^3$ is polyhedral cone with defining normals

$$\nu_1 = (1, p-q-1, p-q), \quad \nu_2 = (1,1,0), \quad \nu_3 = (1,0,0), \quad \nu_4 = (1, p, p)$$

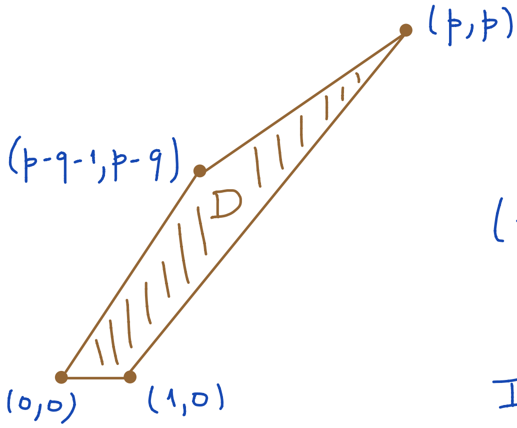
$[c_1 = 0]$

$$\Downarrow$$

Family of contact structures $\xi_{p,q}$ on $S^2 \times S^3$ that are all homotopic as hyperplane fields. Are they equivalent contact structures?

• [A. – Macarini (2012, ..., 2018)]

Toric diagram D



$\boxed{\text{Volume}(D) \text{ is a contact invariant}}$

(≡ mean Euler characteristic of (cylindrical) contact homology)

Independent of q, but dependent of $p \in \mathbb{N}$.

$\Rightarrow$ $\boxed{\infty\text{-many inequivalent contact structures with } c_1 = 0 \text{ on } S^2 \times S^3}$

• [A. – Macarini – Moreira (...)]

$\boxed{\text{Erhart polynomial of toric diagram}} \xleftrightarrow{1:1} \boxed{\text{Betti numbers of (cylindrical) contact homology}}$