Recent advances in minimal surface theory in $\mathbb{R}^3$

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(All surfaces are orientable)

$\mathcal{M}_C = \{M \subset \mathbb{R}^3$ complete embedded minimal surface $\mid g(M) < \infty\}$
$\mathcal{M}_C(g) = \{M \in \mathcal{M}_C \mid g(M) = g\}$
$\mathcal{M}_P = \{M \in \mathcal{M}_C \mid \text{proper}\}$, $\mathcal{M}_P(g) = \mathcal{M}_P \cap \mathcal{M}_C(g)$

Main goals:

1. Examples; special families
2. Conformal structure
3. Asymptotics
4. Classification
5. Properness vs completeness
6. Limits

$M \in \mathcal{M}_C \Rightarrow M$ noncompact $\Rightarrow \mathcal{E}(M) = \{\text{ends of } M\} \neq \emptyset$.

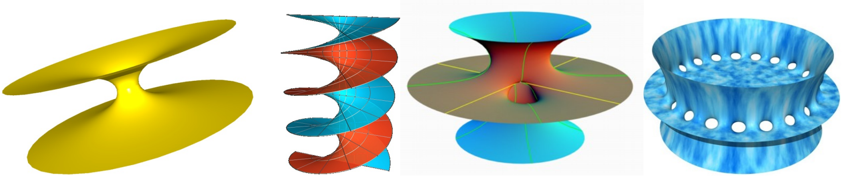
Definition 1

$\mathcal{A} = \{\alpha \colon [0,\infty) \to M$ proper arc$\}$.
$\alpha_1 \sim \alpha_2$ if $\forall C \subset M$ cpt set, α_1, α_2 lie eventually in the same compnt of $M - C$.
$\mathcal{E}(M) = \mathcal{A}/_\sim \longleftarrow$ set of ends of M.
$E \subset M$ proper subdomain, ∂E cpt.
E represents $[\alpha] \in \mathcal{M}(E)$ if $\alpha[t_0,\infty) \subset E$ for some t_0.

$\mathcal{M}_C(g,k) = \{M \in \mathcal{M}_C(g) \mid \#\mathcal{E}(M) = k\}$, $\quad k \in \mathbb{N} \cup \{\infty\}$
$\mathcal{M}_P(g,k) = \mathcal{M}_P \cap \mathcal{M}_C(g,k)$.

Surfaces with finite topology $(\#\mathcal{E}(M) < \infty)$

"Classical" examples:



plane catenoid (1744) helicoid (1776) Costa (1982) Hoffman-Meeks (1990)

Theorem 1 (Colding-Minicozzi, Annals 2008)

$M \in \mathcal{M}_C,\ \#\mathcal{E}(M) < \infty \ \Rightarrow\ M \in \mathcal{M}_P.$

Calabi-Yau problem:

$\mathcal{M}_C = \mathcal{M}_P$?

$\#\mathcal{E}(M) = 1$ (one-ended surfaces)

Theorem 2 (Meeks-Rosenberg, Annals 2005)

$\mathcal{M}_P(0,1) = \{$*plane, helicoid* $\}$ *(conformally* $\mathbb{C}$*).*

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Theorem 3 (Bernstein-Breiner' Commentarii 2011, Meeks-P)

$M \in \mathcal{M}_P(g,1),\ g \geq 1 \Rightarrow M$ *asymptotic to helicoid* *(conformally parabolic)*

M parabolic $\overset{\text{def}}{\Leftrightarrow} \nexists f \in C^\infty(M)$ nonconstant s.t. $f \leq 0$, $\Delta f \geq 0$.

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$\mathcal{M}_P(1,1) \neq \emptyset$ *(existence of a genus 1 helicoid).*

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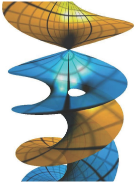
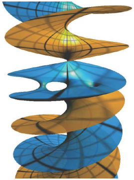
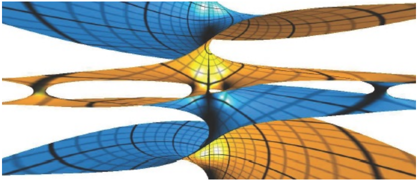
Theorem 5 (Hoffman-Traizet-White, Acta 2016)

$\forall g \in \mathbb{N}$, $\mathcal{M}_P(g,1) \neq \emptyset$ *(existence of a genus g helicoid).* *Uniqueness?*







$2 \leq \#\mathcal{E}(M) = k < \infty$

Theorem 6 (Collin, Annals 1997)

$M \in \mathcal{M}_P(g,k)$, $2 \leq k < \infty \;\Rightarrow\;$ *finite total curvature* $\quad (\int_M K > -\infty)$

Consequence: $M \overset{conf.}{\cong} \mathbb{M}_g - \{p_1, \ldots, p_k\}$, *ends asymptotic to planes or half-catenoids, Gauss map extends meromorphically through the* p_i *(Osserman)*

Theorem 7 (Schoen, JDG 1983)

$M \in \mathcal{M}_C(g,2)$ + *finite total curvature* $\;\Rightarrow\;$ *catenoid.*

Theorem 8 (López-Ros, JDG 1991)

$M \in \mathcal{M}_C(0,k)$ + *finite total curvature* $\;\Rightarrow\;$ *plane, catenoid.*

Theorem 9 (Costa, Inventiones 1991)

$M \in \mathcal{M}_C(1,3)$ + *finite total curvature* $\;\Rightarrow\;$ M *deformed Costa-Hoffman-Meeks (1-parameter family).*

$2 \leq \#\mathcal{E}(M) = k < \infty$: The Hoffman-Meeks Conjecture

Conjecture 1

If $M \in \mathcal{M}_C(g,k)$ + finite total curvature (FTC) $\implies k \leq g+2$.

Theorem 10 (Meeks-P-Ros, 2016)

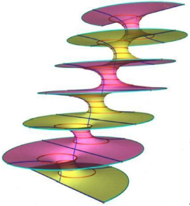
Given $g \in \mathbb{N}$, $\exists C = C(g) \in \mathbb{N}$ s.t. $k \leq C(g)$, $\forall M \in \mathcal{M}_C(g,k)$.

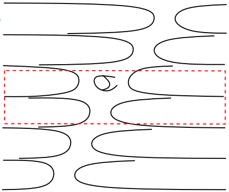
$M \subset \mathbb{R}^3$ minimal surface, $f \in C_0^\infty(M) \Rightarrow \frac{d^2}{dt^2}\Big|_0 \text{Area}(M + tfN) = -\int_M f Lf \, dA$,
$L = \Delta - 2K$ (Jacobi operator).
$\Omega \subset\subset M$. Index$(\Omega) = \#\{$negative eigenvalues of L for Dirichlet problem on $\Omega\}$
Index(M) $= \sup\{\text{Index}(L,\Omega) \mid \Omega \subset\subset M\}$.
If M complete, then FTC $\Leftrightarrow$ Index$(M) < \infty$ (Fischer-Colbrie)
If $M \in \mathcal{M}_C(g,k)$ FTC $\Rightarrow$Index(M)=*Index*$(\Delta + \|\nabla N\|^2)$ on compactification $\mathbb{M}_g$
$\phi \colon \mathbb{M} \to \mathbb{S}^2$ holom map on $\mathbb{M}$ cpt $\Rightarrow$ Index$(\Delta + \|\nabla \phi\|^2) < 7.7 \deg(\phi)$ (Tysk)
If $M \in \mathcal{M}_C(g,k)$ has FTC $\Rightarrow \deg(N) = g + k - 1$ (Jorge-Meeks)

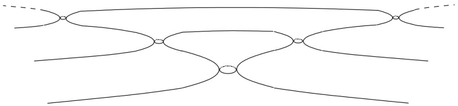
Corollary 1 (Meeks-P-Ros, 2016)

Given $g \in \mathbb{N}$, $\exists C_1 = C_1(g) \in \mathbb{N}$ s.t. Index$(M) \leq C_1(g)$, $\forall M \in \mathcal{M}_C(g,k)$.

$\#\mathcal{E}(M) = \infty$: EMS with infinite topology







Riemann (1867) Hauswirth-Pacard (2007) Traizet (2012) $g = \infty$

Definition 2

$\mathcal{E}(M) \hookrightarrow [0,1]$ embedding. $\mathbf{e} \in \mathcal{E}(M)$ simple end if $\mathbf{e}$ isolated in $\mathcal{E}(M)$.
$\mathbf{e} \in \mathcal{E}(M)$ limit end if not isolated.

Theorem 11 (Collin-Kusner-Meeks-Rosenberg, JDG 2004)

If $M \in \mathcal{M}_P(g,\infty) \Rightarrow M$ has at most two limit ends (top and/or bottom).

Theorem 12 (Hauswirth-Pacard, Inventiones 2007)

If $1 \leq g \leq 37 \Rightarrow \mathcal{M}_P(g,\infty) \neq \emptyset$ ($g \geq 38$ Morabito IUMJ 2008).

$\#\mathcal{E}(M) = \infty$: EMS with infinite topology

Theorem 13 (Meeks-P-Ros, Inventiones 2004)

If $M \in \mathcal{M}_P(g, \infty)$, $g < \infty \Rightarrow$ M cannot have just 1 limit end.

Theorem 14 (Meeks-P-Ros, Annals 2015)

$\mathcal{M}_P(0, \infty) = \{$Riemann minimal examples$\}$.
If $M \in \mathcal{M}_P(g, \infty)$, $g < \infty$ (two limit ends) $\Rightarrow$ simple (middle) ends are asymptotic to planes, and limit ends are asymptotic to Riemann limit ends (conformally parabolic)

Theorem 15 (Traizet, IUMJ 2012)

$\exists M \subset \mathbb{R}^3$ CEMS with infinite genus and 1 limit end, all whose simple ends are asymptotic to half-catenoids.

Theorem 16 (Meeks-P-Ros, 2018, Calabi-Yau for finite genus)

If $M \in \mathcal{M}_C(g, \infty)$ countably many limit ends $\Rightarrow$ $M \in \mathcal{M}_P$, exactly 2 limit ends, conformally parabolic.

Limits of EMS

$\{M_n \subset A \overset{\text{open}}{\subset} \mathbb{R}^3\}_n$ emb min surf (EMS), ∂M_n cpt (possibly empty).

Classical limits (Arzelá-Ascoli)

Locally bded curvature + Area(M_n) locally unifly bded + $\exists$ accumulation point $\Rightarrow \{M_n\}_n \overset{\text{subseq}}{\to} M_\infty$ EMS inside A, with finite multiplicity.

Theorem 17 (Lamination limits, Meeks-Rosenberg, Annals 2005)

Locally bded curv + $\exists$ accum point $\Rightarrow \{M_n\}_n \overset{subseq}{\to} \mathcal{L}_\infty$ minimal lamination of A (closed union of disjoint EMS, called leaves).

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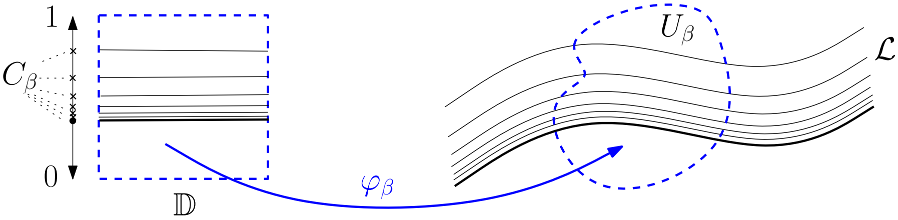

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$$\widehat{\mathcal{S}} = \left\{ x \in A \mid \sup |K_{M_n \cap \overline{\mathbb{B}}(x,r)}| \to \infty, \ \forall r > 0 \right\}.$$

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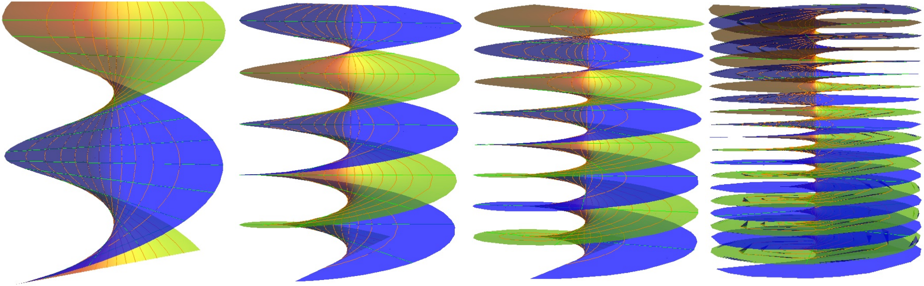
Theorem 18 (Colding-Minicozzi, Annals 2004)

$M_n \subset \mathbb{B}(R_n)$, $\partial M_n \subset \partial \mathbb{B}(R_n)$ emb min disks, $R_n \to \infty$.

If $\widehat{\mathcal{S}} \cap \overline{\mathbb{B}}(1) \neq \emptyset \Rightarrow \{M_n\}_n \overset{subseq}{\rightarrow} \mathcal{F}_\infty$ foliation of $\mathbb{R}^3$ by planes, outside $S(\mathcal{L}) = \{1 \text{ line}\}$ (singular set of convergence) $\longleftarrow$ Meeks, Duke 2004

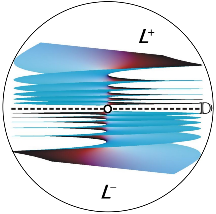
In particular, no singularities for limit lamination. Example: $\frac{1}{n}$ helicoid.



H $\quad$ $\frac{1}{2}H$ $\quad$ $\frac{1}{4}H$ $\quad$ $\frac{1}{16}H$

Limits of EMS

(Colding-Minicozzi, 2003):
Singular minimal lamination
$\mathcal{L} = L^+ \cup L^- \cup \mathbb{D} = \lim_n M_n$
$M_n \subset \mathbb{B}(1)$ emb min disks,
$\partial M_n \subset \partial \mathbb{B}(1)$.



($\vec{0}$ = isolated singularity)

When does a minimal lamination extend across an isolated singularity?

Theorem 19 (Local Removable Sing Thm, Meeks-P-Ros, JDG 2016)

$\mathcal{L} \subset \overline{\mathbb{B}}(1) - \{\vec{0}\}$, $\vec{0} \in \overline{\mathcal{L}}$.
$\mathcal{L}$ extends to a minimal lamination of $\overline{\mathbb{B}}(1)$ $\Leftrightarrow$ $|K_{\mathcal{L}}|(x) \cdot |x|^2$ bded on $\mathcal{L}$.

Valid in a Riemannian 3-mfd (N, g): $\mathcal{L} \subset \overline{B}_N(p, r) - \{p\}$, $p \in \overline{\mathcal{L}}$.
$\mathcal{L}$ extends to a minimal lamination of $\overline{B}_N(p, r)$ $\Leftrightarrow$ $|\sigma_{\mathcal{L}}|(x) \cdot d_N(p, x)$ bded on $\mathcal{L}$.

Theorem 20 (Quadratic Curv Decay Thm, Meeks-P-Ros, JDG 2016)

If $\mathcal{L} \subset \mathbb{R}^3 - \{\vec{0}\}$ minimal lamination with $|K_{\mathcal{L}}|(x) \cdot |x|^2$ bded on $\mathcal{L}$ $\Rightarrow$
$\mathcal{L} = \{M\}$, $M \subset \mathcal{M}_P$ with FTC (in particular, $|K_M|(x) \cdot |x|^4$ bded on M).

Limits of EMS: Locally simply connected sequences

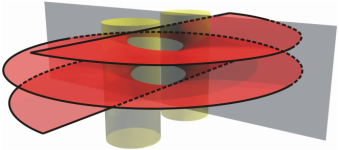
$\{M_n\}_n$ locally simply connected (LSC) in $A \overset{\text{open}}{\subset} \mathbb{R}^3 \overset{\text{(def)}}{\Leftrightarrow} \forall q \subset A$, $\exists \varepsilon_q > 0$ s.t. $\mathbb{B}(q, \varepsilon_q) \subset A$ and for n suf large, $M_n \cap \mathbb{B}(q, \varepsilon_q)$ consists of disks $D_{n,m}$ with $\partial D_{n,m} \subset \partial \mathbb{B}(q, \varepsilon_q)$.

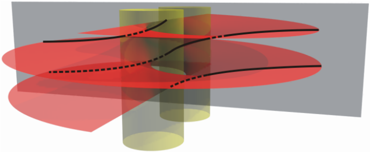
Theorem 21 (Meeks-P-Ros, 2016)

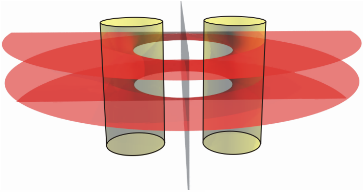
$W \overset{\text{closed}}{\subset} \mathbb{R}^3$ countable, $\{M_n\}_n$ EMS, LSC in $A = \mathbb{R}^3 - W$, ∂M_n cpt (or $\emptyset$), $g(M_n) \leq g$. Then:

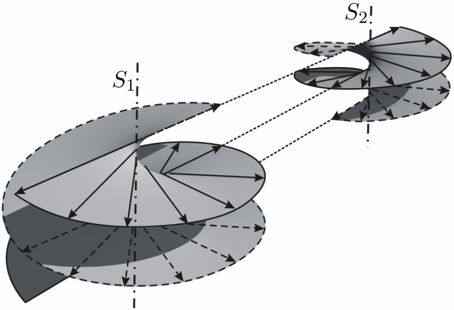
$\exists \mathcal{L} \subset \mathbb{R}^3$ minimal lamination, $\exists \mathcal{S}(\mathcal{L}) \overset{\text{closed}}{\subset} \mathcal{L} - W$ s.t. $\{M_n\}_n \overset{\text{(subseq)}}{\to} \mathcal{L}$ on cpt subsets of $A - S(\mathcal{L})$. Furthermore:

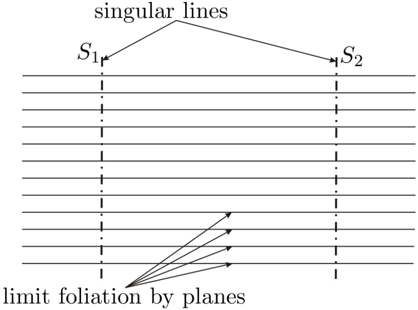
1. *If $S(\mathcal{L}) \neq \emptyset \Rightarrow \mathcal{L}$ foliation of $\mathbb{R}^3$ by planes, $S(\mathcal{L}) = \{$ 1 or 2 lines $\}$ (limit parking garage structure). In part: no singularities for $\mathcal{L}$.* FIGURE
2. *If $\exists L \in \mathcal{L}$ nonflat leaf $\Rightarrow S(\mathcal{L}) = \emptyset$, $\mathcal{L} = \{L\}$, $L \in \mathcal{M}_P$ and $g(L) \leq g$. Furthermore, L lies in one of three cases:*
   1. *$L \in \mathcal{M}_P(g(L), 1)$ (helicoid with handles)*
   2. *$L \in \mathcal{M}_P(g(L), k)$, $k \geq 2$ (finite total curvature)*
   3. *$L \in \mathcal{M}_P(g(L), \infty)$ (two limit ends)*








S_2
S_1


singular lines
S_1
S_2
limit foliation by planes

Back to the Calabi-Yau problem

Theorem 22 (Min Lam Closure Thm, Meeks-Rosenberg, DMJ 2006)

$M \subset \mathbb{R}^3$ CEMS, ∂M cpt (or $\emptyset$). If $I_M \geq \delta(\varepsilon) > 0$ outside of some intrinsic ε-neighb of ∂M (I_M = inj radius fct) $\Rightarrow$ M proper.

Valid in a Riemannian 3-mfd (N, g) with the conclusion: $\overline{M}$ = min lamin of N

Sketch of proof of Thm 16

Take $M \in \mathcal{M}_C(g, \infty)$ with countably many limit ends. Baire's Thm $\Rightarrow$ isolated points in $\mathcal{E}_{limit}(M)$ (simple limit ends) are dense. So it suffices to show:

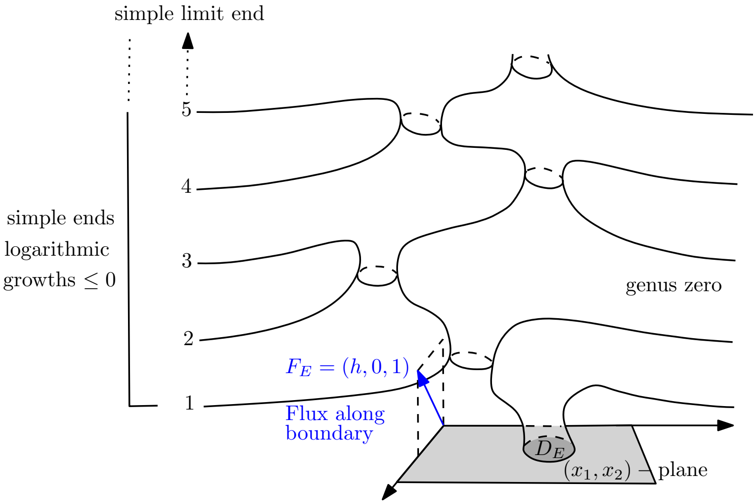
1. If M has 2 simple limit ends $\Rightarrow$ M proper.
2. M cannot have 3 simple limit ends (Thm 13 discards 1 limit end).

Proposition 1 (Christmas tree picture)

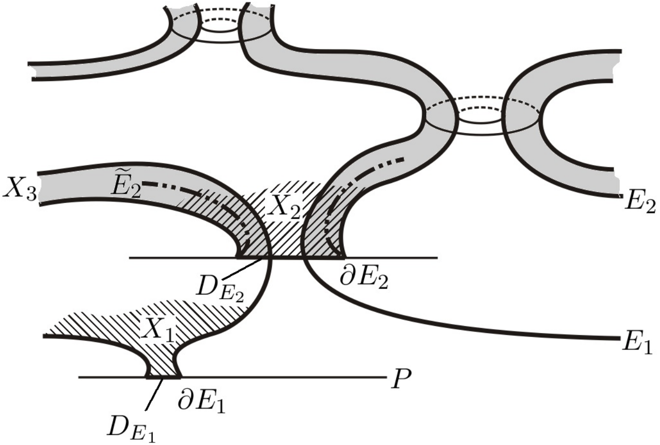
E simple limit end of $M \subset \mathbb{R}^3$ CEMS, $g(E)=0$ $\Rightarrow$ E proper and after passing to a smaller end representative, translation, rotation & homothety:

(1) Simple ends of E have FTC & $\log \leq 0$
(2) The limit end of E is the top end
(3) $\partial E = \partial D$, $D \overset{cnvx}{\subset} \{x_3 = 0\}$, $\overset{\circ}{D} \cap E = \emptyset$

(4) $\exists f : \mathcal{R}_+ \to E$ orient preserving diffeo ($\mathcal{R}_+$ = top half of a Riemann min example)



(Christmas tree picture)



(Discarding 3 simple limit ends)