Homogeneous Ricci curvature and the beta operator

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$CURVATURE \rightleftarrows TOPOLOGY$

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$\swarrow \nearrow$

SYMMETRY

Lie groups

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$\searrow\nwarrow$ $\swarrow\nearrow$

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Ricci

$\searrow\nwarrow$ $\swarrow\nearrow$

SYMMETRY

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CURVATURE
Ricci
Einstein, Ricci solitons, ...
Ricci pinching
Ricci < 0

$\rightleftarrows$

TOPOLOGY

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SYMMETRY
Lie groups
Structure
Uniqueness
Classification
Compact quotients

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- [L 10] F increases along the Ricci flow. Ricci solitons are therefore the only (local) global maxima of F on the set of all left-invariant metrics on a nilpotent Lie group.

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Which are the local and global maxima of $F : \mathcal{C}_S \longrightarrow \mathbb{R}$??

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Gradient of $F|_{\mathcal{C}_S}$ and the second variation; moment map for the conjugation of matrices.

Definition of the beta operator (switch to GIT)

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Actually a maximum attained at a single optimal direction in $\mathrm{Dg}(m)$:

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$(G/K, g)$, $\dim G/K = n$, G non-semisimple, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, nilradical $\mathfrak{n} \subset \mathfrak{p}$, $\dim \mathfrak{n} = m$,

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$(G/K, g)$ is Einstein $\Leftrightarrow$ in addition $\boxed{S(\mathrm{ad}_{\mathfrak{p}}\, H) = |\,\mathrm{scal}_N\,|\, D_+|_{\mathfrak{p}}}$.