Engel-type subgroups and length parameters of finite groups

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Abstract

Let g be an element of a finite group G. For a positive integer n, let $E_n(g)$ be the subgroup generated by all commutators $[...[[x,g],g],\dots,g]$ over $x\in G$, where g is repeated n times. By Baer's theorem, if $E_n(g)=1$, then g belongs to the Fitting subgroup $F(G)$. We generalize this theorem in terms of certain length parameters of $E_n(g)$.

For soluble G we prove that if, for some n, the Fitting height of $E_n(g)$ is equal to k, then g belongs to the $(k+1)$th Fitting subgroup $F_{k+1}(G)$.

For nonsoluble G the results are in terms of nonsoluble length and generalized Fitting height. The generalized Fitting height $h^*(H)$ of a finite group H is the least number h such that $F^*_h(H)=H$, where $F^*_0(H)=1$, and $F^*_{i+1}(H)$ is the inverse image of the generalized Fitting subgroup $F^*(H/F^*_i(H))$. Let m be the number of prime factors of $|g|$ counting multiplicities. It is proved that if, for some n, the generalized Fitting height of $E_n(g)$ is equal to k, then g belongs to $F^*_{f(k,m)}(G)$, where $f(k,m)$ depends only on k and m.

The nonsoluble length $\lambda(H)$ of a finite group H is defined as the minimum number of nonsoluble factors in a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups. It is proved that if $\lambda(E_n(g))=k$, then g belongs to a normal subgroup whose nonsoluble length is bounded in terms of k and m.

We also state conjectures of stronger results independent of m and show that these conjectures reduce to a certain question about automorphisms of direct products of finite simple groups.

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