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Polynomial integrable Hamiltonian systems and symmetric powers of $\mathbb{C}^2$

We give a construction of polynomial integrable systems in $\mathbb{C}^{2N}$ (or on $\mathbb{R}^{2N}$, if the base field is $\mathbb{R}$) using the algebra-geometric structure of the space $Sym^N(\mathbb{C}^2)$. It is based on a canonical transformation $\varphi : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ from variables $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{2N}$, $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathbf{y} = (y_1, \ldots, y_N)$ to $\mathbf{q} = (q_1, \ldots, q_N)$, $\mathbf{p} = p_1, \ldots, p_N)$ given by the generating function

$$G = \sum_{i,n=1}^{N} \frac{1}{n} x_i^n p_n \;\Rightarrow\quad q_n = \frac{\partial G}{\partial p_n} = \frac{1}{n} \sum_{i=1}^{N} x_i^n, \quad y_i = \frac{\partial G}{\partial x_i} = \sum_{n=1}^{N} x_i^{n-1} p_n\,.$$

The canonical transformation φ can be decomposed in the projection $\pi : \mathbb{C}^{2N} \to Sym^N(\mathbb{C}^2)$ and a bi-rational isomorphism $Sym^N(\mathbb{C}^2) \to \mathbb{C}^{2N}$. The projection π gives a branching covering of $Sym^N(\mathbb{C}^2)$.

With any polynomial $F(x,y) \in \mathbb{C}[x,y]$ such that $\partial_y F(x,y) \neq 0$ we associate N compatible Stäkel type integrable Hamiltonian systems in $\mathbb{C}^{2N}$

$$\frac{d\,x_i}{d\,t_k} = \frac{\partial\,H_k(\mathbf{x},\mathbf{y})}{\partial\,y_i}, \quad \frac{d\,y_i}{d\,t_k} = -\frac{\partial\,H_k(\mathbf{x},\mathbf{y})}{\partial\,x_i}, \qquad i,k \in \{1,\ldots,N\},$$

where $H_k(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{N} W_{k,i} F(x_i, y_i)$ and $W_{k,i}$ is the inverse Vandermonde matrix. The intersection of the level sets $H_s(\mathbf{x},\mathbf{y}) = h_s$, $h_s \in \mathbb{C}$, $s = 1,\ldots,N$, is a quasi-projective algebraic variety in $\mathbb{C}^{2N}$

$$\mathcal{G} = \{(\mathbf{x},\mathbf{y}) \in \mathbb{C}^{2N} \,|\, x_i \neq x_j \text{ if } i \neq j, \text{and } F(x_i,y_i) = \sum_{s=1}^{N} h_s x_i^{s-1},\; i = 1,\ldots,N\}.$$

which is S_N invariant with the free action of S_N.

We show that the functions $\mathcal{H}_k(\mathbf{q},\mathbf{p})$, $k = 1,\ldots,N$, defined by $\phi^* \mathcal{H}_k(\mathbf{q},\mathbf{p}) = H_k(\mathbf{x},\mathbf{y})$ are polynomials. They are functionally independent. It leads us to one of our main result:
In the space $\mathbb{C}^{2N}$ there are N commuting *polynomial* Hamiltonian systems correponding to the Hamiltonians $\mathcal{H}_1(\mathbf{q},\mathbf{p}),\ldots,\mathcal{H}_N(\mathbf{q},\mathbf{p})$.

It follows from the Liouville theorem that all Hamiltonian systems obtained are completely integrable. In the results obtained we do not impose any condition on the genus of the curve

$$\Gamma = \{(x,y) \in \mathbb{C}^2 \,|\, F(x,y) = \sum_{s=1}^{N} h_s x^{s-1}\}$$

neither request that the curve Γ is regular.

Application of this construction to N–th symmetric power of a plane algebraic curve Γ of genus g leads to N integrable Hamiltonian systems on $\mathbb{C}^{2N}$. In the case of a non-singular hyperelliptic curves Γ of genus g and $N = g$ our systems represent integrable hierarchies of equations which had been discovered in the theory of finite gap solutions (algebra-geometric integration) of the Korteweg-de-Vrise equation.

For $N = 2, 3$ and $g = 1, 2, 3$ we present explicit examples of our polynomial systems and discuss the problem of their integration. These results were announced in:

V. M. Buchstaber and A. V. Mikhailov, *Polynomial integrable Hamiltonian systems on symmetric powers of plane algebraic curves*, UMN, December (2018).