Rigidity of Rings and Invariants of the Weyl Algebra I

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Introducing the notion of rigidity

All base fields k will have *char* $= 0$. All automorphisms will be k-algebra automorphisms.
A natural and important question in invariant theory is the following:

Question

(Galois Embedding) Given a finitely generated k algebra A, are there non-trivial finite groups of automorphisms G such that $A^G \cong A$?

In the commutative world, this question has been famously adressed by the Chevalley-Shephard-Todd theorem. In non-commutative algebras, it is quite common that the so called “rigidity"phenomena happens:

- *(Rigidity)* A^G is never isomorphic to A, for any finite group of automorphisms.

One such case, our main interest, is the Weyl Algebra $A_n(k)$ - proved rigid by Alev and Polo in 1995 (for k algebraically closed). In 2017, Tikaradze settled an old conjectured and proved even more:

Theorem

(Tikaradze, 2017) There is no $\mathbb{C}$-domain Γ with a non-trivial finite group of automorphisms G such that $\Gamma^G \cong A_n(\mathbb{C})$.

Nonetheless, we have the following surprising:

... and the surreal softness it hides

Theorem

(Alev, Dumas, 1997) Let G be any finite group of automorphisms of $A_1(\mathbb{C})$, and extend its action to $W_1(\mathbb{C})$, the total quotient field of the Weyl algebra. Then we allways have $W_1(\mathbb{C})^G \cong W_1(\mathbb{C})$

We will use the notation $W_n(k)$ to denote the *Weyl Fields*, the field o fractions of the Weyl Algebras.

What is behind this surprising phenomena? The idea comes from algebraic geometry. Classifying objects up to isomorphism is a too hard problem, so they are studied up to birational equivalence. In case of affine varieties this means: $\mathit{Specm}\, A \cong \mathit{Specm}\, B$ if and only if $A \cong B$; they are birationally equivalent if and only if $\mathit{Frac}\, A \cong \mathit{Frac}\, B$. We have the GIT quotient $(\mathit{Specm}\, A)/G = \mathit{Specm}\, A^G$, so in the geometric case Question 1 asks: $(\mathit{Specm}\, A)/G \cong (\mathit{Specm}\, A)$? The birational version, then, is:

Question

(Birational Galois Embedding) Let A be an Ore domain an G a finite group of automorphisms of A. Use Q() to denote the ring of fractions. When $Q(A^G) = Q(A)^G \cong Q(A)$*?*

In the commutative case we have the famous

Problem

(Noether's Problem, 1913) Given a finite group G acting linearly on the rational function field $k(x_1, \ldots, x_n)$, when $k(x_1, \ldots, x_n)^G \cong k(x_1, \ldots, x_n)$?

The n-th Weyl $A_n(k)$ algebra is generated by the canonical Weyl generators, denoted here by $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_n$, that satisfy the canonical Weyl relations. It is also the ring of differential operators on the polynomial algebra, and any action of a finite group G by linear automorphisms on the polynomial algebra $P_n(k)$ can be extended to the Weyl Algebra (as seen this way) by conjugation with differential operators: $g.D(f) = g(D(g^{-1}(f))), g \in G, D \in A_n(k), f \in P_n(k)$. Such group automorphisms of the Weyl Algebra are called linear.

Following Gelfand-Kirillov philosophy that the Weyl Fields are an important non-commutative analogue to the field of rational functions, in 2006 Alev in Dumas introduced the Noncommutative Noether's Problem:

Problem

(Noncommutative Noether's Problem) Let G be a finite group of linear automorphisms of $A_n(k)$. When we have $W_n(k)^G \cong W_n(k)$?

As we shall see, there is a striking similarity for the solutions between the original and noncommutative versions of Noether's Problem. So, despite the rigidity of the Weyl algebra, there is still a lot of good structure theory, resembling the original Weyl algebra, when we consider an adequate notion of birational equivalence.

The same happens for the representation theory. Remember the following well-known result from commutative algebra:

Proposition

Let $A \subset B$ be an integral extension of two k-algebras. Consider the induced map $\Phi : Specm\, B \to Specm\, A$.

- *The fibers are never empty.*
- *In case B is also a finite algebra over A, the fibers are all finite;*
- *In case $A = B^G$ the number of fibers of the map is uniformely bounded (G a finite group of automorphisms of B).*

We shall see that a similar phenomena happens for many invariant rings of the Weyl algebra (and other rings of differential operators). We use the theory of Galois Algebras and Orders (Futorny, Ovsienko, 2010, 2014), which provide an adequate theoretical framework the categories like the Gelfand-Tsetlin one for $U(gl_n)$. This involves a pair of algebra U and commutative subalgebra Γ. This involes embedding U in a skew monoid ring over $Frac\,\Gamma$, and we obtain a map $\Phi : left - Specm\, U \to Specm\, \Gamma$ with properties which are similar as those above. Again, despite the non-isomorphism given by the rigidity result, the Weyl algebra and their invariants are similar in the structure of their categories of (Gelfand-Tsetlin) modules.

A review of the classical commutative case with a geometric bias

Let G be a finite group of automorphism of $GL_n(k)$ acting on the polynomial algebra $P_n(k) = k[x_1, \ldots, x_n]$ by linear automorphisms. Every such action arises in the following way: we have G a finite group of $GL(V)$ for a finite dimensional vector space V of dimension n and we make it act in the algebra of polynomial functions on V, $S(V^*)$, in the standard way: $g.f(v) = f(g^{-1}(v))$, $f \in S(V^*), g \in G, v \in V$.
Let's recall the precise statement of Chevalley-Shephard-Todd Theorem.

Theorem

(Chevalley-Shephardd-Tod) Let G be a finite group acting linearly on $P_n(k)$. Then are equivalent:

- *G, seem as a subgroup of $GL_n(k)$, is a pseudo-reflection group in it's natural representation.*
- *$P_n(k)^G \cong P_n(k)$ (geometric interpretation: $\mathbb{A}^n/G \cong \mathbb{A}^n$).*
- *$P_n(k)$ is finitely generated free/projective/flat module over $P_n(k)^G$ (geometric interpretation: the projection map $\pi : \mathbb{A}^n \to \mathbb{A}^n/G$ is a flat morphism.)*
- *$P_n(k)^G$ is a regular ring (geometric interpretation: $\mathbb{A}^n/G$ is smooth).*

We asked in the Chevalley-Shephard-Todd, in the geometric interpretation: when $\mathbb{A}^n/G \cong \mathbb{A}^n$ for a finite group G acting linearly? When we consider the weaker relation of birational equivalence, we can expect a more rich situation.
This is Noether's Problem:

Problem

(Noether's Problem, 1913) Given a finite group G acting linearly on the rational function field $k(x_1, \ldots, x_n)$, when $k(x_1, \ldots, x_n)^G \cong k(x_1, \ldots, x_n)$?

These are some of most important cases of positive solution:

- When $n = 1$ and 2; and when k is algebraically closed, for $n = 3$.
- When G is a group of pseudo-reflections (by Chevalley-Shephard-Todd Theorem).
- When the natural representation of G decomposes as a direct sum of one dimensional representations (Fischer).
- For finite abelian groups acting by transitive permutations of the variables $x_1, \ldots, x_n$, the problem is settled by the work of Lenstra.
- For $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the symmetric group S_n permutes the variables y_i, x_i simultaneously (Mattuck).
- For the alternating groups A_3, A_4, and A_5 (Maeda).

Counter-examples are also known. The case of permutation actions is particularly important for it's relation to constructive aspects of the Inverse Galois Problem.

Rigidity in Algebras

Given the study realized for the polynomial algebra, it was natural to search for analogues to the Chevalley-Shephard-Todd theorem for other kinds of algebras. Given that the polynomial algebra is the relatively free algebra in the variety determined by the identity $[x, y] = 0$, a natural first step was to study invariants of free and relativiely free algebras in varieties.

Theorem

- *Let* $k\langle x_1, \ldots, x_n\rangle$ *be the free associative algebra and G a finite group acting linearly. Then the subalgebra of invariants is allways free (Lane, Kharchenko, 1976, 1978). However, the rank behaves badly (it can be of infinite rank of instance). The rank is the same only if the group is trivial (Dicks, Formanek, 1982).*
- *The ring of generic matrices is rigid (Guralnick, 1985).*
- *Using the result above, it can be shown that the ring of invariants of an relavitively free algebra is allways rigid, unless the Jacobson(=prime) radical of it's T-ideal is the one generated by* $([x, y])$.

In 1995 Alev and Polo proved the following theorems (k is suposed algebraically closed):

Theorem

- *Let g be a semisimple Lie algebra of finite dimension, and G a finite group of automorphisms of $U(g)$. If $U(g)^G \cong U(g')$ for another semisimple lie algebra g', then $g \cong g'$ and G is trivial. In particular the algebra is rigid.*
- *For any finite group of automorphisms of $A_n(k)$, G, we can't have $A_n(k)^G \cong A_n(k)$. The proof is much simpler if the action of G is linear.*

In 2008, E. Kirkman, J. Kuzmanovich and J.J. Zhang discovered more classes of rigid algebras, and noticing work of Alev and Dumas such as the mentioned above, they proposed the following two questions: For a rigid Ore domain, study when $Q(A)^G \cong Q(A)$, or then $Q(B)$, where B is a Artin-Schelter regular algebra. Noncommutative Noether's Problem is a natural approach to the first question; we will have some words to say about the second one also.

Gelfand-Kirillov Philosophy

We said before that, following the work of Gelfand and Kirillov, it has become usual to consider the Weyl Fields as canonical representatives of isomorphisms classes of $Q(A)$, for many Noetherian domains A in certain families of algebras. In this sense, rationality is understood as having as total ring of quotients a Weyl Field.

The origin of this way of thinking is the remarkable

Conjecture

(Gelfand-Kirillov Conjecture, 1966) Let L be a finite dimensional Lie algebra over an algebraically closed field. Then $Q(U(L))$ is isomorphic to $W_n(K)$, where K is a purely transcedental extension of k (of finite transcendence degree).

The most important cases of positive solution are:

- $L = gl_n, sl_n$, L nilpotent (Gelfand,Kirillov, 1966).
- L solvable. (Borho, Joseph, McConnell. 1973)
- L has dimension at most 8 (Alev, Ooms, Van den Bergh, 2000).

The first counter-example to this conjecture was found by Alev, Ooms, Van den Bergh (1996). For simple Lie algebras, the question was almost completely solved by Premet (2010) : true for algebras of type A, unknown for type C and G_2, and false for all others.
However, the Conjecture is true after a small modification for all simple Lie algebras: for $U(L) \otimes_{Z(U(L))} \tilde{Z}$, where $\tilde{Z}$ is an adequate extension of the center (Gelfand, Kirillov, 1969). It is also true after the necessary modifications for the maximal primitive quotients of their envelopping algebras (Conze, 1974).

There are many other cases of algebras, infinite dimensional over their centers, where the Gelfand-Kirillov philosophy is applied - for instance, the Sympletic Reflection Algebras of Etingof and Ginzburg (2002); and many Galois Algebras. We are going to see these in a moment. They all are all connected to Noncommutative Noether's Problem.

Given the development of quantum groups, a quantum Gelfand-Kirillov Conjecture was also introduced. Loosely speaking:

Conjecture

(Quantum Gelfand-Kirillov Conjecture) Let A be a "quantum group- such as quantized coordinate ring, or a quantized Weyl algebra, or a quantized envelopping algebra. Then $Q(A)$ is isomorphic to a the total quotient ring of a quantum affine space over a purely transcedental extension (of finite transcendence degree) of the base field (suposed algebraically closed).

A full discussion can be found in K. Brown, K. Goodearl, "Lectures on Algebraic Quantum Groups". We will adress this question again - and point it has also a connection to a quantized version of Noether's Problem.

Back to Noncommutative Noether's Problem

Now to Noncommutative Noether's Problem proper. Let's recall it.

Problem

(Noncommutative Noether's Problem) Let G be a finite group of linear automorphisms of $A_n(k)$. When we have $W_n(k)^G \cong W_n(k)$?

The following cases of positive solution were obtained by Alev and Dumas (2006)

- When $n = 1, 2$ and G is arbitrary.
- When the natural representation of G decomposes as a direct sum of one dimensional representations.

In 2010 Futorny, Molev and Ovsienko proved the positive solution for the problem when $G = S_n$ act on a space of dimension n permuting the base as usual, and k is algebraically closed.

New Results!

In 2017, elaborating on a preliminar work of 2006 by Futorny and Ovsienko, Futorny and Schwarz obtained a elementary proof (a much simplified version of the 2006's one) of the same statement for any field.

Theorem

(Eshmatov, Futorny, Schwarz, 2015 arxiv, 2017 Proc. Amer. Math. Society) When, in it's natural representation, G acts as an unitary reflection group, we have $W_n(\mathbb{C})^G \cong W_n(\mathbb{C})$

The key igredient for the proof is the following: call $\Lambda = \mathbb{C}[x_1, \ldots, x_n]$. By restriction of domain, the inclusion $\Lambda^G \to \Lambda$ induces an injective map $D(\Lambda)^G \to D(\Lambda^G) \cong D(\Lambda)$ that, however, is not injective, because the action of G is not free. Removing the fixed hyperplanes is equivalent to localize Λ by an G invariant polynomial Δ, and obtain a Zariski open set where the action of G restrict and is free.

Then we can use:

Theorem

(Main Technical Result, from Cannings and Holland, 1994) Let X be an affine irreducible algebraic variety and G a finite group acting on it. Let V be the open subset of X where G acts freely. If X satisfies Serre condition S_2 (in particular, if it is normal) and $\mathrm{codim}_X(X - V) \geq 2$ then $D(X/G) \cong D(X)^G$. The same holds if G acts freely.

From this follows clearly:

Proposition

Consider the inclusion $\Lambda_\Delta^W \to \Lambda_\Delta$. By restriction of domain $(\dagger)\, \phi : D(\Lambda_\Delta)^W \to D(\Lambda_\Delta^W)$ is an isomorphism.

This is the main ingredient in the proof.

In the last two years, Futorny and Schwarz obtained the following generalization of this result. We use the same notation as before. The difference is that k is an arbitrary field of zero characteristic. We would like to thanks Jacques Alev for his great discussions, which contributed to the final form of this theorem.

Theorem

(Futorny, Schwarz, 2018) Noncommutative Noether's Problem has a positive solution for any pseudo-reflection group over any field k: if G acts by pseudo-reflections, $W_n(k)^G \simeq W_n(k)$. Morover, we have an algorithm to find the Weyl generators inside $W_n(k)^G$ - and in fact, they can be found inside a localization of $A_n(k)^G$ by a single polynomial.

Here is an example of the use of the algorithm for the isomorphism $W_3(k)^{S_3} \to W_3(k)$.

$$x_1 + x_2 + x_3 \to X_1,\ x_1x_2 + x_2x_3 + x_1x_3 \to X_2,\ x_1x_2x_3 \to X_3;$$

$$\frac{x_1^2(x_2 - x_3)}{J}\partial_1 + \frac{x_2^2(x_3 - x_1)}{J}\partial_2 + \frac{x_3^2(x_1 - x_2)}{J}\partial_3 \to Y_1;$$

$$\frac{x_1(x_3 - x_2)}{J}\partial_1 + \frac{x_2(x_1 - x_3)}{J}\partial_2 + \frac{x_3(x_2 - x_1)}{J}\partial_3 \to Y_2;$$

$$\frac{(x_2 - x_3)}{J}\partial_1 + \frac{(x_3 - x_1)}{J}\partial_2 + \frac{(x_1 - x_2)}{J}\partial_3 \to Y_3.$$

where $J = (x_1 - x_2)(x_1 - x_3)(x_3 - x_2)$.

The idea of the proof is as follows:

- Use invariant theory theory of arbitrary pseudo-reflection groups to find good invariant polynomials to localize (mimicking the proof for the complex case).
- Use the well-known (but not easy to find) fact that pseudo-reflection groups decompose as a direct product of irreducible factors (like the finite coxeter groups case).
- Very hard results regarding the field of definition of the representation theory of pseudo-reflection groups, to reduce the problem to the cases where we can apply ideas from algebraic geometry.

Conjecture

(Schwarz, 2018) The proof strongly suggest the following statement: Let G be a Weyl group. Then $FracA_n(k)^G \cong FracA_n(k)$, where k is field of prime characteristic, for char k $>> 0$.

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A Conjecture relating Noether's Problem and it's Noncommutative Analogue

The following is an elaboration of a heuristic conjecture made by Schwarz in 2015 (S. Paulo Journal Math. Sci.).

Conjecture

Let k be any field of 0 char. Let G be a finite group acting linearly on a vector space V of dimension n. Then if classical Noether's Problem holds for $S(V^*)^G$ then the Noncommutative one holds for $D(S(V^*))^G$.

As we saw, the conjecture holds for pseudo-reflection groups; from the work of Alev and Dumas, we have more coincidences: when *dim* $V = 1, 2$, or when the representation of G is a direct sum of one dimensional representations.

Under technical conditions, the conjecture allways holds. Let k be algebraically closed.

Theorem

(Futorny, Schwarz, 2017) Supose G satisfies the conditions of the Main Technical Result (since the affine space is a normal variety, the conditions are only on the group action). Then the Conjecture holds for G.

This theorem has a very technical assumption. However, with it we can show the amazing:

Theorem

(Futorny, Schwarz, 2018) Let k be algebraically closed, and let G be any group acting by permutations on Noether's Problem with a positive solution. Then the Noncommutative Noether's Problem also has a positive solution. This includes, for instance, the positive part of Lenstra work, Mattuck result, and the alternating groups.

As surprising this result might be, ideally the result should hold for the same fields as the original actions in the original Noether's Problem. In a certain sense, being algebraically closed trivializes things, as it reduces things to algebraic geometry (if there is something like trivial algebraic geometry to begin with).
Indeed we have:

Theorem

(Futorny, Schwarz, 2018) Let k be any field of zero characteristic. If A_3 acts on $W_3(k)$ permuting the variables as usual, then Noncommutative Noether's Problem holds: $W_3(k)^{A_3} \cong W_3(k)$. Similary, $W_4(k)^{A_4} \cong W_4(k)$.

Open Problems

Problem

Find Counter-Examples to Noncommutative Noether Problem.

Problem

In view of our Conjecture and the strong evidence for it, find an example of a linear action of a finite group such that the Classical Noether's Problem fails but the Noncommutative does hold.

A question to the audience before they start making questions to me

Have you heard about this result?

- Let k be any field of zero characteristic and G *any* finite group of automorphisms of $k[x, y]$. Then $k(x, y)^G \cong k(x, y)$.

I am curious because of Alev and Dumas result about the first Weyl Field, among the same lines. Note that is *essential* in the case of the polynomial ring to suppose k not algebraically closed - otherwise the result is trivially true because of Zariski-Castelnuovo Theorem.