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RIEMANNIAN SUBMERSIONS

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this short note

Abstract. In these short notes we introduce and study a few basic concepts concerning Riemannian submersions. We exhibit some geometric features of these geometric objects by looking at the properties of two tensor fields defined in terms of vertical and horizontal sections. All geometric information concentrated on a Riemannian submersion may be described by those tensor fields as it can be seen when working with curvature invariants of the ambient spaces.

Contents

1. Introduction

It is well known that immersions and submersions, both of them viewed as special tools in differential geometry, also play an important role in Riemannian geometry, especially when the involved manifolds carry an additional structure. Although submersions are, in a certain sense, a counterpart of immersions, the corresponding theories are quite different, even from a historical point of view. The theory of isometric immersions, started with the work of Gauss on surfaces in the Euclidean 3-dimensional space, is classical and strongly explained in many books, whereas the theory of Riemannian submersion goes back to four decades ago, when B. O'Neill in [8, 9] and A. Gray in [1], independently, formulated the base of such theory, which has hugely been developed in the last two decades.

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Riemannian submersions also appear in physics providing several applications. For instance, they are useful to explain extensions of important aspects of theoretical particle physics in the presence of non-Abelian gauge theories. An evidence of this phenomena was given by B. Watson who studied the relations between Riemannian submersions and instantons, the latter of which are critical functionals of the Yang–Mills action; see [11]. Other applications in physics where Riemannian submersions are widely used are generalized nonlinear sigma models in curved spaces, the Dirac monopole, Einstein equations, among others. See for example [2, c. 8] as well as the references therein.

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One of our main reasons that encouraged us to study Riemannian submersions is because they are needed to define a notion of Riemannian stack; visit [5, 6]. These geometric objects generalize both notions of Riemannian manifold and Riemannian orbifold, thus allowing the study of Riemannian geometry over more general singular spaces. It can be evidenced with the recent study of geodesics on Riemannian stacks [4].

The notes are divided as follows. In Section 2 we define Riemannian submersions and introduce the basic terminology we will be using throughout this work. We also exhibit here three classical examples. In the spirit of O'Neill seminal papers [8, 10], in Section 3 we define two tensor fields in terms of vertical and horizontal sections which will essentially determine the geometry that brings us a Riemannian submersion as well as the foliation determined by its vertical distribution. To describe the curvature relations of the involved spaces where we are working on we strongly study horizontal geodesics and holonomy fields. Finally, Section 4 is devoted to cite a few classical results appearing in this setting. The main references that we will be using to develop these notes are [3, 2]. We mainly adopt both notation and approach from [3].

2. Basic notions and examples

Let us start by defining the geometric object this work is concerned with. Let (M, g^M) and (B, g^B) be two Riemannian manifolds.

Definition 2.1. A surjective submersion $\pi : M \to B$ is said to be *Riemannian* if $d\pi_p$ is a linear isometry from $(\ker d\pi_p)^\perp$ onto $T_{\pi(p)}M$ for all $p \in M$. Here $(\ker d\pi_p)^\perp$ denotes the orthogonal complement of $\ker d\pi_p$ with respect to g^M.

For every $p \in M$ we define the *vertical* V and *horizontal* H distributions in TM respectively as

$$V_p = \ker d\pi_p \qquad \text{and} \qquad H_p = (\ker d\pi_p)^\perp, \qquad p \in M.$$

Note that they are such that $TM = V \oplus H$ for which each element $X \in TM$ may be written as $X = X^v + X^h$ where $X^v \in V$ and $X^h \in H$ are respectively called the *vertical* and *horizontal components* of X. It is worth noticing that as every surjective submersion is transverse to any submanifold in its base then $\pi \pitchfork \{\pi(p)\}$ which implies that $\pi^{-1}(\pi(p))$ is a submanifold in M of codimension $\dim(B)$ and

$$T_p\pi^{-1}(\pi(p)) = d\pi_p^{-1}(T_{\pi(p)}\{\pi(p)\}) = d\pi_p^{-1}(0) = \ker d\pi_p.$$

Therefore, our vertical distribution V is integrable since it is tangent to the regular foliation $\mathcal{F}_\pi = \{\pi^{-1}(x) : \ x \in B\}$. This is regular because π is a surjective submersion. Nevertheless, the horizontal distribution may be not integrable.

A vector field $X \in \mathfrak{X}(M)$ is said to be *vertical* (resp. *horizontal*) if it is section of the canonical vector bundle $V \to M$ (resp. $H \to M$). That is, $X_p \in V_p$ (resp. $X_p \in H_p$) for all $p \in M$. Let us denote by $\mathfrak{X}^v(M)$ and $\mathfrak{X}^h(M)$ the set of vertical and horizontal vector fields on M, respectively. Since V is integrable then as consequence of the Frobenius' theorem we have that the vertical distribution is involutive so that $\mathfrak{X}^v(M)$ is actually a Lie subalgebra of $\mathfrak{X}(M)$.

A vector field $X \in \mathfrak{X}(M)$ is called *projectable* if there exists a vector field $\overline{X} \in \mathfrak{X}(B)$ such that X and $\overline{X}$ are π-related. Moreover, X is defined to be *basic* if it is horizontal and projectable. It is important to notice that for all $\overline{Y} \in \mathfrak{X}(B)$ there exists a unique horizontal vector field Y^h such that Y^h and $\overline{Y}$ are π-related. This vector field Y^h is called *horizontal lift* of $\overline{Y}$ and it is defined by using the linear isomorphisms $d\pi_p|_{H_p} : H_p \to T_{\pi(p)}B$ as

$$Y^h_p = d\pi_p^{-1}(\overline{Y}_{\pi(p)}), \qquad p \in M. \tag{1}$$

$(d\pi_p|_{H_p})^{-1}$

It is simple to check that Y^h is indeed smooth by rewriting (1) in local coordinates and by using the fact that every surjective submersion locally looks like a linear projection; see for instance [3, p. 10]. Observe that the set of projectable vector fields is a subalgebra of $\mathfrak{X}(M)$ since if X_1 and X_2 are projectable over $\overline{X}_1$ and $\overline{X}_2$, respectively, then the identity $d\pi \circ [X_1, X_2] = [\overline{X}_1, \overline{X}_2] \circ \pi$ holds true. Furthermore, $\mathfrak{X}^v(M)$ is actually a Lie algebra ideal of this Lie algebra of projectable vector fields since any vertical vector field clearly is π-related with the zero vector field on B.

Let us denote by $\mathcal{B}$ the set of basic vector fields on M. Using the horizontal lift operation it is simple to see that $\mathcal{B}$ and $\mathfrak{X}(B)$ are isomorphic only as vector spaces since H may be not integrable and

$$[\mathcal{B}, \mathfrak{X}^v(M)] \subset \mathfrak{X}^v(M) \iff [X, U]^h = 0, \qquad X \in \mathcal{B},\ U \in \mathfrak{X}^v(M). \tag{2}$$

We also denote by $v : TM \to V$ and $h : TM \to H$ the canonical vertical and horizontal projections with respect to the splitting $TM = V \oplus H$, respectively. Clearly, these two maps induce canonical vertical and horizontal projections on section $v : \mathfrak{X}(M) \to \mathfrak{X}^v(M)$ and $h : \mathfrak{X}(M) \to \mathfrak{X}^h(M)$ which shall be equally denoted by v and h only if there is no risk of confusion. Finally, let $g^h = g^M \circ (h \times h)$ denote the horizontal component of the metric tensor g^M. We are now in conditions to give an answer to the question: given a surjective submersion with total space a Riemannian manifold, under what conditions is there a Riemannian metric on its base making of such submersion a Riemannian submersion?

Proposition 2.2. *Let $\pi : M \to B$ be a surjective submersion with connected fibers where (M, g^M) is a Riemannian manifold. Then there exists a Riemannian metric on B for which π becomes a Riemannian submersion if and only if the Lie derivative $\mathcal{L}_U g^h$ vanishes at any vertical direction U.*

Proof. Let us first suppose that $\mathcal{L}_U g^h = 0$ for all $U \in \mathfrak{X}^v(M)$. If $\overline{X}, \overline{Y} \in \mathfrak{X}(B)$ with respective horizontal lifts $X, Y \in \mathfrak{X}(M)$ then we set $g^B(\overline{X}, \overline{Y}) = g^M(X, Y) \circ \pi$. More precisely, for all $x \in B$ we define

$$g^B_x(\overline{X}_x, \overline{Y}_x) = g^M_p(X_p, Y_p), \quad \text{for some } p \in \pi^{-1}(x).$$

To see that this expression is well defined we have to prove that $g^M(X, Y)$ is constant along the fibers of π. Note that as consequence of Identity (2) we obtain

$$\begin{aligned} 0 &= (\mathcal{L}_U g^h)(X, Y) \\ &= U \cdot g^h(X, Y) - g^h(\mathcal{L}_U(X), Y) - g^h(X, \mathcal{L}_U(Y)) \\ &= U \cdot g^M(X, Y) - g^M([U, X]^h, Y) - g^M(X, [U, Y]^h) \\ &= U \cdot g^M(X, Y). \end{aligned}$$

since X and Y are horizontal vector fields. Therefore, $U \cdot g^M(X, Y) = 0$ holds true for every $U \in \mathfrak{X}^v(M)$ but the fibers of π are connected so that we get that $g^M(X, Y)$ is constant along the fibers. This implies that g^B is a well defined Riemannian metric on B and from the defining formula of it we easily see that π is a Riemannian submersion. Conversely, if $\pi : (M, g^M) \to (B, g^B)$ is a Riemannian submersion then for all $p, q \in M$ belonging to the same fiber of π and for all pair of horizontal vector fields $X, Y \in \mathfrak{X}(M)$ we have that

$$g^M_p(X_p, Y_p) = g^B_{\pi(p)}(d\pi_p(X_p), d\pi_p(X_p)) = g^B_{\pi(q)}(d\pi_q(X_q), d\pi_q(X_q)) = g^M_q(X_q, Y_q),$$

since $\pi(p) = \pi(q)$. On the other hand, from Equation (2) we get that for every $U \in \mathfrak{X}^v(M)$ it holds $(\mathcal{L}_U g^h)(X, Y) = U \cdot g^M(X, Y)$. But $g^M(X, Y)$ is constant along the fibers of π, thus obtaining that $\mathcal{L}_U g^h = 0$. □

Some interesting examples come in order.

Example 2.3 (Hopf fibration)**.** If we think of $S^3(1) \subset \mathbb{C}^2$ and $S^2(\frac{1}{2}) \subset \mathbb{C} \times \mathbb{R}$ then the smooth map $\pi : S^3(1) \to S^2(\frac{1}{2})$ defined by

$$\pi(z, w) = \left(\frac{1}{2}(\|w\|^2 - \|z\|^2), z\overline{w}\right),$$

provides us with an example of Riemannian submersion. Indeed, note that the fiber containing (z, w) consists of the points $(e^{it}z, e^{it}w)$ for which $i(z, w)$ is tangent to the fiber. Thus, the tangent vectors that are perpendicular to those vectors are of the form $\lambda(-\overline{z}, \overline{w})$ with $\lambda \in \mathbb{C}$. Let $\alpha(t) = (z(t), w(t))$ be a smooth curve in $S^3(1)$ such that $\alpha(0) = (z, w)$ and $\alpha'(0) = \lambda(-\overline{w}, \overline{z})$. Then

$$\begin{aligned} d\pi_{(z,w)}(\lambda(-\overline{z}, \overline{w})) &= \frac{d}{dt}|_{t=0}\pi(\alpha(t)) = \left(\mathrm{Re}(\langle w'(0), w(0)\rangle - \langle z'(0), z(0)\rangle), z'(0)\overline{w(0)} + z(0)\overline{w'(0)}\right) \\ &= (\mathrm{Re}(\langle \lambda\overline{z}, w\rangle + \langle \lambda\overline{w}, z\rangle), -\lambda\overline{w}^2 + \lambda z^2) = (2\mathrm{Re}(\overline{\lambda}zw), -\lambda\overline{w}^2 + \lambda z^2). \end{aligned}$$

It is simple to check that the previous expression and $\lambda(-\overline{w}, \overline{z})$ have the same length $|\lambda|$ as we required.

The Hopf fibration is a particular case of a more general example.

Example 2.4 (Equivariant setting)**.** Let (M, g) be a Riemannian manifold and assume that there exists a Lie group G acting freely, properly, and by isometries on (M, g). If G is compact the properness of the action is immediate. It is well known that as consequence of Godement's theorem there exists a unique structure of smooth manifold for the orbit space M/G making of the canonical projection $\pi : M \to M/G$ a surjective submersion. Besides, as the action is by isometries we have that M/G can be equipped with a Riemannian metric turning $\pi : M \to M/G$ into a Riemannian submersion. In particular, if H is a closed subgroup[1] of G then we know that G/H is a manifold. If we endow G with a metric such that right translation by elements in H act by isometries, then there is a unique Riemannian metric on G/H making of the projection $G \to G/H$ a Riemannian submersion. If in addition the metric is also left invariant then G acts by isometries on G/H (by the left) thus making of G/H a homogeneous space. For instance, if $M = S^{2n+1}$ is equipped with its canonical Riemannian metric which is induced from $\mathbb{C}^{n+1}$ and $G = S^1$ is acting on S^{2n+1} by complex multiplication then $S^{2n+1}/S^1 = \mathbb{C}P^n$ admits a Riemannian metric, called the *Fubini–Study* metric, such that $S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion. The case $n = 1$ is precisely the Hopf fibration.

Example 2.5 (Vertical warping)**.** Any Riemannian submersion can be used to generate new ones by deforming the metric in the vertical direction. More precisely, let $\pi : (M, g) \to B$ be a Riemannian submersion and let $\mu : M \to \mathbb{R}$ be a smooth function. We define a new metric g^μ on M by setting

$$g^\mu(X, Y) = e^{2\mu(p)}g(X^v, Y^v) + g(X^h, Y^h), \qquad X, Y \in T_pM,\ p \in M.$$

Note that the horizontal metric g^h is unchanged so that $\pi : (M, g^\mu) \to B$ is still a Riemannian submersion called *vertical warping* of π with respect to μ.

An exhaustive study of the geometry immersed in the previous example as well as more interesting examples can be found for instance in [3, c. 2].

[1] By the closed-subgroup Cartan's theorem we know that H is a Lie subgroup of G.

Lemma 3.1. Perhaps easier to compute, using Koszul's formula, that

$$\nabla^{M\,v}_{X} Y = \nabla^{B}_{\overline{X}} \overline{Y} + \frac{1}{2} [X,Y]^{v}.$$

3. The fundamental equations of a submersion

In this section we will see that there are two tensor fields measuring the complexity of a Riemannian submersion. They will allow us to find relations between the curvature invariants of the ambient spaces as well as to study the so called *horizontal geodesics* and *holonomy fields*, the latter of which are special cases of projectable Jacobi fields. If $\pi : (M, g^M) \to (B, g^B)$ is a Riemannian submersion then we denote by ∇^M and ∇^B the Levi–Civita connections on (M, g^M) and (B, g^B), respectively. Let us start by establishing a relation between ∇^M and ∇^B.

Lemma 3.1. *If X and Y are basics vector fields then so is $(\nabla^M_X Y)^h$.*

Proof. Let $\overline{X}$ and $\overline{Y}$ be vector fields on B that are π-related with X and Y, respectively. We claim that $(\nabla^M_X Y)^h$ and $\nabla^B_{\overline{X}}\overline{Y}$ are π-related. Firstly, note that $g^M(X,Y) = g^B(\overline{X},\overline{Y}) \circ \pi$ since π is a Riemannian submersion. Furthermore, it follows that $[X,Y]^h$ is a basic vector field with $d\pi \circ [X,Y]^h = [\overline{X},\overline{Y}] \circ \pi$. This is because

$$d\pi \circ [X,Y]^h = d\pi \circ ([X,Y]^v + [X,Y]^h) = d\pi \circ [X,Y] = [\overline{X},\overline{Y}] \circ \pi.$$

Secondly, observe that for another basic vector field Z that is π-related with $\overline{Z}$ we obtain

$$g^B(d\pi \circ (\nabla^M_X Y)^h, \overline{Z} \circ \pi) = g^B(d\pi \circ \nabla^M_X Y, d\pi \circ Z) = g^M(\nabla^M_X Y, Z).$$

Thus, by using the Koszul formula we get

$$\begin{aligned} 2g^B(d\pi \circ (\nabla^M_X Y)^h, \overline{Z} \circ \pi) &= 2g^M(\nabla^M_X Y, Z) \\ &= X \cdot g^M(Y,Z) + Y \cdot g^M(Z,X) - Z \cdot g^M(X,Y) \\ &+ g^M(Z,[X,Y]) + g^M(Y,[Z,X]) - g^M(X,[Y,Z]). \end{aligned}$$

However, we may rewrite

$$X \cdot g^M(Y,Z) = X(g^B(\overline{Y},\overline{Z}) \circ \pi) = d\pi \circ X \cdot g^B(\overline{Y},\overline{Z}) = \overline{X} \cdot g^B(\overline{Y},\overline{Z}) \circ \pi,$$

and

$$g^M(Z,[X,Y]) = g^B(d\pi \circ Z, d\pi \circ [X,Y]) = g^B(\overline{Z} \circ \pi, [\overline{X},\overline{Y}] \circ \pi) = g^B(\overline{Z},[\overline{X},\overline{Y}]) \circ \pi.$$

Therefore, after replacing these identities in the expression above we have that

$$\begin{aligned} 2g^B(d\pi \circ (\nabla^M_X Y)^h, \overline{Z} \circ \pi) &= \{\overline{X} \cdot g^B(\overline{Y},\overline{Z}) + \overline{Y} \cdot g^B(\overline{Z},\overline{X}) - \overline{Z} \cdot g^B(\overline{Y},\overline{X}) \\ &+ g^B(\overline{Z},[\overline{X},\overline{Y}]) + g^B(\overline{Y},[\overline{Z},\overline{X}]) - g^B(\overline{X},[\overline{Y},\overline{Z}])\} \circ \pi \\ &= 2g^B(\nabla^B_{\overline{X}}\overline{Y},\overline{Z}) \circ \pi = 2g^B(\nabla^B_{\overline{X}}\overline{Y} \circ \pi, \overline{Z} \circ \pi). \end{aligned}$$

Hence, as g^B is nondegenerate we obtain that $d\pi \circ (\nabla^M_X Y)^h = \nabla^B_{\overline{X}}\overline{Y} \circ \pi$ as desired. □

If $\nabla = \nabla^M$ denotes the Levi–Civita connection on (M, g^M) then:

Lemma 3.2. *The expression $A : H \times H \to V$ given by*

$$A_X Y = \nabla^v_X Y = \frac{1}{2}[X,Y]^v \qquad X, Y \in \mathfrak{X}^h(M),$$

defines a skew-symmetric tensor field on M.

No need!

Proof. Let $X, Y \in \mathfrak{X}^h(M)$ and $U \in \mathfrak{X}^v(M)$ be arbitrary. From the Koszul formula we get that

$$\begin{aligned} 2g^M(\nabla^v_X Y, U) + 2g^M(\nabla^h_X Y, U) &= 2g^M(\nabla_X Y, U) \\ &= X \cdot g^M(Y,U) + Y \cdot g^M(U,X) - U \cdot g^M(X,Y) \\ &+ g^M(U,[X,Y]) + g^M(Y,[U,X]) - g^M(X,[Y,U]). \end{aligned}$$

$$\langle \nabla^v_X Y, U\rangle = \langle \nabla_X Y, U\rangle = -\langle Y, \nabla_X U\rangle = -\langle Y, \nabla_U X\rangle \quad (\because [X,U]^h = 0)$$

$$= -U\underbrace{\langle X, Y\rangle}_{\text{const}} + \langle \nabla_U Y, X\rangle = \langle \nabla_Y U, X\rangle = -\langle U, \nabla_Y X\rangle$$

$$= -\langle U, \nabla_Y X\rangle$$

Now $\nabla^v_X Y - \nabla^v_Y X = [X,Y]^v \Rightarrow \nabla^v_X Y = \frac{1}{2}[X,Y]^v$

$A_X Y = \frac{1}{2}\nabla^v_X Y$ $C^\infty(M)$-linear on X ok. shows skew-symmetry

Skew-symmetry gives the same for Y

Thus, as consequence of Formula (2) and the fact that $g^M(X,Y)$ is constant along the fibers of π we conclude that

$$2g^M(\nabla^v_X Y, U) = g^M([X,Y],U) = g^M([X,Y]^v, U).$$

In particular, for all $Z \in \mathfrak{X}(M)$ we have that $2g^M(\nabla^v_X Y, Z) = g^M([X,Y]^v, Z)$ so that $\nabla^v_X Y = \frac{1}{2}[X,Y]^v$ since g^M is nondegenerate. Now, note that the map $[\cdot,\cdot]^v : \mathfrak{X}^h(M) \times \mathfrak{X}^h(M) \to \mathfrak{X}^v(M)$ is $C^\infty(M)$-bilinear

$$[fX,Y]^v = (f[X,Y] + Y(f)X)^v = f[X,Y]^v + Y(f)X^v = f[X,Y]^v,$$

for all $f \in C^\infty(M)$ because of having that X is horizontal. The formula $[X,fY]^v = f[X,Y]^v$ follows from the last one since $[\cdot,\cdot]^v$ is skew-symmetric. Analogously, by using the same trick with the Leibniz formula of ∇, it is simple to check that $\nabla^v : \mathfrak{X}^h(M) \times \mathfrak{X}^h(M) \to \mathfrak{X}^v(M)$ is $C^\infty(M)$-bilinear. So, the expression $A_X Y = \nabla^v_X Y = \frac{1}{2}[X,Y]^v$ actually gives us a skew-symmetric tensor field on M. □

We are now in conditions of setting up the following definition.

Definition 3.3. Let $\pi : M \to B$ be a Riemannian submersion. The *A-tensor* of π is defined to be

$$A_X Y = \nabla^v_X Y = \frac{1}{2}[X,Y]^v \qquad X, Y \in \mathfrak{X}^h(M).$$

The *S-tensor* of π is the tensor field $S : H \times V \to V$ given as

$$S_X U = -\nabla^v_U X \qquad X \in \mathfrak{X}^h(M),\ U \in \mathfrak{X}^v(M).$$

Firstly, note that the horizontal distribution H associated to π is integrable if and only if $A \equiv 0$ in which case the respective foliation is said to be *flat*. Secondly, by arguing with the Leibniz formula of ∇ as we did above it is easy to see that S is indeed a tensor field. Observe that S_X is of course just the second fundamental tensor of a fiber of π in direction X. In particular, $S \equiv 0$ if and only the fibers of π are totally geodesic[2]; see for instance [10, p. 104]. Thirdly, it is important to notice that the definition of A-tensor that we are taking under consideration here differs from that initially introduced in [8] where the author defines

$$A_E F = \nabla^v_{E^h} F^h + \nabla^h_{E^h} F^v.$$

Note that the two expressions only agree on horizontal vector fields. This reference also introduces a tensor field T instead of S defined by

$$T_E F = \nabla^h_{E^v} F^v + \nabla^v_{E^v} F^h.$$

We will study the tensor fields A and S from Definition 3.3 since they essentially determine the geometry that brings us a Riemannian submersion as well as the foliation determined by its vertical distribution. So, let us start by exhibiting a relation between the curvature tensors $R := R^M$ and R^B associated to $\nabla := \nabla^M$ and ∇^B, respectively. Given $A_X : H \to V$ we denote by $A^*_X : V \to H$ its adjoint map with respect to g^M, that is, the unique map verifying the formula

$$g^M(A^*_X U, Y) = g^M(U, A_X Y), \qquad X, Y \in H,\ U \in V.$$

Lemma 3.4. *If X is basic then the following identity holds true*

$$A^*_X U = -\nabla^h_U X = -\nabla^h_X U.$$

[2] A submanifold $\iota : N \hookrightarrow (M, g^M)$ is named to be *totally geodesic* if any geodesic in $(N, \iota^* g^M)$ is also a geodesic in (M, g^M).

Proof. Recall that Identity (2) implies that $[X, U]$ is vertical for all $U \in V$. Thus, as ∇ is torsion free we get that $\nabla^h_X U = \nabla^h_U X$. Therefore, if Y is horizontal

$$\begin{aligned} g^M(A^*_X U, Y) &= g^M(U, A_X Y) = g^M(U, \nabla^v_X Y) = g^M(U, \nabla^h_X Y + \nabla^v_X Y) \\ &= g^M(\nabla_X Y, U) = X \cdot g^M(Y, U) - g^M(\nabla_X U, Y) \\ &= -g^M(\nabla^h_X U + \nabla^v_X U, Y) = -g^M(\nabla^h_U X, Y). \end{aligned}$$

Hence, if $Y \in TM$ is arbitrary then we have obtained that

$$g^M(A^*_X U, Y) = g^M(A^*_X U, Y^h) = -g^M(\nabla^h_U X, Y^h) = -g^M(\nabla^h_U X, Y),$$

so that $A^*_X U = -\nabla^h_U X = -\nabla^h_X U$ since g^M is nondegenerate. □

With the previous formulas in mind we get:

Proposition 3.5. [1],[8] *For every $p \in M$ and $x, y, z \in H_p$ we have*

$$d\pi R(x, y)z = R^B(d\pi x, d\pi y)d\pi z + d\pi(2A^*_z A_x y - A^*_x A_y z - A^*_y A_z x).$$

Proof. If we extend x, y and z locally to basic vector fields X, Y and Z, respectively, and denote by $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ their π-related vector fields on B, then the fact that $V = \ker(d\pi)$ implies

$$\begin{aligned} d\pi \circ R(X, Y)Z &= d\pi \circ (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\ &= d\pi \circ (\nabla^h_X \nabla_Y Z - \nabla^h_Y \nabla_X Z - \nabla^h_{[X,Y]^h + [X,Y]^v} Z) \\ &= d\pi \circ \{\nabla^h_X \nabla^h_Y Z + \nabla^h_X \nabla^v_Y Z - \nabla^h_Y \nabla^h_X Z \\ &- \nabla^h_Y \nabla^v_X Z - \nabla^h_{[X,Y]^h} Z + \nabla^h_{[X,Y]^v} Z\}. \end{aligned}$$

On the one hand, as consequence of Lemma 3.1 we have that each term of the form $d\pi(\nabla^h_X \nabla^h_Y Z)$ agrees with $(\nabla^B_{\overline{X}} \nabla_{\overline{Y}} \overline{Z}) \circ \pi$. On the other hand, from both Lemma 3.2 and Lemma 3.4 we obtain

$$\nabla^h_X \nabla^v_Y Z = \nabla^h_X A_Y Z = -A^*_X A_Y Z,$$

and

$$\nabla^h_{[X,Y]^v} Z = -A^*_Z [X, Y]^v = -2A^*_Z A_X Y.$$

So, the result follows after replacing these formulas in the first expression we got above. □

3.1. **Geodesics and Jacobi fields.** Before going further we will introduce some interesting results addressing relations between geodesics as well as Jacobi fields when we are working with Riemannian submersions. It is well known that if $c : I \to B$ is a smooth curve then for any $t_0 \in I$ and any $p \in \pi^{-1}(c(t_0))$ there exist $\epsilon > 0$ and a horizontal lift[3] $\overline{c} : [t_0, t_0 + \epsilon) \to M$ of $c|_{[t_0, t_0+\epsilon)}$ with $\overline{c}(t_0) = p$. Moreover, any two of such lifts agree in their common domain. Namely, let us assume that c is regular[4] and let us choose a vector field X on B such that $X \circ c = \dot{c}$ on some neighborhood of t_0. Consider the horizontal lift $\overline{X}$ on M of X. As $\overline{X}$ take values in H we have that the integral curves $\overline{c}$ of it are horizontal and moreover they project down over integral curves of X since

$$(\pi \dot{\circ} \overline{c})(t) = d\pi_{\overline{c}(t)}(\dot{\overline{c}}(t)) = d\pi_{\overline{c}(t)}(\overline{X}(\overline{c}(t))) = X(\pi(\overline{c}(t))).$$

[3] By horizontal lift we mean a smooth curve $\overline{c}$ on M such that $\pi \circ \overline{c} = c$ and $\dot{\overline{c}}(t) \in H_{\overline{c}(t)}$ for all t where it is defined.

[4] We may assume it since if it is not so we consider the graph curve $c_1 : I \to I \times B$ which is defined by $c_1(t) = (t, c(t))$. It is clear that c_1 is always regular and a $TI \oplus H$-horizontal lift of c_1 for the Riemannian submersion $(\mathrm{id}_I, \pi) : I \times M \to I \times B$ has the form $\overline{c_1}(t) = (t, \overline{c}(t))$ where $\overline{c}(t)$ is the desired lift of c.

Also follows from local form of a submersion.

Therefore, as the restriction of c and $\pi \circ \bar{c}$ are integral curves of X with $c(t_0) = \pi(p) = \pi(\bar{c}(t_0))$ they locally agree by uniqueness.

So, we set up our next result which tells us that if M is complete then any Riemannian submersion is a locally trivial fiber bundle.

Theorem 3.6. [9] *Let $\pi : M \to B$ be a Riemannian submersion. If $c : I \to M$ is a geodesic with $c(t_0) \in H$ for some $t_0 \in I$, then $c(t) \in H$ for all $t \in I$ and $\pi \circ c$ is a geodesic in B. Furthermore, if M is complete, then*

1. *B is complete,*
2. *π is a submetry meaning that π maps the closure of the metric ball $B_r(p)$ onto the closure of $B_r(\pi(p))$ for any $p \in M$,*
3. *fibers of π are equidistant, that is, for any two fibers F_0 and F_1 with $p \in F_0$, the distance between p and F_1 equals that between F_0 and F_1,*
4. *π is a locally trivial fiber bundle meaning that any point b in B has a neighborhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$ where $F = \pi^{-1}(b)$.*

Proof. Let us suppose that $c : I \to M$ is a geodesic and $t_0 \in I$ is such that $\dot{c}(t_0) \in H$. On the one hand, we know that there exists a geodesic c_B in B defined on some interval $J \subseteq I$ containing t_0 such that $c_B(t_0) = \pi(c(t_0))$ and $\dot{c}_B(t_0) = d\pi_{c(t_0)}(\dot{c}(t_0))$. On the other hand, there also exists a horizontal lift c_M in M of c_B with $c_M(t_0) = c(t_0)$ defined on some subinterval $J' \subseteq J$. If $\alpha : [a, b] \to M$ is another curve on M with the same endpoints as c_M and we decompose $\dot{\alpha} = \dot{\alpha}^v + \dot{\alpha}^h$ then $g^M(\dot{\alpha}, \dot{\alpha}) \geq g^M(\dot{\alpha}^h, \dot{\alpha}^h)$ so that

$$L(\alpha) = \int_a^b \|\dot{\alpha}\| \geq \int_a^b \|\dot{\alpha}^h\| = \int_a^b \|d\pi\dot{\alpha}^h\| = \int_a^b \|d\pi\dot{\alpha}\| = L(\pi \circ \alpha) \geq L(c_B),$$

locally

where last inequality holds true because c_B is a geodesics and thus it is length-minimizing. Note that the fact that c_M is a horizontal lift of c_B plus the fact that π is a Riemannian submersion imply that $L(c_B) = L(c_M)$. Thus, $L(\alpha) \geq L(c_M)$ meaning that c_M is length-minimizing. As consequence, we get that c_M is a geodesic and by uniqueness of geodesics $c_M = c|_{J'}$. This immediately implies that c is horizontal and that $\pi \circ c = \pi \circ c_M = c_B$ is a geodesic.

Let us assume now that M is complete. Firstly, it is clear that items (1) and (2) follows directly from what we just proved. Secondly, if $b_0 = \pi(F_0)$ and $b_1 = \pi(F_1)$ then there exists a minimizing geodesic c joining b_0 and b_1 since B is complete. We may assume that such geodesic is parametrized by arc length. Note that π is distance-decreasing since it is a Riemannian submersion. Namely, this follows from the inequality

$$g^M(\dot{\alpha}, \dot{\alpha}) \geq g^M(\dot{\alpha}^h, \dot{\alpha}^h) = g^B(d\pi\dot{\alpha}^h, d\pi\dot{\alpha}^h).$$

Therefore, the distance between any point of F_0 and any point of F_1 is at least as large as the length of c and then $d(F_0, F_1) \geq L(c)$. Furthermore, for any p in F_0 we have that the horizontal lift of c starting at p is a curve that ends at some point of F_1 and has the same length as c so that $d(F_0, F_1) \leq d(p, F_1) \leq L(c)$. In consequence,

$$d(F_0, F_1) = d(p, F_1).$$

Finally, let us take $b \in B$ and denote $F = \pi^{-1}(b)$ the fiber of π at b. Consider a normal neighborhood U_b of b in B such that $\exp_b^B|_{B_b(0,\epsilon)} : B_b(0, \epsilon) \subseteq T_bB \to U_b \subseteq B$ is a diffeomorphism for some $\epsilon > 0$. For each $x \in B_b(0, \epsilon)$ we consider its unique horizontal lift X and set $h : F \times B_b(0, \epsilon) \to \pi^{-1}(U_b)$ as $h(p, x) = \exp_p^M(X)$. As consequence of (1) this map is well defined and moreover differentiable. Given $q \in \pi^{-1}(U_b)$ we consider the unique minimizing geodesic

$c:[0,a]\to B$ from $\pi(q)$ to b. If c_M denotes the horizontal lift of c starting at q, then $c_M(a)$ is well defined by completeness of M, and $\pi(c_M(a))=c(a)=b$ so that $p:=c_M(a)\in F$. Note that $c(a-t)$ is a geodesic from b to $\pi(q)$ verifying $x=\dot c(a-t)|_{t=0}=-\dot c(a)$ thus obtaining $X=\dot c_M(a-t)|_{t=0}=-\dot c_M(a)$. Then $h(p,x)=\exp_p^M(-\dot c_M(a))=c_M(a-t)|_{t=a}=c_M(0)=q$ concluding that h is surjective. Furthermore, h is injective because of the uniqueness of horizontal lifts. As h is defined in terms of the exponential map it follows that h has maximal rank so that the composition of $\mathrm{id}_F\times\exp_b^B : F\times B_b(0,\epsilon)\to F\times U_b$ and $h^{-1}:\pi^{-1}(U_b)\to F\times B_b(0,\epsilon)$ is a diffeomorphism and we are done. □

A geodesic curve verifying first part of Theorem 3.6 is called *horizontal geodesic*. From now on we assume that M is complete so that every Riemannian submersion $\pi:M\to B$ is a locally trivial fiber bundle. Last property allows us to speak about *holonomy*. Namely, if $c:[0,1]\to B$ is a piecewise smooth curve then we define the *holonomy diffeomorphism* associated to c as the map $h_c:\pi^{-1}(c(0))\to\pi^{-1}(c(1))$ sending a point p in the first fiber to the endpoint of the horizontal lift of c that starts at p. Note that the inverse of h_c is h_{-c} where $-c(t)=c(1-t)$.

Definition 3.7. The *holonomy group* $\mathrm{Hol}(b)$ of a Riemannian submersion $\pi:M\to B$ at $b\in B$ is the group of holonomy diffeomorphisms $h_c:\pi^{-1}(b)\to\pi^{-1}(b)$ of the fiber over b where c is a piecewise smooth closed curve at b.

It is worth noticing that if b_0 and b_1 are points in B and c is a curve joining b_0 to b_1, then the map $\mathrm{Hol}(b_0)\to\mathrm{Hol}(b_1)$ defined as $h_\alpha\mapsto h_c\circ h_\alpha\circ h_{-c}$ is an isomorphism of holonomy groups.

Lemma 3.8. *Let $\pi:M\to B$ denote a Riemannian submersion and $h:F_0\to F_1$ the holonomy diffeomorphism induced by the geodesic $c:[0,1]\to B$ where $c(0)=\pi(F_0)$ and $c(1)=\pi(F_1)$. Take $p\in F_0$ and let c_p denote the horizontal lift of c starting at p. Then for $u\in T_pF_0$ we have*

$$dh(u)=J(1)$$

where J is the Jacobi field along c_p with $J(0)=u$ and $J'(0)=-A^*_{\dot c_p(0)}u-S_{\dot c_p(0)}u$.

Proof. Set $x:=\dot c(0)$ and denote by $X=\dot c_p$. Let $\gamma:I\to F_0$ be a curve defined on some neighborhood I of 0 such that $\dot\gamma(0)=u$ and consider the variation along the geodesic c_p

$$H:[0,1]\times I\to M,\qquad H(s,t)=\exp^M(s(X\circ\gamma)(t)).$$

Note that $(h\circ\gamma)(t)=\exp^M((X\circ\gamma)(t))=H(1,t)$ so that

$$dh_p(u)=\frac{d}{dt}|_{t=0}(h\circ\gamma)(t)=\frac{d}{dt}|_{t=0}H(1,t)=dH_{(1,0)}\left(\frac{\partial}{\partial t}\right)=dH\left(\frac{\partial}{\partial t}\right)(1,0)=\frac{\overline{\partial}}{\partial t}(1,0).$$

Since the variation H is by geodesics we have that its associated variational vector field along c_p

$$J(s)=dH\left(\frac{\partial}{\partial t}\right)(s,0)=\frac{\overline{\partial}}{\partial t}|_{t=0},$$

is a Jacobi field along c_p verifying $dh_p(u)=J(1)$. Furthermore,

$$\frac{\nabla}{ds}J(0)=J'(0)=\overline{\nabla}_{\frac{\partial}{\partial s}}\frac{\overline{\partial}}{\partial t}|_{t=0}=\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\overline{\partial}}{\partial s}|_{t=0}=\nabla_u\dot c_p=\nabla_uX,$$

and from Definition 3.3 and Lemma 3.4 we get $\nabla_uX=\nabla_u^hX+\nabla_u^vX=-A^*_{\dot c_p(0)}u-S_{\dot c_p(0)}u$ as required. □

Previous result motivates the following definition.

Definition 3.9. A Jacobi field J along a horizontal geodesic $c : [0, a] \to M$ that is vertical at 0 and satisfies $J'(0) = -A^*_{\dot{c}(0)}J(0) - S_{\dot{c}(0)}J(0)$ is called a *holonomy field*.

It is important to notice that a holonomy field is always vertical and is identically zero if it vanishes at only one point. This is because the holonomy transformations defining it, according with Lemma 3.8, are diffeomorphisms[5]. Furthermore, observe that as consequence of this for $t_0 \in (0, a)$, the restriction $J|_{[t_0,a]}$ is again a holonomy field along $c|_{[t_0,a]}$ so that

$$J'(t) = -(A^*_{\dot{c}(t)} + S_{\dot{c}(t)})J(t), \qquad t \in [0, a]. \tag{3}$$

Recall that a submanifold $\iota : N \hookrightarrow (M, g^M)$ is named to be *totally geodesic* if any geodesic in (N, ι^*g^M) is also a geodesic in (M, g^M). So,

Lemma 3.10. *If $\pi : M \to B$ is a Riemannian submersion with totally geodesic fibers and M is complete then the holonomy diffeomorphisms between fibers are isometries.*

Proof. Consider a holonomy field J along a geodesic $c : [0, 1] \to M$ with $J(0) = u$ as above. Since J is vertical and the fibers are totally geodesic (i.e. $S \equiv 0$) then Formula (3) implies that

$$\frac{d}{dt}g^M(J, J) = 2g^M(J, J') = 2g^M(J, (J')^v) = -2g^M(J, S_{\dot{c}}J) = 0,$$

thus obtaining that $\|J\|$ is constant. If h is the associated holonomy diffeomorphism, then the previous fact implies that

$$\|dh(u)\| = \|J(1)\| = \|J(0)\| = \|u\|.$$

□

Theorem 3.11. *Let $\pi : M \to B$ be a Riemannian submersion with M complete and such that the fibers are totally geodesic. If the A-tensor is identically zero then π splits. More precisely, each point $b \in B$ has a neighborhood U such that $\pi^{-1}(U)$ is isometric to a metric product $U \times F$ where $F := \pi^{-1}(b)$. Moreover, if $\Phi : U \times F \to \pi^{-1}(U)$ denotes such a isometry, then $\pi \circ \Phi : U \times F \to U$ is projection onto the first factor.*

Proof. Let U be a normal neighborhood around b. Since U is the diffeomorphic image under $\exp^B_b$ of some ball around 0 in T_bB them we directly assume that U is simply connected. Let us consider the local trivialization $(\pi, \phi) : \pi^{-1}(U) \to U \times F$ from Theorem 3.6 which is defined as $q \mapsto (\pi(q), c_q(1))$ where $c_q : [0, 1] \to F$ is the shortest geodesic from q to F. It is worth noticing that if $\hat{b} \in U$ then from what we proved above we know that $\phi|_{\pi^{-1}(\hat{b})}$ is a holonomy diffeomorphism so that, as consequence of Lemma 3.10, it is an isometry. As the A-tensor is identically zero we have that the horizontal distribution H is integrable. Therefore, if $p \in \pi^{-1}(U)$, then the restriction of π to the connected component $V \subset \pi^{-1}(U)$ of the leaf of the associated foliation on M induced by H that contains p is a covering map and hence a diffeomorphism since U is simply connected and π is a surjective submersion. This implies that V intersects F in exactly one point, namely, $\phi(p)$. Fix $x \in H_p$ and take a horizontal geodesic γ such that $\dot{\gamma}(0) = x$. As γ is contained in V then $\phi \circ \gamma = \phi(p)$. So,

$$d\phi_p(x) = \frac{d}{dt}|_{t=0}(\phi \circ \gamma)(t) = \frac{d}{dt}|_{t=0}\phi(p) = 0.$$

Let $z \in T_pM = H_p \oplus V_p$ be arbitrary. On the one hand, as π is a Riemannian submersion we get that that $\|d\pi_p(z)\| = \|d\pi_p(z^h)\| = \|z^h\|$. On the other hand, as $\phi|_{\pi^{-1}(\hat{b})}$ is an isometry then

[5]Uniqueness of a Jacobi field given certain initial conditions

the above identity implies that $\|d\phi_p(z)\| = \|d\phi_p(z^v)\| = \|z^v\|$. Therefore, $\|d(\pi,\phi)_p(z)\| = \|z\|$ meaning that our trivialization (π,ϕ) is an isometry. $\square$

An interesting fact derived from the previous result is the following:

Corollary 3.12. *Assume that $\pi : M \to B$ is a Riemannian submersion with M complete and of nonpositive sectional curvature. Then, M splits locally as a metric product. In particular, a negatively curved manifold admits no Riemannian submersions with totally geodesic fibers.*

Proof. If J is a Jacobi field along a geodesic c in M, then

$$g^M(J,J)'' = 2(g^M(J'',J) + g^M(J',J')) = 2(\|J'\|^2 - g^M(R(J,\dot c)\dot c, J)) \geq 0.$$

If the fibers of π are totally geodesic, then by Lemma 3.10 the holonomy Jacobi fields have constant norm so that the preceding inequality implies that they are parallel. Thus, from Equation (3) we get that that $A^*_{\dot c}J = 0$ for a holonomy field along a horizontal geodesic c. Since $\dot c$ and J are arbitrary taken we get that $A = 0$ and therefore M splits locally as a metric product. $\square$

Remark 3.13. A similar result holds true if one removes the sectional curvature condition on M and assumes instead that the fibers are compact and have negative Ricci curvature. A theorem of Bochner asserts that such a fibers cannot admit nontrivial Killing fields and the previous assertion follows by proving that A_XY is always a Killing vector field. Indeed, as consequence of Identity (2), if U is a vertical field then we have that the Lie derivative $\mathcal{L}_XU = (\mathcal{L}_XU)^v = \nabla^v_XU$ since $\nabla^v_UX = -S_XU = 0$ because of having totally geodesic fibers. Let us further assume that U is unitary. So, we have to prove that $g^M(\nabla_UA_XY, U) = 0$ or, equivalently, since the A-tensor is skew-symmetric and ∇ is torsion free, we may check that

$$g^M(\mathcal{L}_{[X,Y]^v}U,U) = 2(g^M(\nabla_{A_XY}U,U) - g^M(\nabla_UA_XY,U)) = -2g^M(\nabla_UA_XY,U) = 0.$$

Form the first comment regarding the Lie derivative we obtain that $g^M(\mathcal{L}_{[X,Y]^h}U,U) = 0$ so that

$$\begin{aligned}
g^M(\mathcal{L}_{[X,Y]^v}U,U) &= g^M(\mathcal{L}_{[X,Y]}U,U) = g^M(\mathcal{L}_X\mathcal{L}_YU,U) - g^M(\mathcal{L}_Y\mathcal{L}_XU,U)\\
&= g^M(\nabla^v_X\nabla^v_YU - \nabla^v_Y\nabla^v_XU, U)\\
&= g^M(R(X,Y)U,U) - g^M(\nabla^v_X\nabla^h_YU - \nabla^v_Y\nabla^h_XU,U).
\end{aligned}$$

Last equality is obtained by similar computations as those we did before. But $g^M(R(X,Y)U,U) = g^M(R(U,U)X,Y) = 0$ and therefore

$$\begin{aligned}
g^M(\mathcal{L}_{[X,Y]^v}U,U) &= -g^M(\nabla^v_X\nabla^h_YU - \nabla^v_Y\nabla^h_XU,U)\\
&= g^M((P-P^*)(U),U) = g^M(P(U),U) - g^M(U,P(U)) = 0,
\end{aligned}$$

where $-P(U) = \nabla^v_X\nabla^h_YU = A_X\nabla^h_YU = -A_XA^*_Y(U)$, that is, $P = A_XA^*_Y$ with $P^* = A_YA^*_X$. In conclusion, $g^M(\nabla_UA_XY,U) = 0$ thus obtaining that A_XY is a Killing vector field.

Let us return to our initial purpose. In order to investigate the curvature relations for one or more vertical vector fields we extend the tensor fields A and S on all TM by setting

$$A_EF = A_{E^h}F^h \quad \text{and} \quad S_EF = S_{E^h}F^v, \qquad E,F \in \mathfrak{X}(M).$$

The covariant derivative $\frac{\nabla}{dt}B$ of a tensor field B along a curve c will be just denoted by B'. Thus, for example, in the particular case that B is a tensor field of type $(1,1)$ and E is a vector field, both of them along c, we shall use below the Leibniz rule $B'E = (BE)' - BE'$ without making any special mention. To relax notation we denote by $g := g^M$. Having in mind Formula (3), for a holonomy Jacobi field J and for a vertical vector field T, both of them along a horizontal geodesic c, we get

$$\begin{aligned} g(R(T,\dot c)\dot c, J) &= g(R(J,\dot c)\dot c, T) = -g(T, J'') = g(T,(A^*_{\dot c}J)') + g(T,(S_{\dot c}J)') \\ &= g(T, A^*_{\dot c}J)' - g(T', A^*_{\dot c}J) + g(T, S_{\dot c}J)' - g(T', S_{\dot c}J) \\ &= -g(\nabla_{\dot c}T, A^*_{\dot c}J) + g(S_{\dot c}T, J)' - g(S_{\dot c}T', J) \\ &= g(-\nabla^v_{\dot c}T, A^*_{\dot c}J) + g((S_{\dot c}T)', J) + g(S_{\dot c}T, J') - g(S_{\dot c}T', J) \\ &= g(A^*_{\dot c}T, A^*_{\dot c}J) + g(S'_{\dot c}T, J) - g(S_{\dot c}T, A^*_{\dot c}J + S_{\dot c}J) \\ &= g((A_{\dot c}A^*_{\dot c} + S'_{\dot c} - S^2_{\dot c})T, J). \end{aligned}$$

Given any t_0 we assume that the holonomy Jacobi fields may be chosen so that they form an orthonormal basis of the vertical space at $c(t_0)$ which immediately implies that the vertical component of $R(T,\dot c)\dot c$ is given by

$$R^v(T,\dot c)\dot c = (A_{\dot c}A^*_{\dot c} + \nabla^v_{\dot c}S_{\dot c} - S^2_{\dot c})T.$$

Equivalently, as the geodesic we are working with is horizontal, we obtain the formula

$$R^v(u,x)x = (A_xA^*_x + (\nabla^v_x S)_x - S^2_x)u, \qquad x \in H,\ u \in V.$$

With similar computations as we did above, if X is a horizontal vector field along c then

$$g(R(X,\dot c)\dot c, J) = g(X,(A^*_{\dot c}J)') + g(X,(S_{\dot c}J)').$$

On the one hand,

$$\begin{aligned} g(X,(S_{\dot c}J)') &= g(X, S_{\dot c}J)' - g(X', S_{\dot c}J) = -g(X', S_{\dot c}J) = -g(\nabla^v_{\dot c}X, S_{\dot c}J) \\ &= -g(A_{\dot c}X, S_{\dot c}J) = -g(S_{\dot c}A_{\dot c}X, J). \end{aligned}$$

On the other hand, from Identity (3)

$$\begin{aligned} g(X,(A^*_{\dot c}J)') &= g(X, A^*_{\dot c}J)' - g(X', A^*_{\dot c}J) = g(A_{\dot c}X, J)' - g(A_{\dot c}X', J) \\ &= g((A_{\dot c}X)', J) + g(A_{\dot c}X, J') - g(A_{\dot c}X', J) \\ &= g(A'_{\dot c}X, J) - g(A_{\dot c}X, A^*_{\dot c}J + S_{\dot c}J) \\ &= g((\nabla^v_{\dot c}A_{\dot c} - S_{\dot c}A_{\dot c})X, J). \end{aligned}$$

Therefore, by arguing as above we obtain $R^v(X,\dot c)\dot c = (\nabla^v_{\dot c}A_{\dot c} - 2S_{\dot c}A_{\dot c})X$ or equivalently

$$R^v(y,x)x = ((\nabla^v_x A)_x - 2S_xA_x)y, \qquad x, y \in H.$$

It is simple to check that $\nabla^v_x A$ is skew-symmetric meaning that $(\nabla^v_x A)_y z = (\nabla^v_x A)_z y$. Thus, by applying polarization identities together with the first Bianchi identity, from the previous expression we obtain

$$\begin{aligned} 3R^v(x,y)z &= R^v(x,y+z)(y+z) - R^v(y,x+z)(x+z) - R^v(x,y)y - R^v(x,z)z \\ &+ R^v(y,x)x + R^v(y,z)z \\ &= (\nabla^v_y A)_z x - (\nabla^v_x A)_z y + (\nabla^v_z A)_y x - (\nabla^v_z A)_x y - 2S_yA_zx + 2S_xA_zy + 4S_zA_xy. \end{aligned}$$

As consequence of the Jacobi identity for the Lie bracket of vector fields, by looking at $0 = \frac{1}{2} \circlearrowleft [X,[Y,Z]]^v = 0$ after extending $x, y, z \in H_p$ to basic fields X, Y, Z with vanishing horizontal Lie bracket at p, it is simple to derive the formula $\circlearrowleft (\nabla^v_x A)_y z + \circlearrowleft S_xA_yz = 0$ so that

$$\begin{aligned} 3R^v(x,y)z &= \circlearrowleft (\nabla^v_x A)_y z - (\nabla^v_z A)_x y + 2(\nabla^v_z A)_y x - 2S_yA_zx + 2S_xA_zy + 4S_zA_xy \\ &= -3(\nabla^v_z A)_x y + 3S_zA_xy - 3S_yA_zx - 3S_xA_yz. \end{aligned}$$

Therefore

$$R^v(x,y)z = -(\nabla^v_z A)_x y + S_z A_x y - S_y A_z x - S_x A_y z, \qquad x,y,z \in H.$$

Let us now compute $R^v(x,u)y$ for $x,y \in H$ and $u \in V$. As usual, we extend these vectors to basics fields X, Y and vertical field U, respectively. Then, by using Formula (2) and covariant derivative of tensor fields we get

$$\begin{aligned}
R^v(X,U)Y &= \nabla^v_X \nabla_U Y - \nabla^v_U \nabla_X Y - \nabla^v_{[X,U]} Y \\
&= \nabla^v_X \nabla^v_U Y + \nabla^v_X \nabla^h_U Y - \nabla^v_U \nabla^v_X Y - \nabla^v_U \nabla^h_X Y - \nabla^v_{[X,U]^v} Y \\
&= -\nabla^v_X S_Y U - \nabla^v_X A^*_Y U - \nabla^v_U A_X Y + S_{\nabla^h_X Y} U + S_Y [X,U]^v \\
&= -\nabla^v_X S_Y U - A_X A^*_Y U - \nabla^v_U A_X Y + S_{\nabla^h_X Y} U + S_Y \nabla^v_X U - S_Y \nabla^v_U X \\
&= -A_X A^*_Y U - (\nabla^v_X S)_Y U + S_Y S_X U - \nabla^v_U A_X Y.
\end{aligned}$$

Observe that

$$-(\nabla^v_U A)_X Y = -\nabla^v_U A_X Y + A_{\nabla^v_U X} Y + A_X \nabla^v_U Y = -\nabla^v_U A_X Y + A_Y A^*_X U - A_X A^*_Y U,$$

since A is skew-symmetric. Hence,

$$R^v(X,U)Y = -(\nabla^v_U A)_X Y - A_Y A^*_X U - (\nabla^v_X S)_Y U + S_Y S_X U.$$

Using this expression together the first Bianchi identity $R^v(x,y)u = R^v(x,u)y - R^v(y,u)x$ and the fact that $\nabla^v_u A$ is skew-symmetric we obtain

$$R^v(x,y)u = -2(\nabla^v_u A)_x y - (\nabla^v_x S)_y u + (\nabla^v_y S)_x u + [S_x, S_y]u + (A_x A^*_y - A_y A^*_x)u,$$

for all $x, y \in H$ and $u \in V$, where $[S_x, S_y]$ denotes the usual commutator. Recall that by mimicking the de Rham differential, the connection ∇ allows us to define an *exterior covariant differential* d^∇ which provides us with a cochain complex if and only if ∇ is flat; see for instance [3]. So,

$$(d^\nabla S)_X Y = \nabla_X(S_Y) - \nabla_Y(S_X) - S_{[X,Y]^h} = (\nabla_X S)_Y + (\nabla_Y S)_X.$$

Summing up, with those computations above we have obtained a complete description of the curvature tensor in terms of the tensor fields A and S.

Theorem 3.14. [1],[8] *Let $\pi : M \to B$ be a Riemannian submersion with $\dim B \geq 2$ and R, R^B, and R^F denoting the curvature tensors of M, B, and a fiber F, respectively. Let $p \in M$, $x,y,z \in H_p$, and $u,v,w \in V_p$. Denote by σ the second fundamental tensor of the fiber $\pi^{-1}(\pi(p))$ at p which is defined as $\sigma(U,V) = \nabla^h_U V$. Then*

$$\begin{aligned}
d\pi R(x,y)z &= R^B(d\pi x, d\pi y) d\pi z + d\pi(2A^*_z A_x y - A^*_x A_y z - A^*_y A_z x) \\
R^v(x,y)z &= -(\nabla^v_z A)_x y + S_z A_x y - S_y A_z x - S_x A_y z \\
R^v(x,u)y &= -(\nabla^v_u A)_x y - A_y A^*_x u - (\nabla^v_x S)_y u + S_y S_x u \\
R^v(x,y)u &= -2(\nabla^v_u A)_x y - ((d^\nabla S)_x y)(u) + [S_x, S_y]u + (A_x A^*_y - A_y A^*_x)u \\
R^F(u,v)w &= R^v(u,v)w + S_{\sigma(v,w)} u - S_{\sigma(u,w)} v \quad \leftarrow \text{Gauss eqn.} \\
R^v(u,w)x &= (\nabla^v_w S)_x u - (\nabla^v_u S)_x w. \quad \leftarrow \text{Codazzi eqn.}
\end{aligned}$$

Last two identities are the so-called *Codazzi equations* for the fibers of π. Using the fact that $\nabla^v_u A$ is skew-symmetric we immediately get the following expressions for the sectional curvatures:

Corollary 3.15. [1],[8] *With the notation of the previous theorem we have*

$$\begin{aligned} K(d\pi x, d\pi y) &= K(x,y) + 3\|A_x y\|^2 \\ K^F(u,v) &= K(u,v) + \sigma(u,u)\sigma(v,v) - \sigma^2(u,v) \\ K(x,u) &= g((\nabla^v_x S)_x u, u) + \|A^*_x u\|^2 - \|S_x u\|^2. \end{aligned}$$

The second equation in the previous corollary is nothing but the Gauss equation.

4. Some miscellany results

In search of completeness, in this short section we cite some other interesting results concerning Riemannian submersions. For more details about the assertions we shall present below the reader is recommended to visit the O'Neill's seminal papers [8, 9].

- [8] Let π and $\overline{\pi}$ be two Riemannian submersions of a connected Riemannian manifold M onto B. If π and $\overline{\pi}$ have the same tensor fields A and S and their derivative maps agree at one point of M, then $\pi = \overline{\pi}$.

Note that this result generalizes the well known case in which is required that both π and $\overline{\pi}$ are local isometries coinciding at least at order 1 at one point of M.

A related result to Theorem 3.11 is the following. The simplest type of submersion is the projection of a Riemannian product manifold on one of its factors. Motivated by what we proved in the previous section we say that a submersion $\pi : M \to B$ is *trivial* if it differs for such a projection only by an isometry of M. Equivalently, $\pi : M \to B$ is trivial if there exists a Riemannian manifold F and a submersion $\phi : M \to F$ *dual* to π in the sense that the horizontal distribution of ϕ is the vertical distribution of π (hence vice-versa). So, we have:

- [8] Let $\pi : M \to B$ be a Riemannian submersion of a complete Riemannian manifold M. Then π is trivial if and only the tensor A and all holonomy groups of π vanish.

Let us now exhibit a comparison of the index forms. Let $\pi : M \to B$ be a Riemannian submersion and let c be a horizontal geodesic in M. If E is a vector field along c then the *derived vector filed* of E is defined to be $D(E) = E'^v + S_{\dot c} E^v + 2A_{\dot c} E^h$; compare [9] with [3, p. 32]. As it was shown in [9], there is a relation of the Jacobi fields along c on M to those along $\pi \circ c$ on B. Namely, a Jacobi field E along c projects to a Jacobi field along $\pi \circ c$ if and only if $A^*_{\dot c}(D(E)) = 0$. Besides:

- [9] Let c be a horizontal geodesic in M. Given a vector field E_* along $\pi \circ c$ and a vertical vector u at $c(0)$ there exists a unique vector field E along c such that: $d\pi(E) = E_*$, $D(E) = 0$, and $E(0) = u$. Furthermore, E is a Jacobi field if and only if E_* is so.

Note that a piecewise differentiable vector field E along a horizontal geodesic $c : [a,b] \to M$ will have projection E_* along $\pi \circ c$ if and only if E is orthogonal to c and vertical at a and b. The set $\mathcal{E}^{v,v}$ of such vector fields constitute the linear space appropriate to the study of c as a geodesic segment joining the fibers F_a and F_b through its end points. Accordingly:

- [9] Let $\pi : M \to B$ be a Riemannian submersion and c a horizontal geodesic in M. If I denotes the index form of M restricted to $\mathcal{E}^{v,v}$ and I_B is the index form of B then

$$I(E,F) = I_B(E_*, F_*) + \int_a^b g(D(E), D(F))\, dt.$$

The numerical invariants concerning both conjugacy and index which may be derived from the previous results are explained in detail in Theorem 4 from [9].

Lots of interesting results that appear when studying Riemannian submersions where the involved manifolds carry an additional structure as for instance of contact, Hermitian, quaternionic type, among others, can be found in [2].

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