

Chapter 2

Surfaces: local theory

2.1 The first fundamental form

Let $S \subset \mathbf{R}^3$ be a surface. The *first fundamental form* of S is just the restriction I of the dot product of \mathbf{R}^3 to the tangent spaces of S :

$$I_p = \langle \cdot, \cdot \rangle|_{T_p S \times T_p S}.$$

If $\varphi : U \subset \mathbf{R}^2 \rightarrow \varphi(U) = V \subset S$ is a parametrization, we already know that $\{\varphi_u(u, v), \varphi_v(u, v)\}$ is a basis of $T_{\varphi(u, v)}S$. Then any tangent vector to S at a point in V can be written $w = a\varphi_u + b\varphi_v$ and thus

$$I(w, w) = a^2 \underbrace{I(\varphi_u, \varphi_u)}_{E(u, v)} + 2ab \underbrace{I(\varphi_u, \varphi_v)}_{F(u, v)} + b^2 \underbrace{I(\varphi_v, \varphi_v)}_{G(u, v)}.$$

E, F, G are smooth functions on U , the so called coefficients of the first fundamental form. If $\{du, dv\}$ denotes the dual basis of $\{\varphi_u, \varphi_v\}$, then $a = du(w)$, $b = dv(w)$ and we can write

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

This is the local expression of I with respect to φ . The matrix associated to this bilinear form is

$$(I) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Examples 2.1 1. A plane can be parametrized by $\varphi(u, v) = p + uw_1 + vw_2$, where p is a point in \mathbf{R}^3 , $\{w_1, w_2\}$ is an orthonormal set of vectors in \mathbf{R}^3 , and $(u, v) \in \mathbf{R}^2$. We have $\varphi_u = w_1, \varphi_v = w_2$, so $E = G = 1, F = 0$ and $I = du^2 + dv^2$.

2. The (right circular) cylinder can be parametrized by $\varphi(u, v) = (\cos u, \sin u, v)$, where $u_0 < u < u_0 + 2\pi$ and $v \in \mathbf{R}$. In this case, $\varphi_u = (-\sin u, \cos u, 0)$, $\varphi_v = (0, 0, 1)$, so $E = G = 1, F = 0$ and $I = du^2 + dv^2$.

3. The helicoid is the union of horizontal lines joining the z -axis to the points of an helix, namely, it is the image of the parametrization $\varphi(u, v) = (v \cos u, v \sin u, au)$ ($a > 0$). We see that $I = (a^2 + v^2)du^2 + dv^2$.

4. Spherical coordinates $\varphi(u, v) = (\sin u, \cos v, \sin u \sin v, \cos u)$, where $0 < u < \pi/2, v_0 < v < v_0 + 2\pi$, yield that $I = du^2 + \sin^2 u dv^2$ on the sphere.

Remark 2.2 If two surfaces S_1, S_2 can be covered by open sets which are images of parametrizations such that the local expressions of the first fundamental forms of S_1, S_2 coincide on corresponding open sets (like in the cases of the plane and the cylinder), then S_1 and S_2 are called *locally isometric*.

Using the first fundamental form, we can define:

Length of a smooth curve $\gamma : (a, b) \rightarrow S$ by

$$L(\gamma) = \int_a^b I(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt.$$

Angle between two vectors $w_1, w_2 \in T_p S$ by

$$\cos \angle(w_1, w_2) = \frac{I_p(w_1, w_2)}{I_p(w_1, w_1)^{1/2} I_p(w_2, w_2)^{1/2}}.$$

Surface integral of a compactly supported function $f : S \rightarrow \mathbf{R}$. If the support D of f is contained in the image of a parametrization $\varphi : U \rightarrow S$, then

$$\iint_D f dS = \iint_{\varphi^{-1}(D)} f(\varphi(u, v)) \|\varphi_u \times \varphi_v\| dudv,$$

where the left hand side is a double integral. Taking another parametrization $\bar{\varphi} : \bar{U} \rightarrow S$, the change of parameters $(u, v) = (\varphi^{-1} \circ \bar{\varphi})(\bar{u}, \bar{v})$ is smooth with Jacobian determinant $\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}$. Note that

$$\|\bar{\varphi}_{\bar{u}} \times \bar{\varphi}_{\bar{v}}\| = \left| \frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})} \right| \|\varphi_u \times \varphi_v\| dudv.$$

The formula of change of variables in the double integral yields that

$$\begin{aligned} & \iint_{\bar{\varphi}^{-1}(D)} f(\bar{\varphi}(\bar{u}, \bar{v})) \|\bar{\varphi}_{\bar{u}} \times \bar{\varphi}_{\bar{v}}\| d\bar{u}d\bar{v} \\ &= \iint_{\varphi^{-1}(D)} f(\bar{\varphi}(\bar{u}, \bar{v})) \|\varphi_u \times \varphi_v\| \left| \frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})} \right| d\bar{u}d\bar{v} \\ &= \iint_{\varphi^{-1}(D)} f(\varphi(u, v)) \|\varphi_u \times \varphi_v\| dudv, \end{aligned}$$

so the definition is independent of the choice of parametrization. In general, one needs to cover the support of f by finitely many parametrizations and

define the surface integral as a sum of double integrals. The proof of independence of the choices involved is more complicated in this case, and we will not go into details. The relation to the first fundamental form is that

$$\|\varphi_u \times \varphi_v\|^2 = \|\varphi_u\|^2 \|\varphi_v\|^2 - \langle \varphi_u, \varphi_v \rangle^2,$$

so

$$\|\varphi_u \times \varphi_v\| = \sqrt{EG - F^2}.$$

Area of a compact domain $D \subset S$ is

$$\text{area}(D) = \iint_D 1 \, dS.$$

In particular, if D is contained in the image of φ ,

$$\text{area}(D) = \iint_{\varphi^{-1}(D)} \sqrt{EG - F^2} \, dudv.$$

2.2 The Gauss map and the second fundamental form

Let $S \subset \mathbf{R}^3$ be a surface. For each $p \in S$, we want to assign a unit vector $\nu(p)$ which is normal to $T_p S$; note that there are exactly two possible choices. If it is possible to make such an assignment continuously along the whole of S , we say that S is *orientable*. The resulting map

$$\nu : S \rightarrow S^2$$

into the unit sphere is called the *Gauss map*. We will always assume that the Gauss map is continuous.

Examples 2.3 1. If $\varphi : U \rightarrow S$ is a parametrization, then we can take

$$\nu = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}.$$

This construction shows that every surface is locally orientable. It also shows that the Gauss map is smooth.

2. If S is given as the inverse image under F of regular value, then we can take

$$\nu = \frac{\nabla F}{\|\nabla F\|}.$$

The differential

$$d\nu_p : T_p S \rightarrow T_{\nu(p)} S^2 = T_p S$$

is an operator on $T_p S$, since $T_q S^2$ is always normal to q , for $q \in S^2$. The operator

$$-d\nu_p : T_p S \rightarrow T_p S$$

is called the *Weingarten operator*.

Proposition 2.4 *The Weingarten operator is symmetric:*

$$\langle d\nu_p(w_1), w_2 \rangle = \langle w_1, d\nu_p(w_2) \rangle,$$

where $w_1, w_2 \in T_p S$.

Proof. By linearity, it suffices to check the relation for a basis of $T_p S$. Let $\varphi : U \rightarrow S$ be a parametrization; then $\{\varphi_u, \varphi_v\}$ is a tangent frame. Set $N = \nu \circ \varphi$. Then

$$d\nu(\varphi_u) = d\nu\left(\frac{\partial\varphi}{\partial u}\right) = \frac{\partial}{\partial u}(\nu \circ \varphi) = \frac{\partial N}{\partial u}.$$

Similarly $d\nu(\varphi_v) = \frac{\partial N}{\partial v}$. Since $\langle \frac{\partial\varphi}{\partial v}, N \rangle = 0$ on U , differentiating with respect to u ,

$$\left\langle \frac{\partial^2\varphi}{\partial u\partial v}, N \right\rangle + \left\langle \frac{\partial\varphi}{\partial v}, \frac{\partial N}{\partial u} \right\rangle = 0,$$

and similarly

$$\left\langle \frac{\partial^2\varphi}{\partial v\partial u}, N \right\rangle + \left\langle \frac{\partial\varphi}{\partial u}, \frac{\partial N}{\partial v} \right\rangle = 0,$$

Taking the difference of these equations,

$$\left\langle \frac{\partial\varphi}{\partial v}, \frac{\partial N}{\partial u} \right\rangle - \left\langle \frac{\partial\varphi}{\partial u}, \frac{\partial N}{\partial v} \right\rangle = 0,$$

which says that

$$\langle \varphi_v, d\nu(\varphi_v) \rangle - \langle \varphi_u, d\nu(\varphi_u) \rangle = 0,$$

as we wished. \square

The associated symmetric bilinear form

$$II_p(w_1, w_2) = -\langle d\nu_p(w_1), w_2 \rangle$$

is called the *second fundamental form*.

Proposition 2.5 *Let $\gamma : (-\epsilon, \epsilon) \rightarrow S$ be a smooth curve parametrized by arc-length, $\gamma(0) = p$, $\gamma'(0) = w \in T_p S$. Then*

$$\langle \gamma''(0), \nu(p) \rangle = II_p(w, w).$$

Proof. Start with the equation $\langle \gamma'(s), \nu(\gamma(s)) \rangle = 0$ and differentiate it at $s = 0$ to obtain

$$\langle \gamma''(0), \nu(p) \rangle + \left\langle w, \frac{d}{ds}\Big|_{s=0} \nu(\gamma(s)) \right\rangle = 0.$$

Then $\langle \gamma''(0), \nu(p) \rangle = -\langle w, d\nu_p(w) \rangle = II_p(w, w)$, as desired. \square

There is a geometric interpretation of last proposition (Meusnier). Given a unit vector $w \in T_p S$, the affine plane through p spanned by w and $\nu(p)$ meets S transversally along a curve which is called the *normal section of S along w* . If γ

is a parametrization by arc-length of this normal section as in the proposition, then the curvature of γ at $p = \gamma(0)$ is

$$\kappa_w = \langle \gamma''(0), \nu(p) \rangle = II_p(w, w),$$

where we view γ as a plane curve and we view its supporting plane as oriented by $\{w, \nu(p)\}$.

Since the Weingarten operator $-d\nu_p$ is symmetric, there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p S$ such that

$$-d\nu_p(e_1) = \kappa_1 e_1, \quad -d\nu_p(e_2) = \kappa_2 e_2.$$

The eigenvalues κ_1, κ_2 are called *principal curvatures* at p , and the eigenvectors e_1, e_2 are called *principal directions* at p . Of course, $II(e_1, e_1) = \kappa_1$, $II(e_2, e_2) = \kappa_2$, $II(e_1, e_2) = 0$. It follows that for a unit vector $w = \cos \theta e_1 + \sin \theta e_2 \in T_p S$ we have *Euler's formula*:

$$\kappa_w = I_p(w, w) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta.$$

Since this a convex linear combination ($\sin^2 \theta + \cos^2 \theta = 1$), Euler's formula also shows that κ_1, κ_2 are the extrema of the curvatures of the normal sections through p .

In general, a smooth curve γ in S is called a *line of curvature* if $\dot{\gamma}(t)$ is a principal direction at $\gamma(t)$ for all t . Note that if $\kappa_1(p) \neq \kappa_2(p)$, then the principal directions at p are uniquely defined, but not otherwise. If $\kappa_1(p) = \kappa_2(p)$, we say that p is an *umbilic point* of S .

Proposition 2.6 *If all the points of a connected surface S are umbilic, then S is contained in a plane or a sphere.*

Proof. We first prove the result in case S is the image V of a parametrization $\varphi : U \rightarrow V$. Set $N = \nu \circ \varphi$. By assumption, $d\nu = \lambda \cdot I$, where $\lambda : V \rightarrow \mathbf{R}$ is a smooth function. It follows that

$$\begin{aligned} N_u = d\nu(\varphi_u) &= (\lambda \circ \varphi)\varphi_u, \\ N_v = d\nu(\varphi_v) &= (\lambda \circ \varphi)\varphi_v. \end{aligned}$$

We differentiate the first (resp. second) of these equations with respect to v (resp. u) to get

$$\begin{aligned} N_{uv} &= (\lambda \circ \varphi)_v \varphi_u + (\lambda \circ \varphi)\varphi_{uv}, \\ N_{vu} &= (\lambda \circ \varphi)_u \varphi_v + (\lambda \circ \varphi)\varphi_{vu}. \end{aligned}$$

Taking the difference,

$$0 = (\lambda \circ \varphi)_v \varphi_u - (\lambda \circ \varphi)_u \varphi_v.$$

Since $\{\varphi_u, \varphi_v\}$ is linearly independent, the partial derivatives of $\lambda \circ \varphi$ on U are zero. Since U is connected, $\lambda \circ \varphi$ is constant.

Next, we consider two cases. If $\lambda \circ \varphi = 0$, then $N_u = N_v = 0$ so N is constant. This implies $\frac{\partial}{\partial u} \langle \varphi, N \rangle = \frac{\partial}{\partial v} \langle \varphi, N \rangle = 0$ and hence V is contained in an affine plane parallel to $\langle N \rangle^\perp$. On the other hand, if $\lambda \circ \varphi = R \neq 0$, then $\varphi - \frac{1}{R}N$ is a constant $q \in \mathbf{R}^3$ and hence V is contained in the sphere of center q and radius $1/|R|$.

In the case of arbitrary S , fix a point $p \in S$ and a parametrized neighborhood V_0 of p . By the previous case, V_0 is contained in a plane or a sphere. Given $x \in S$, by connectedness of S , there exists a continuous curve $\gamma : [0, 1] \rightarrow S$ joining p to x (S is locally arcwise connected, so it is arcwise connected). For any $t \in [0, 1]$, there exists a parametrized neighborhood of $\gamma(t)$ which is contained in a plane or a sphere. By compactness of $\gamma([0, 1])$, it is possible to cover it by open sets V_0, V_1, \dots, V_n such that each V_i is contained in a plane or a sphere and $V_i \cap V_{i+1} \neq \emptyset$ (check this!); the latter condition implies that V_{i+1} is contained in the same plane or sphere that contains V_i . The result follows. \square

2.3 Curvature of surfaces

Let $S \subset \mathbf{R}^3$ and consider its Weingarten operator $-d\nu_p : T_p S \rightarrow T_p S$. Recall that $-d\nu_p$ is symmetric and its eigenvalues $\kappa_1(p), \kappa_2(p)$ are the principal curvatures of S at p . We define:

$$\begin{aligned} \text{Gaussian curvature : } K(p) &= \det(-d\nu_p) = \kappa_1(p) \cdot \kappa_2(p), \\ \text{Mean curvature : } H(p) &= \frac{1}{2} \text{trace}(-d\nu_p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p)). \end{aligned}$$

Note that $\kappa_1, \kappa_2 = H \pm \sqrt{H^2 - K}$; we will soon see that H and K are smooth functions on S , and so it follows from this equation that κ_1, κ_2 are continuous functions on S which are smooth away from umbilic points (points characterized by $H^2 = K$).

If we change ν to $-\nu$, then H is changed to $-H$ but K is unchanged. The next example analyses the meaning of the sign of K .

Examples 2.7 1. Let us compute the Gaussian curvature of the graph S of a smooth function $f : U \rightarrow \mathbf{R}$, where $U \subset \mathbf{R}^2$ is open. In general, for a parametrization φ and $N = \nu \circ \varphi$,

$$II(\varphi_u, \varphi_u) = -\langle d\nu(\varphi_u), \varphi_u \rangle = -\langle N_u, \varphi_u \rangle = \langle N, \varphi_{uu} \rangle.$$

Similarly,

$$II(\varphi_u, \varphi_v) = -\langle N_u, \varphi_v \rangle = \langle N, \varphi_{uv} \rangle$$

and

$$II(\varphi_v, \varphi_v) = -\langle N_v, \varphi_v \rangle = \langle N, \varphi_{vv} \rangle.$$

In our case,

$$\begin{aligned}\varphi_u &= (1, 0, f_u), \\ \varphi_v &= (0, 1, f_v), \\ \varphi_{uu} &= (0, 0, f_{uu}), \\ \varphi_{uv} &= (0, 0, f_{uv}), \\ \varphi_{vv} &= (0, 0, f_{vv}), \\ N &= \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}.\end{aligned}$$

Hence

$$\begin{aligned}(II) &= \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} \\ &= \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} (\text{Hess}(f)).\end{aligned}$$

We specialize to the case $p = (0, 0, 0) = f(0, 0)$ and $T_p S$ is the xy -plane. Then $f_u(0, 0) = f_v(0, 0) = 0$ and $II_p = \text{Hess}_{(0,0)}(f)$. In particular, if $f(u, v) = au^2 + bv^2$, then

$$(II) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}.$$

It follows that $K(p) = 4ab$ is positive (resp. negative) if a and b have the same sign (resp. opposite signs), and it is zero if one of a, b is zero.

Another interesting case is $f(u, v) = u^4 + v^4$. We get $II = 0$.

2. Consider the sphere $S^2(R)$ of radius $R > 0$. We can take $\nu(p) = -\frac{1}{R}p$, so $-d\nu_p = \frac{1}{R} \text{id}_{T_p S}$ and $K(p) = \frac{1}{R^2} > 0$, $H(p) = \frac{1}{R}$.

A point p in a surface S is called *elliptic* (resp. *hyperbolic*, *parabolic*) if $K(p) > 0$ (resp. $K(p) < 0$, $K(p) = 0$).

2.4 Local expressions for K, H

Fix a parametrization φ of S . Then $\{\varphi_u, \varphi_v\}$ is a tangent frame with respect to which we consider the matrices of the fundamental forms and the Weingarten operator and introduce a new (index) notation for the coefficients.

$$\begin{aligned}(I) &= \begin{pmatrix} \langle \varphi_u, \varphi_u \rangle & \langle \varphi_u, \varphi_v \rangle \\ \langle \varphi_v, \varphi_u \rangle & \langle \varphi_v, \varphi_v \rangle \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \\ (II) &= \begin{pmatrix} \langle N, \varphi_{uu} \rangle & \langle N, \varphi_{uv} \rangle \\ \langle N, \varphi_{vu} \rangle & \langle N, \varphi_{vv} \rangle \end{pmatrix} = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \\ &(-d\nu) = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}.\end{aligned}$$

We have that

$$h_{11} = II(\varphi_u, \varphi_u) = -\langle d\nu(\varphi_u), \varphi_u \rangle = \langle h_1^1 \varphi_u + h_1^2 \varphi_v, \varphi_u \rangle,$$

so

$$h_{11} = h_1^1 g_{11} + h_1^2 g_{21}.$$

Similarly,

$$h_{12} = h_1^1 g_{12} + h_1^2 g_{22},$$

$$h_{21} = h_2^1 g_{11} + h_2^2 g_{21},$$

$$h_{22} = h_2^1 g_{12} + h_2^2 g_{22}.$$

In matrix form.

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Recall that (I) is invertible since it is positive definite. Thus

$$\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}.$$

Back to the classical notation,

$$\begin{aligned} (-d\nu) &= \frac{1}{EG - F^2} \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} \ell G - mF & -\ell F + mE \\ mG - nF & -mF + nE \end{pmatrix}. \end{aligned}$$

Hence

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{\det(II)}{\det(I)}$$

and

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)}.$$

It follows from these expressions that K, H are smooth on S .

2.5 Surfaces of revolution

Consider the parametrized surface of revolution

$$\varphi(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

where $(u, v) \in (a, b) \times (v_0, v_0 + 2\pi)$ and the geratrix $\gamma(s) = (f(s), 0, g(s))$ is parametrized by arc length. Then

$$\varphi_u = (f' \cos v, f' \sin v, g'), \quad \varphi_v = (-f \sin v, f \cos v, 0),$$

so

$$E = f'^2 + g'^2 = 1, \quad F = 0, \quad G = f^2.$$

Moreover,

$$\begin{aligned} \varphi_{uu} &= (f'' \cos v, f'' \sin v, g''), \\ \varphi_{uv} &= (-f' \sin v, f' \cos v, 0), \\ \varphi_{vv} &= (-f \cos v, -f \sin v, 0). \end{aligned}$$

We compute the coefficients of II .

$$\begin{aligned} \ell &= \langle N, \varphi_{uu} \rangle = \frac{\langle \varphi_u \times \varphi_v, \varphi_{uu} \rangle}{\|\varphi_u \times \varphi_v\|} = \frac{1}{EG - F^2} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ f'' \cos v & f'' \sin v & g'' \end{vmatrix} \\ &= f'g'' - f''g'. \end{aligned}$$

Similarly,

$$m = \langle N, \varphi_{uv} \rangle = \frac{1}{f} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ -f' \sin v & f' \cos v & 0 \end{vmatrix} = 0$$

and

$$n = \langle N, \varphi_{vv} \rangle = \frac{1}{f} \begin{vmatrix} f' \cos v & f' \sin v & g \\ -f \sin v & f \cos v & 0 \\ -f \cos v & -f \sin v & 0 \end{vmatrix} = fg'.$$

Note that $f'g'' - f''g'$ is the signed curvature κ_γ of γ , for $\kappa_\gamma = \langle \gamma'', (-g', 0, f') \rangle$.
Now

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{(f'g'' - f''g')g'}{f} = \kappa_\gamma \frac{g'}{f} \quad (2.8)$$

and

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = \frac{1}{2} \left(\kappa_\gamma + \frac{g'}{f} \right). \quad (2.9)$$

It follows that the principal curvatures

$$\kappa_1 = \kappa_\gamma, \quad \kappa_2 = \frac{g'}{f}.$$

The identities $F = m = 0$ mean that the fundamental forms are diagonalized in the frame $\{\varphi_u, \varphi_v\}$. In particular, φ_u, φ_v are always principal directions and thus the curves u -constant (parallels) and v -constant (meridians) are lines of curvature.

We can also derive some other useful formulas for K, H . Differentiation of $f'^2 + g'^2 = 1$ gives $f'f'' + g'g'' = 0$. Substituting this identity into (2.8) yields

$$K = \frac{-f''}{f}.$$

The identity also gives $\kappa_1 = \kappa_\gamma = \frac{g''}{f'}$ if $f' \neq 0$, so

$$H = \frac{1}{2} \left(\frac{g''}{f'} + \frac{g'}{f} \right) = \frac{(fg')'}{(f^2)'}. \quad (2.10)$$

We use this formula to prove the following theorem. A surface satisfying $H \equiv 0$ is called *minimal*. This terminology will be explained in section ??.

Theorem 2.11 *The only minimal surfaces of revolution are the plane and the catenoid (the surface of revolution generated by a catenary, which is the graph of hyperbolic cosine).*

Proof. We use the above notation. If $f' = 0$ on an interval, then eqn. (2.9) gives $g' = 0$, which is a contradiction to the fact that γ is regular. Therefore we can assume that f' is never zero. By formula (2.10), we need to solve the equation $fg' = k$, where k is a constant. Using $g' = \pm\sqrt{1-f'^2}$, we get

$$f' = \pm\sqrt{1-(k/f)^2}.$$

Note that $|f| \geq |k|$ is a necessary condition. This equation can be easily integrated by rewriting it as

$$\frac{f df}{\sqrt{f^2 - k^2}} = \pm ds.$$

We get

$$f(s) = \pm\sqrt{k^2 + (s + c_1)^2}.$$

The constant c_1 can be chosen to be zero by redefining the instant $s = 0$, and we recall that $f > 0$, so we have

$$f(s) = \sqrt{k^2 + s^2}.$$

If $k = 0$ then $f(s) = \pm s$ and g is constant, which corresponds to the case of the plane. Suppose $k \neq 0$ and integrate $g' = k/f$ to get

$$g(s) = k \log(s + \sqrt{k^2 + s^2}) + c_2.$$

We choose the constant $c_2 = -k \log |k|$ so that $\gamma(0) = (|k|, 0, 0)$. Changing the sign of k is equivalent to changing the sign of g , which corresponds to a reflection on the plane $z = 0$, so we may assume $k > 0$. Finally, we make the change of variable

$$t = g(s) = k \log(\sqrt{1 + (s/k)^2} + s/k)$$

to get

$$\gamma(t) = (k \cosh(t/k), 0, t)$$

which is a catenary. □

2.6 Ruled surfaces

A ruled surface is a surface generated by a smooth one-parameter family of lines. More precisely, a (nonnecessarily regular) parametrized surface $\varphi : U \subset \mathbf{R}^2 \rightarrow S$ is called a *ruled surface* if there exist a smooth curves $\gamma : I \rightarrow \mathbf{R}^3$ and $w : I \rightarrow S^2$ such that

$$\varphi(u, v) = \gamma(u) + v w(u)$$

where $(u, v) \in I \times \mathbf{R} = U$. The curve γ is called a *directrix* and the lines $\mathbf{R}w(u)$ are called the *rulings*.

Obvious examples of ruled surfaces are planes, cylinders and cones. Other examples are the helicoid, the one-sheeted hyperboloid and the hyperbolic paraboloid (given by the equation $z = xy$ in \mathbf{R}^3).

We make some local considerations. Assume that $w'(u) \neq 0$ for all u , in other words, w is regular; this condition is sometimes expressed by saying that the ruled surface is *noncylindrical*. Then it is possible to introduce the so called *standard parameters* on S .

Proposition 2.12 *There exists a unique reparametrization*

$$\tilde{\varphi}(\tilde{u}, \tilde{v}) = \tilde{\gamma}(\tilde{u}) + \tilde{v} \tilde{w}(\tilde{u})$$

such that $\|\tilde{w}'\| = 1$ and $\langle \tilde{\gamma}', \tilde{w}' \rangle = 0$.

Proof. Since w is regular, we can introduce arc-length parameter \tilde{u} so that $\tilde{w}(\tilde{u}) = w(u(\tilde{u}))$, and then $\|\tilde{w}'\| = 1$. Next, we write $\tilde{\gamma}(\tilde{u}) = \gamma(\tilde{u}) - \tilde{v}(\tilde{u})\tilde{w}(\tilde{u})$ and impose the condition $\langle \tilde{\gamma}', \tilde{w}' \rangle = 0$ to get $\tilde{v}(\tilde{u}) = -\langle \frac{d}{d\tilde{u}}\gamma, \tilde{w}'(\tilde{u}) \rangle$. \square

The curve $\tilde{\gamma}$ is called the *striction line* of the surface and its points are called *central points* of the surface; note that $\tilde{\gamma}$ is not necessarily regular.

Using the standard parametrization, we can compute the curvature of a ruled surface

$$\varphi(u, v) = \gamma(u) + v w(u)$$

where $\|w\| = \|w'\| = 1$, $\langle \gamma', w' \rangle = 0$. We have

$$\varphi_u = \gamma' + v w', \quad \varphi_v = w,$$

so

$$E = \|\gamma'\|^2 + v^2, \quad F = \langle \gamma', w \rangle, \quad G = 1.$$

Since w' is orthogonal to w and γ' , there exists a smooth function $\lambda = \lambda(u)$, called the *distribution parameter*, such that

$$\gamma' \times w = \lambda w'. \tag{2.13}$$

It follows that

$$\|\varphi_u \times \varphi_v\| = \sqrt{EG - F^2} = \|\gamma'\|^2 - \langle \gamma', w \rangle^2 + v^2 = \lambda^2 + v^2.$$

In particular, the singular points of φ occur along the striction line ($v = 0$) precisely when $\lambda(u) = 0$.

Next, we compute the coefficients of II . We have

$$\varphi_{uu} = \gamma'' + v w', \quad \varphi_{uv} = w', \quad \varphi_{vv} = 0.$$

This implies

$$m = \frac{\langle \varphi_u \times \varphi_v, \varphi_{uv} \rangle}{\|\varphi_u \times \varphi_v\|} = \frac{\langle \lambda w' + v w \times w', w' \rangle}{\sqrt{\lambda^2 + v^2}} = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}$$

and

$$n = 0,$$

what is sufficient to get the formula for the Gaussian curvature:

$$K = \frac{\ell n - m^2}{EG - F^2} = -\frac{\lambda(u)^2}{(\lambda(u)^2 + v^2)^2}.$$

Note that $K \leq 0$, and $K = 0$ precisely along the rulings that meet the striction line at a singular point ($\lambda(u) = 0$), except of course the singular point itself ($v \neq 0$). If $\lambda(u) \neq 0$, this formula also shows the striction line is characterized by the property that the maximum of K along each ruling occurs exactly at the central point.

The computation of ℓ is more involved. We have

$$\begin{aligned} \langle \varphi_u \times \varphi_v, \varphi_{uu} \rangle & \\ = \lambda \langle w', \gamma'' \rangle + \lambda v \langle w', w'' \rangle + v \langle w' \times w, \gamma'' \rangle + v^2 \langle w' \times w, w'' \rangle. \end{aligned} \quad (2.14)$$

We analyse separately the four terms on the right-hand side. Introduce the parameter

$$J = \langle w \times w', w'' \rangle.$$

Since $\{w, w', w \times w'\}$ is an orthonormal frame,

$$\gamma' = \langle \gamma', w \rangle w + \langle \gamma', w \times w' \rangle w \times w' = Fw + \lambda w \times w'$$

and

$$\langle w', \gamma'' \rangle = -\langle w'', \gamma' \rangle = -F \langle w'', w \rangle - \lambda \langle w'', w \times w' \rangle = F - \lambda J,$$

where we have used $\langle w'', w \rangle = -\langle w', w' \rangle = -1$. Equation $\|w'\| = 1$ also implies $\langle w', w'' \rangle = 0$. In order to analyse the third term in eqn. (2.14), differentiate eqn. (2.13) to get

$$\gamma'' \times w + \gamma' \times w' = \lambda' w' + \lambda w'',$$

and multiply through by w' to write

$$\langle \gamma'' \times w, w' \rangle = \lambda'.$$

Now eqn. (2.14) is

$$\langle \varphi_u \times \varphi_v, \varphi_{uu} \rangle = -Jv^2 - \lambda'v + \lambda(F - \lambda J)$$

and

$$\ell = \frac{-\lambda^2 J + \lambda' v + \lambda(F - \lambda J)}{\sqrt{\lambda^2 + v^2}}.$$

Hence

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = -\frac{Jv^2 + \lambda'v + \lambda(\lambda J + F)}{2(\lambda^2 + v^2)^{3/2}}.$$

Example 2.15 The standard parametrization of the helicoid has $\gamma(u) = (0, 0, bu)$ and $w(u) = (\cos u, \sin u, 0)$. Since $\gamma' \times w = bw'$, the distribution parameter $\lambda = b$ is constant and $K = -b^2/(b^2 + v^2)^2$. Note that $F = 0$ and, since $w'' = -w$, we also have $J = 0$. Hence $H = 0$.

Proposition 2.16 *The only minimal ruled surfaces are the plane and the helicoid.*

Proof. We have just seen that $H = 0$ says that

$$Jv^2 + \lambda'v + \lambda(\lambda J + F) = 0.$$

This is a quadratic polynomial in v whose coefficients, being functions depending only on u , must vanish. It follows that λ is constant and $J = \lambda F = 0$.

Since $J = 0$, w'' is a linear combination of w , w' . But $\langle w'', w' \rangle = 0$ and $\langle w'', w \rangle = -1$, so $w'' = -w$ and w is a circle.

If $\lambda = 0$ then $II = 0$, which corresponds to the case of the plane. Suppose $\lambda \neq 0$. Then $F = 0$ implying that $\gamma' = \lambda w \times w'$. Differentiation of this equation yields $\gamma'' = 0$. It follows that γ is a line perpendicular to the circle defined by w . Hence the surface is the helicoid. \square

2.7 Minimal surfaces

Let $S \subset \mathbf{R}^3$ be a surface. We say that S is a *minimal surface* if the mean curvature $H \equiv 0$. Historically speaking, this concept is related to the problem of characterizing the surface with smallest area spanned by a given boundary, a problem raised by Lagrange in 1760. The question of showing the existence of such a surface is called the *Plateau problem*, in honor of the Belgian physicist who performed experiments with soap films around 1850, and it was solved completely only in 1930, independently by Jesse Douglas and Tibor Radó.

In order to explain the relation between mean curvature and minimization of surface area, consider a parametrization $\varphi : U \subset \mathbf{R}^2 \rightarrow S$, $N = \nu \circ \varphi$ the induced unit normal, and a smooth function $f : U \rightarrow \mathbf{R}$. Then we can introduce the *normal variation* of φ along f :

$$\varphi^\epsilon = \varphi + \epsilon f N.$$

Let us compute the first fundamental form of φ^ϵ :

$$\varphi_u^\epsilon = \varphi_u + \epsilon(f_u N + f N_u), \quad \varphi_v^\epsilon = \varphi_v + \epsilon(f_v N + f N_v).$$

Since $\langle \varphi_u, N \rangle = \langle \varphi_v, N \rangle = 0$ and $\langle \varphi_u, N_u \rangle = -\ell$, $\langle \varphi_u, N_v \rangle = \langle \varphi_v, N_u \rangle = -m$, $\langle \varphi_v, N_v \rangle = -n$, we obtain

$$\begin{aligned} E^\epsilon &= E - 2\epsilon f\ell + O(\epsilon^2), \\ F^\epsilon &= F - 2\epsilon fm + O(\epsilon^2), \\ G^\epsilon &= G - 2\epsilon fn + O(\epsilon^2), \end{aligned}$$

where $O(\epsilon^2)$ denotes a continuous function satisfying $\lim_{\epsilon \rightarrow 0} O(\epsilon^2)/\epsilon = 0$. It follows that

$$\begin{aligned} E^\epsilon G^\epsilon - (F^\epsilon)^2 &= EG - F^2 - 2\epsilon f(\ell G + nE - 2mF) + O(\epsilon^2) \\ &= (EG - F^2)(1 - 4\epsilon fH) + O(\epsilon^2). \end{aligned}$$

Let now $D \subset U$ be a compact domain and introduce

$$A(\epsilon) = \text{area}(\varphi^\epsilon(D)) = \iint_D \sqrt{E^\epsilon G^\epsilon - (F^\epsilon)^2} \, dudv.$$

We have

$$\begin{aligned} A'(0) &= \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \iint_D (E^\epsilon G^\epsilon - (F^\epsilon)^2)^{1/2} \, dudv \\ &= \iint_D \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \frac{E^\epsilon G^\epsilon - (F^\epsilon)^2}{2(EG - F^2)^{1/2}} \, dudv. \end{aligned}$$

Hence

$$A'(0) = -2 \iint_D fH \sqrt{EG - F^2} \, dudv.$$

This formula is called *first variation of surface area*. As a corollary, we obtain the following characterization of minimal surfaces as critical points of the area functional.

Proposition 2.17 *A surface S is minimal if and only if $A'(0) = 0$ for every parametrization $\varphi : U \rightarrow S$, every normal variation of φ , and every compact domain $D \subset U$.*

Proof. If $H(p) \neq 0$ for some $p \in S$, we choose a compact neighborhood \tilde{D} of p in S such that H does not vanish on \tilde{D} and \tilde{D} is contained in the image of a parametrization $\varphi : U \rightarrow S$, set $D = \varphi^{-1}(\tilde{D})$, and take $f = H|_U$. We get

$$A'(0) = -2 \iint_D H^2 \sqrt{EG - F^2} \, dudv < 0,$$

so the given condition is sufficient for the minimality of S . That it is also necessary is obvious. \square

2.7.1 Isothermal parameters

When studying minimal surfaces, it is useful to introduce special parameters. A parametrized surface $\varphi : U \rightarrow S$ is called *isothermal* if

$$E = G = \lambda^2, \quad F = 0,$$

where $\lambda \geq 0$ is a smooth function on U ; in this case, the parameters $(u, v) \in U$ are also called *isothermal*. Note that φ is regular if and only if $\lambda > 0$. An isothermal parametrization φ is also called *conformal* or *angle preserving* because angles between curves in the surface are equal to the angles between the corresponding curves in the parameter plane.

Note that the mean curvature expressed in terms of isothermal parameters becomes

$$H = \frac{\ell G - 2mF + nE}{2(EG - F^2)} = \frac{\ell + n}{2\lambda^2}. \quad (2.18)$$

Proposition 2.19 *If φ is isothermal, then $\Delta\varphi = 2\lambda^2HN$ (here $N = \nu \circ \varphi$ is the unit normal along φ).*

Proof. Here Δ denotes the Laplacian operator and $\Delta\varphi = \varphi_{uu} + \varphi_{vv}$. Consider the equations $\langle \varphi_u, \varphi_u \rangle = \langle \varphi_v, \varphi_v \rangle$ and $\langle \varphi_u, \varphi_v \rangle = 0$; differentiating the first one with respect to u and the second one with respect to v , we obtain

$$\langle \varphi_{uu}, \varphi_u \rangle = \langle \varphi_{vu}, \varphi_u \rangle \quad \text{and} \quad \langle \varphi_{uv}, \varphi_v \rangle + \langle \varphi_u, \varphi_{vv} \rangle = 0.$$

Putting these together yields $\langle \Delta\varphi, \varphi_u \rangle = 0$. Similarly, differentiating the first equation with respect to v and the second one with respect to u , we get that $\langle \Delta\varphi, \varphi_v \rangle = 0$. This shows that $\Delta\varphi$ is a normal vector. Finally,

$$\langle \Delta\varphi, N \rangle = \langle \varphi_{uu}, N \rangle + \langle \varphi_{vv}, N \rangle = \ell + n = 2\lambda^2H$$

by eqn. (2.18). □

Corollary 2.20 *An isothermal regular parametrized surface $\varphi : U \rightarrow S$ is minimal if and only if the coordinate functions of φ are harmonic functions on U .*

Isothermal parameters exist around any point in a surface. In the next section, we present a proof of their existence in the case of minimal surfaces.

Theorem 2.21 *There exist no compact minimal surfaces in \mathbf{R}^3 .*

Proof. Suppose, to the contrary, that S is a compact minimal surface. Without loss of generality, we may assume S is connected. Consider the coordinate function $x : \mathbf{R}^3 \rightarrow \mathbf{R}$. There exists a point $p \in S$ where the restriction $x|_S$ attains its maximum. Let $\varphi : U \rightarrow \varphi(U)$ be an isothermal parametrization around p with U connected. Then $x \circ \varphi$ is a harmonic function on U which attains its maximum at an interior point $p \in U$. By the maximum principle, $x \circ \varphi$ is a constant function on U , or, $x \equiv x(p)$ on $\varphi(U)$.

Next, let $q \in S$ be arbitrary and choose a continuous curve $\gamma : [0, 1] \rightarrow S$ joining p to q . Cover $\gamma([0, 1])$ by finitely many connected open sets $V_0 = \varphi(U), V_1, \dots, V_n$, each V_i equal to the image of an isothermal parametrization φ_i , such that $V_i \cap V_{i+1} \neq \emptyset$ for all $i = 0, 1, \dots, n$ and $q \in V_n$. Since $x|_S$ attains its maximum value along $V_0 \cap V_1 \neq \emptyset$, the maximum principle applied to $x \circ \varphi_1$ yields that this function is constant along V_1 , namely, $x \equiv x(p)$ on V_1 . Proceeding by induction, we get that $x \equiv x(p)$ on V_n and hence $x(q) = x(p)$. Since q is arbitrary, this argument proves that $x|_S$ is a constant function. The same argument applied to the other coordinate functions $y, z : \mathbf{R}^3 \rightarrow \mathbf{R}$ finally shows that S must be a point, a contradiction. \square

2.7.2 The Enneper-Weierstrass representation

We discuss now an unexpected connection between minimal surfaces and theory of functions of one complex variable. Let $\varphi : U \rightarrow S$ be a parametrized surface. Denote by $x_1, x_2, x_3 : U \rightarrow \mathbf{R}$ the coordinate functions of φ . We introduce the complex functions ($j = 1, 2, 3$):

$$f_j(\zeta) = \frac{\partial x_j}{\partial u} - \frac{\partial x_j}{\partial v}, \quad \text{where } \zeta = u + iv. \quad (2.22)$$

The function f_j is smooth as a real function $U \subset \mathbf{R}^2 \rightarrow \mathbf{R}^2$, so a necessary and sufficient condition for f_j to be holomorphic is given by the Cauchy-Riemann equations $\frac{\partial}{\partial u} \Re f_j = \frac{\partial}{\partial v} \Im f_j, \frac{\partial}{\partial v} \Re f_j = -\frac{\partial}{\partial u} \Im f_j$. We deduce that

- (a) f_j is holomorphic in ζ if and only if x_j is harmonic in u, v .

Note also the identities:

$$\begin{aligned} f_1^2 + f_2^2 + f_3^2 &= \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial u} \right)^2 - \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial v} \right)^2 - 2i \sum_{j=1}^3 \frac{\partial x_j}{\partial u} \cdot \frac{\partial x_j}{\partial v} \\ &= E - G - 2iF \end{aligned}$$

and

$$|f_1|^2 + |f_2|^2 + |f_3|^2 = \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial u} \right)^2 + \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial v} \right)^2 = E + G. \quad (2.23)$$

It follows from these identities that

- (b) φ is isothermal if and only if

$$f_1^2 + f_2^2 + f_3^2 = 0. \quad (2.24)$$

- (c) If φ is isothermal, then φ is regular if and only if

$$|f_1|^2 + |f_2|^2 + |f_3|^2 \neq 0. \quad (2.25)$$

Proposition 2.26 *Let $\varphi : U \rightarrow S$ be an isothermal regular parametrized minimal surface. Then the functions f_j defined by (2.22) are holomorphic and satisfy (2.24) and (2.25). Conversely, if f_1, f_2, f_3 are holomorphic functions defined on a simply-connected domain U which satisfy (2.24) and (2.25), then ($j = 1, 2, 3$)*

$$x_j(\zeta) = \Re \int_{\zeta_0}^{\zeta} f_j(z) dz, \quad \zeta \in U,$$

(for fixed $\zeta_0 \in U$) are the coordinates of an isothermal regular parametrized minimal surface $\varphi : U \rightarrow S$ such that eqns. (2.22) are valid.

Proof. One direction follows from assertions (a), (b), (c) above and Corollary 2.20. For the converse, note that $\zeta \mapsto \int_{\zeta_0}^{\zeta} f_j(z) dz$ is well defined because U is simply connected and f_j is holomorphic, and yields a holomorphic function on U for which we can apply the Cauchy-Riemann equations:

$$\begin{aligned} \frac{d}{d\zeta} \int_{\zeta_0}^{\zeta} f_j &= \frac{\partial}{\partial u} \Re \int_{\zeta_0}^{\zeta} f_j + i \frac{\partial}{\partial u} \Im \int_{\zeta_0}^{\zeta} f_j \\ &= \frac{\partial}{\partial u} \Re \int_{\zeta_0}^{\zeta} f_j - i \frac{\partial}{\partial v} \Im \int_{\zeta_0}^{\zeta} f_j, \end{aligned}$$

so eqns. (2.22) are valid; the rest now follows from (a), (b), (c) and Corollary 2.20 applied in the opposite direction. \square

Note that the functions x_j in the preceding proposition are defined up to an additive constant so that the surface is defined up to a translation.

Thus we see that the local study of minimal surfaces in \mathbf{R}^3 is reduced to solving equations (2.24) and (2.25) for triples of holomorphic functions. We next explain how this can be done. Rewrite (2.24) as

$$(f_1 + if_2)(f_1 - if_2) = -f_3^2. \quad (2.27)$$

Except in case $f_1 \equiv if_2$ and $f_3 \equiv 0$ (which is easily seen to correspond to the case of a plane), the functions

$$f = f_1 - if_2, \quad g = \frac{f_3}{f_1 - if_2}$$

are such that f is holomorphic and g is meromorphic. Clearly, $f_3 = fg$, and it follows from eqn. (2.27) that

$$f_1 + if_2 = \frac{-f_3^2}{f_1 - if_2} = -fg^2. \quad (2.28)$$

Hence

$$f_1 = \frac{1}{2}f(1 - g^2), \quad \text{and} \quad f_2 = \frac{i}{2}f(1 + g^2).$$

By (2.28), fg^2 is holomorphic and this says that at every pole of g , f has a zero of order at least twice the order of the pole. Further, eqn. (2.25) says that $f_1, f_2,$

f_3 cannot vanish simultaneously, and this means that f can only have a zero at a pole of g , and then the order of its zero must be exactly twice the order of the pole of g . We summarize this discussion as follows.

Theorem 2.29 (The Enneper-Weierstrass representation) *Every minimal surface which is not a plane can be locally represented as*

$$\begin{aligned}x_1 &= \Re \int \frac{1}{2} f(\zeta)(1 - g^2(\zeta)) d\zeta \\x_2 &= \Re \int \frac{i}{2} f(\zeta)(1 + g^2(\zeta)) d\zeta \\x_3 &= \Re \int f(\zeta)g(\zeta) d\zeta,\end{aligned}$$

where: f is a holomorphic function on a simply-connected domain U , g is meromorphic on U , f vanishes only at the poles of g , and the order of its zero at such a point is exactly twice the order of the pole of g .

Conversely, every pair functions f, g satisfying these conditions define an isothermal regular parametrized minimal surface via the above equations.

Examples 2.30 1. The catenoid is given by $f(z) = -e^{-z}$, $g(z) = -e^z$.

2. The helicoid is given by $f(z) = -ie^{-z}$, $g(z) = -e^z$.

3. The minimal surface of Enneper (discovered in 1863) is given by $f(z) = 1$, $g(z) = z$. Solving for the parametrization, we obtain $x_1 = u - \frac{1}{3}u^3 + uv^2$, $x_2 = -v - u^2v + \frac{1}{3}v^3$, $x_3 = u^2 - v^2$.

4. The minimal surface of Scherk (discovered in 1834) is given by $f(z) = 4/(1 - z^4)$, $g(z) = iz$. It can also be parametrized as the graph of $(x, y) \mapsto \log \frac{\cos x}{\cos y}$.

The Enneper-Weierstrass representation not only allows us to construct a great variety of minimal surfaces having interesting properties, but also serve to prove general theorems about minimal surfaces by translating the statements into corresponding statements about holomorphic functions. Unfortunately, developing this philosophy would take us beyond the scope of these notes, so we content ourselves with a small remark. Let us express the basic geometric quantities of an isothermal regular parametrized minimal surface $\varphi : U \rightarrow S$ in terms of f, g . We have

$$E = G = \lambda^2, \quad F = 0,$$

where

$$\begin{aligned}\lambda^2 &= \frac{1}{2} \sum_{j=1}^3 |f_j|^2 \quad \text{by (2.23)} \\&= \frac{1}{4} |f|^2 |1 + g|^2 + \frac{1}{4} |f|^2 |1 - g|^2 + |fg|^2 \\&= \left(\frac{|f|(1 + |g|^2)}{2} \right)^2.\end{aligned}$$

Moreover,

$$\begin{aligned}\varphi_u \times \varphi_v &= (\Im\{f_2\bar{f}_3\}, \Im\{f_3\bar{f}_1\}, \Im\{f_1\bar{f}_2\}) \\ &= \frac{|f|^2(1+|g|^2)}{4} (2\Re g, 2\Im g, |g|^2 - 1),\end{aligned}$$

and

$$\|\varphi_u \times \varphi_v\| = \sqrt{EG - F^2} = \lambda^2,$$

so

$$N = \left(\frac{2\Re g}{|g|^2 + 1}, \frac{2\Im g}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right).$$

Recall that *stereographic projection* $\pi : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbf{C}$ is the map

$$\pi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3},$$

and its inverse is

$$\pi^{-1}(z) = \left(\frac{2\Re z}{|z|^2 + 1}, \frac{2\Im z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Hence

$$N = \pi^{-1} \circ g. \quad (2.31)$$

Proposition 2.32 *Let $\varphi : U \rightarrow S$ be an isothermal regular parametrized minimal surface, where U is the entire ζ -plane. Then either S lies in a plane, or the image of the Gauss map takes on all values with at most two exceptions.*

Proof. If S does not lie in a plane, we can construct the function $g(\zeta)$ which is meromorphic on the entire ζ -plane; by Picard's theorem, it either takes all values with at most two exceptions, or else it is constant. Eqn. (2.31) shows that the same alternative applies to N , and in the latter case S lies in a plane. \square

2.7.3 Local existence of isothermal parameters for minimal surfaces

Lemma 2.33 *Let S be a minimal surface. Then every point of S lies in the image of an isothermal parametrization of S .*

Proof. Let $p \in S$. First all, we can find a neighborhood of p in which S is the graph of a smooth function which, by relabeling coordinates, can be assumed in the form $z = h(x, y)$ for $(x, y) \in U$ (Check!). The minimal equation for graphs is easily computed to be

$$(1 + h_y^2)h_{xx} - 2h_x h_y h_{xy} + (1 + h_x^2)h_{yy} = 0.$$

We then have equation

$$\frac{\partial}{\partial x} \frac{1 + h_y^2}{W} = \frac{\partial}{\partial y} \frac{h_x h_y}{W}.$$

satisfied on U , where $W = \sqrt{1 + h_x^2 + h_y^2}$. Taking U simply-connected, this implies that we can find a smooth function Φ on U with

$$\frac{\partial \Phi}{\partial x} = \frac{h_x h_y}{W}, \quad \frac{\partial \Phi}{\partial y} = \frac{1 + h_y^2}{W}.$$

Introduce new coordinates

$$\bar{x} = x, \quad \bar{y} = \Phi(x, y).$$

One checks

$$\frac{\partial x}{\partial \bar{x}} = 1, \quad \frac{\partial x}{\partial \bar{y}} = 0, \quad \frac{\partial y}{\partial \bar{x}} = -\frac{h_x h_y}{1 + h_y^2}, \quad \frac{\partial y}{\partial \bar{y}} = \frac{W}{1 + h_y^2},$$

and the coefficients of the second fundamental form with respect to \bar{x}, \bar{y} are

$$\bar{E} = \bar{G} = \frac{W^2}{1 + h_y^2}, \quad \bar{F} = 0,$$

as desired. □