Chapter 1

Surfaces: basic definitions

A *regular parametrized surface* is a smooth mapping $\varphi : U \to \mathbf{R}^3$, where U is an open subset of $\mathbf{R}^2$, of maximal rank. This is equivalent to saying that the rank of φ is 2 or that φ is an immersion. Such a φ is called a *parametrization*.

Let (u, v) be coordinates in $\mathbf{R}^2$, (x, y, z) be coordinates in $\mathbf{R}^3$. Then

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$x(u, v)$, $y(u, v)$, $z(u, v)$ admit partial derivatives of all orders and the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank two. This is equivalent to

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(y, z)}{\partial(u, v)} \neq 0 \quad \text{or} \quad \frac{\partial(z, x)}{\partial(u, v)} \neq 0.$$

A *surface* is a subset S of $\mathbf{R}^3$ satisfying:

(1) $S = \cup_{i \in I}$, where V_i is an open subset of S and $\varphi_I : U_i \subset \mathbf{R}^2 \to \varphi_i(U_i) = V_i$ is a parametrization. In other words, every point $p \in S$ lies in an open subset $W \subset \mathbf{R}^3$ such that $W \cap S$ is the image of a smooth immersion of an open subset of $\mathbf{R}^2$ into $\mathbf{R}^3$.

(2) Each $\varphi_i : U_i \to V_i$ is a homeomorphism. The continuity of φ_i^{-1} means that for any V_i, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\varphi_i^{-1}(\underbrace{B(q, \delta)}_{\text{bola em } \mathbf{R}^3} \cap V_i) \subset \underbrace{B(\varphi_i^{-1}(q), \epsilon)}_{\text{bola em } \mathbf{R}^2}.$$

1.1 Examples

1. The graph of a smooth function $f : U \to \mathbf{R}$, where $U \subset \mathbf{R}^2$ is open, is a regular parametrized surface, where the parametrization is given by $\varphi(u,v) = (u, v, f(u,v))$. Note that

$$(d\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$

has rank two.

2. If $S \subset \mathbf{R}^3$ is a union of graphs as in (1) (with respect to any one of the three coordinate planes), then S is a surface. It only remains to check that the parametrizations constricted in (1) are homeomorphisms. But this follows from the fact that $\varphi^{-1} = \pi|_{\varphi(U)}$ is continuous, where $\pi(x,y,z) = (x,y)$ is continuous.

3. The unit sphere is defined as

$$S^2 = \{(x,y,z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

It is clearly a union of (six) graphs: $z = \pm\sqrt{1 - x^2 - y^2}$, $y = \pm\sqrt{1 - x^2 - z^2}$, $x = \pm\sqrt{1 - y^2 - z^2}$.

1.2 Inverse images of regular values

Let $F : W \to \mathbf{R}^3$ be a smooth map, where $W \subset \mathbf{R}^3$ is open. A point $p \in W$ is called a *critical point* of F is $dF_p = 0$; otherwise, it is called a *regular point.* A point $q \in \mathbf{R}$ is called a *critical value* of F if there exists a critical point of F in $F^{-1}(q)$; otherwise, it is called a *regular value.*

Theorem 1.1 *If q is a regular value of F and $F^{-1}(q) \neq \varnothing$, then $S = F^{-1}(q)$ is a surface.*

Proof. It suffices to show that S is a union of graphs. Let $p = (x_0, y_0, z_0) \in S$. Then $dF_p = \begin{pmatrix} \frac{\partial F}{\partial x}(p) \\ \frac{\partial F}{\partial y}(p) \\ \frac{\partial F}{\partial z}(p) \end{pmatrix} \neq 0$. Without loss of generality, asssume that $\frac{\partial F}{\partial z}(p) \neq 0$. By the implict function theorem, tehre exist open neighborhoods $\tilde{V}$ of (x_0, y_0, z_0) in $\mathbf{R}^3$ and U of (x_0, y_0) in $\mathbf{R}^2$ and a smooth function $f : U \to R$ such that $F(x,y,z) = q$, $(x,y,z) \in \tilde{V}$ if and only if $z = f(x,y)$, $(x,y) \in U$. Hence $p \in V = \tilde{V} \cap S$ is the graph of f. □

1.3 More examples

5. Spheres cam also be seen as inverse images of regular values. Let $F(x,y,z) = x^2 + y^2 + z^2$. Then $(dF_{(x,y,z)})^t = (2x\ 2y\ 2z) = (0\ 0\ 0)$ if and only if $(x,y,z) =$

$(0,0,0)$. Since $(0,0,0) \notin F^{-1}(r^2)$ for $r > 0$, we have that the spehre $F^{-1}(r^2)$ of radius $r > 0$ is a surface. Similarly, the ellipsoides $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (a, b, $c > 0$) are surfaces.

6. The hyperboloids $x^2 + y^2 - z^2 = r^2$ (one sheet) and $x^2 + y^2 - z^2 = -r^2$ (two sheets) are surfaces, $r > 0$. The cone $x^2 + y^2 - z^2 = 0$ is not a surface in a neighborhood of its vertex $(0,0,0)$.

7. The tori of revolution are surfaces given by the equation $z^2 + (\sqrt{x^2 + y^2} - a)^2 = r^2$, where a, $r > 0$.

8. More generally, one can consider surfaces of revolution. Let $\gamma(t) = (f(t), 0, g(t))$ be a regular parametrized curve, $t \in (a,b)$. Define

$$\varphi(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u,v) \in (a,b) \times (v_0, v_0 + 2\pi)$. One can cover the surface by varying v_0 in $\mathbf{R}$. But there are conditions on γ for φ to be an immersion. One has

$$\frac{\partial(x,y)}{\partial(u,v)} = ff', \quad \frac{\partial(y,z)}{\partial(u,v)} = -fg'\cos v, \quad \frac{\partial(z,x)}{\partial(u,v)} = -fg'\sin v,$$

so

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 = f^2 ||\dot{\gamma}||^2,$$

and φ is an immersion if and only if $f > 0$. Note also that φ is injective if and only if γ is injective. One also checks that φ^{-1} is continuous by writing its explicit expression.

1.4 Change of parameters

Theorem 1.2 *Let $S \subset \mathbf{R}^3$ be a surface and let $\varphi : U \to \varphi(U)$, $\psi : V \to \psi(V)$ be two parametrizations of S, where U, $V \subset \mathbf{R}^2$ are open. Then the* change of parameters $h = \varphi^{-1} \circ \psi : \psi^{-1}(\varphi(U)) \to \varphi^{-1}(\psi(V))$ *is a diffeomorphism between open sets of* $\mathbf{R}^2$.

Proof. h is a homeomorphism because it is the composite map of two homeomorphisms. Note that a similar argument cannot be used to say that h is smooth, because it does not make sense (yet) to say that φ^{-1} is smooth.

Let $p = (u_0, v_0) \in U$, $q \in V$, $\varphi(p) = \psi(q)$. Since φ is an immersion, $d\varphi$ has rank two and we may assume WLOG that $\frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0$. Write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in U$ and define

$$\Phi(u,v,w) = (x(u,v), y(u,v), z(u,v) + w),$$

where $(u,v,w) \in U \times \mathbf{R}$. Then $\Phi : U \times \mathbf{R} \to \mathbf{R}^3$ is smooth and

$$\det(d\Phi_{(u_0,v_0,0)}) = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{array} \right|_{(u_0,v_0,0)} = \frac{\partial(x,y)}{\partial(u,v)}(p) \neq 0.$$

Since $d\Phi_{(u_0,v_0,0)}$ is non-singular, by the inverse function theorem, Φ^{-1} is defined and is smooth on some open neighborhood W of $\varphi(p)$ in $\mathbf{R}^3$. Since $\Phi|_{U\times\{0\}} = \varphi$, we have that $\Phi^{-1}_{\varphi(U)\cap W} = \varphi^{-1}|_{\varphi(U)\cap W}$. Since $W \cap \varphi(U)$ is open in S and ψ is a homeomorphism, $\psi^{-1}(W \cap \varphi(U)) \subset V$ is open. Now

$$h|_{\psi^{-1}(W)} = \varphi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))} = \Phi^{-1} \circ \psi|_{\psi^{-1}(W\cap\varphi(U))}$$

is smooth, because it is the composite map of smooth maps.

Similarly, one sees that h^{-1} is smooth by reversing the rôles of φ and ψ in the argument above. Hence h is a diffeomorphism. □

Corollary 1.3 *Let $S \subset \mathbf{R}^3$ be a surface and suppose $f : W \to \mathbf{R}^3$ is a smooth map defined on the open subset $W \subset \mathbf{R}^m$ such that $f(W) \subset S$. Then $\varphi^{-1} \circ f : W \to \mathbf{R}^2$ is smooth for every parametrization $\varphi : U \to \varphi(U)$ of S.*

Proof. If Φ is as in the proof of the theorem, we have that $\varphi^{-1} \circ f = \Phi^{-1} \circ f$ is the composite of smooth maps between Euclidean spaces. □

As an application of the smoothness of change of parameters, we can make the following definition. Let S be a surface. An application $f : S \to \mathbf{R}^n$ is smooth at a point $p \in S$ if $f \circ \varphi : U \to \mathbf{R}^n$ is smooth at $\varphi^{-1}(p) \in U$, for some parametrization $\varphi : U \to \varphi(U)$ of S with $p \in \varphi(U)$. Note that if $\psi : V \to \psi(V)$ is another parametrization of S with $p \in \psi(V)$, then $f \circ \psi$ is smooth at $\psi^{-1}(p)$ if and only if $f \circ \varphi$ is smooth at $\varphi^{-1}(p)$, because

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi^{-1} \circ \psi)$$

and the change of parameters $\varphi^{-1} \circ \psi$ is smooth.

Example 1.4 If S is a surface and $F : \mathbf{R}^3 \to \mathbf{R}$ is smooth, then the restriction $F|_S : S \to \mathbf{R}$ is smooth. In fact, $f \circ \varphi = F \circ \varphi$ is smooth for any parametrization φ of S. As special cases, we can take the *height function* relative to a, $F(x) = \langle x, a\rangle$, where $a \in \mathbf{R}^3$ is a fixed vector; or the *distance function* from q, $F(x) = ||x - q||^2$, where $q \in \mathbf{R}^3$ is a fixed point.

In particular, if $f : S \to \mathbf{R}^3$ is smooth at $p \in S$ and $\tilde{S} \subset \mathbf{R}^3$ is a surface such that $f(S) \subset \tilde{S}$, then we say that $f : S \to \tilde{S}$ is smooth at p.

1.5 Tangent plane

Let $S \subset \mathbf{R}^3$ be a surface. Recall that a smooth curve $\gamma : I \subset \mathbf{R} \to S$ is simply a smooth curve $\gamma : I \to \mathbf{R}^3$ such that $\gamma(I) \subset S$. Fix a point $p \in S$. A *tangent vector* to S at p is the tangent vector $\dot{\gamma}(0)$ to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. The *tangent plane* to S at p is the collection of all tangent vectors to S at p.

Proposition 1.5 *The tangent space T_pS is the image of the differential*

$$d\varphi_a : \mathbf{R}^2 \to \mathbf{R}^3, \tag{1.6}$$

where $\varphi : U \to \varphi(U)$ is any parametrization of S with $p = \varphi(a)$ and $a \in U$.

Proof. Any vector in the image of (1.6) is of the form $d\varphi_a(w_0)$ for some $w_0 \in \mathbf{R}^2$ and therefore is the tangent vector at 0 of the smooth curve $t \mapsto \varphi(a + tw_0)$.

Conversely, suppose $w = \dot\gamma(0)$ is tangent to a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$. By Corollary 1.3, $\eta := \varphi^{-1} \circ \gamma : (-\epsilon, \epsilon) \to U \subset \mathbf{R}^2$ is a smooth curve in $\mathbf{R}^2$ with $\eta(0) = a$. Note that $\gamma = \varphi \circ \eta$. By the chain rule

$$v = d\varphi_a(\dot\eta(0)) \tag{1.7}$$

lies in the image of (1.6). □

Corollary 1.8 *The tangent plane T_pS is a 2-dimensional vector subspace of $\mathbf{R}^3$. For any parametrization $\varphi : U \to \varphi(U)$ of S with $p = \varphi(a)$, $a \in U$,*

$$\{\tfrac{\partial\varphi}{\partial u}(a), \tfrac{\partial\varphi}{\partial v}(a)\} \tag{1.9}$$

is a basis of T_pS.

It is also convenient to write $\varphi_u := \frac{\partial\varphi}{\partial u}$ and $\varphi_v := \frac{\partial\varphi}{\partial v}$.

Consider a tangent vector $w \in T_pS$, say $w = \dot\gamma(0)$ where $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, as in the proof of Proposition 1.5. Then $\eta = \varphi^{-1} \circ \gamma$ is a smooth curve in $\mathbf{R}^2$ which we may write as $\eta(t) = (u(t), v(t))$. Since $\dot\eta(0) = (u'(0), v'(0))$, eqn. (1.7) yields that

$$w = u'(0)\varphi_u(a) + v'(0)\varphi_v(a),$$

namely, $u'(0)$, $v'(0)$ are the coordinates of w in the basis (1.9). This remark also shows that another smooth curve $\bar\gamma : (-\epsilon, \epsilon) \to S$ represents the same w if and only if $\bar\eta(t) = \varphi^{-1} \circ \bar\gamma(t) = (\bar u(t), \bar v(t))$ satisfies $(\bar u'(0), \bar v'(0)) = (u'(0), v'(0))$.

With the same notation as above, suppose now that $f : S \to \tilde S$ is a smooth map at $p \in S$. Note that $f \circ \gamma$ is a smooth curve in $\tilde S$. The *differential* of f at p is the map

$$df_p : T_pS \to T_{f(p)}\tilde S$$

that maps $w = \dot\gamma(0) \in T_pS$ to the tangent vector $\dot{\tilde\gamma}(0)$, where $\tilde\gamma = f \circ \gamma$. We check that $df_p(w)$ does not depend on the choice of curve γ. Let $\varphi : U \to \varphi(U) = V$, $\tilde\varphi : \tilde U \to \tilde\varphi(\tilde U) = \tilde V$ be parametrizations of S, $\tilde S$, resp., with $p = \varphi(a)$, $a \in U$, $f(p) = \tilde\varphi(\tilde a)$, $\tilde a \in \tilde U$, and such that $f(V) \subset \tilde V$. Consider the local representation of f,

$$g = \tilde\varphi^{-1} \circ f \circ \varphi : U \to \tilde U,$$

and write

$$g(u, v) = (g_1(u, v), g_2(u, v))$$

for $(u,v) \in U \subset \mathbf{R}^2$. Then

$$\tilde{\gamma}(t) = \tilde{\varphi}\,(\,g_1(u(t),v(t)), g_2(u(t),v(t))\,)\,,$$

so

$$\dot{\tilde{\gamma}}(0) = \left(\tfrac{\partial g_1}{\partial u}\,u'(0) + \tfrac{\partial g_1}{\partial v}\,v'(0)\right)\tilde{\varphi}_{\tilde{u}} + \left(\tfrac{\partial g_2}{\partial u}\,u'(0) + \tfrac{\partial g_2}{\partial v}\,v'(0)\right)\tilde{\varphi}_{\tilde{v}}.$$

This relation shows that $\dot{\tilde{\gamma}}(0)$ depends only on $u'(0)$, $v'(0)$ and hence has the same value for any smooth curve representing w. This relation can also be rewritten as

$$df_p(w) = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix},$$

which shows that df_p is a linear map whose matrix with respect to the bases $\{\varphi_u, \varphi_v\}$, $\{\tilde{\varphi}_{\tilde{u}}, \varphi_{\tilde{v}}\}$ is the 2 by 2 matrix above.

Example 1.10 If S is a surface given as the inverse image under $F : \mathbf{R}^3 \to \mathbf{R}$ of a regular value, then $T_pS = \ker(dF_p)$ for every $p \in S$. In fact, if $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve with $\gamma(0) = p$, then $F(\gamma(t))$ is constant for $t \in (-\epsilon, \epsilon)$. By the chain rule, $dF_p(\dot{\gamma}(0)) = 0$. This proves the inclusion $T_pS \subset \ker(dF_p)$ and hence the equality by dimensional reasons.