Chapter 3

Surfaces: intrinsic geometry

It is the geometry of objects associated to the surface which depend only on the first fundamental form. Obvious examples are lengths of curves, angles between tangent vectors and areas of regions in the the surface. Less obvious examples are geodesics (yet to be defined) and the Gaussian curvature (Theorema Egregium). We will first discuss local questions.

3.1 Directional derivative

We start by recalling the concept of directional derivative of vector fields on $\mathbf{R}^n$. Let $Y : W \to \mathbf{R}^n$ be a smooth vector field defined on an open subset W of $\mathbf{R}^n$, let $v \in \mathbf{R}^n$ be a fixed vector and $p \in W$. Then the *directional derivative* of Y along v at p is

$$D_vY|_p = dY_p(v) = \lim_{t\to 0} \frac{Y(p+tv) - Y(p)}{t}.$$

Note that if $\gamma : (-\epsilon, \epsilon) \to \mathbf{R}^n$ is a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$\begin{aligned} \lim_{t\to 0} \frac{Y(\gamma(t)) - Y(\gamma(0))}{t} &= \frac{d}{dt}\Big|_{t=0} Y(\gamma(t)) \\ &= dY_{\gamma(0)}(\gamma'(0)) \qquad \text{(by the chain rule)} \\ &= dY_p(v) \\ &= D_vY|p, \end{aligned}$$

in other words, $D_vY|_p$ depends only on the values of Y along a smooth curve through p with velocity v.

As a particular case, consider the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ and denote by $(x^1, \ldots, x^n)$ the standard coordinates in $\mathbf{R}^n$. Then

$$D_{e_i}Y|_p = \tfrac{\partial Y}{\partial x^i}(p)$$

and, if $v = \sum_i v^i e_i$, then

$$D_v Y = \sum_i v^i \frac{\partial Y}{\partial x_i}.$$

A word about notation: If $X : W \to \mathbf{R}^n$ is another smooth vector field, then $X(p)$ is a vector in $\mathbf{R}^n$ and we write

$$D_X Y|_p = D_{X(p)} Y|_p.$$

3.2 Vector fields on surfaces

Let S be a surface in $\mathbf{R}^3$, and let $V \subset S$ be an open subset. A (smooth) *vector field* on V is a (smooth) map $X : V \to \mathbf{R}^3$. We say that X is *tangent to* S (resp. *normal to* S) if $X(p) \in T_pS$ (resp. $X(p) \perp T_pS$) for $p \in V$.

Let $\varphi : U \subset \mathbf{R}^2 \to \varphi(U) = V \subset S$ be a parametrization. In the following, we use (u^1, u^2) to denote coordinates on the parameter plane $\mathbf{R}^2$ and (x^1, x^2, x^3) to denote coordinates in the ambient $\mathbf{R}^3$. According to the above definition,

$$\frac{\partial \varphi}{\partial u^1} \circ \varphi^{-1}, \frac{\partial \varphi}{\partial u^2} \circ \varphi^{-1} : V \to \mathbf{R}^3$$

are vector fields on V tangent to S. For the sake of convenience, henceforth we will abuse terminology and say that $\frac{\partial \varphi}{\partial u^1}$, $\frac{\partial \varphi}{\partial u^2}$ are vector fields tangent to S. Note then that any vector field X on V tangent to S can be written as a linear combination

$$X = a^1 \frac{\partial \varphi}{\partial u^1} + a^2 \frac{\partial \varphi}{\partial u^2}$$

where a^1, $a^2 : V \to \mathbf{R}$. It is an easy exercise to check that X is smooth if and only if a^1, a^2 are smooth functions (do it!). Similarly, any smooth vector field on V normal to S is of the form

$$X = b \frac{\partial \varphi}{\partial u^1} \times \frac{\partial \varphi}{\partial u^2}$$

for a smooth function $b : V \to \mathbf{R}$.

3.3 Covariant derivative

In this section we explain how to differentiate a vector field tangent to a surface along another tangent vector field to obtain a third tangent vector field. Let S be a surface in $\mathbf{R}^3$, $V \subset S$ an open subset, $p \in V$. Consider vector fields X, Y on V such that X is tangent to S. Our previous discussion about the directional derivative shows that $D_X Y|_p$ is well defined, namely, it equals $\frac{d}{dt}|_{t=0} Y(\gamma(t))$ where $\gamma : (-\epsilon, \epsilon) \to S$ is smooth and $\gamma(0) = p$, $\gamma'(0) = v$. The association

$$p \mapsto D_X Y|_p$$

is a vector field on V. As a special case, if $\varphi : U \to V$ is a parametrization of S and $\varphi(u) = p$ for some $u \in U$,

$$D_{\frac{\partial \varphi}{\partial u^i}} Y|_p = \frac{\partial (Y \circ \varphi)}{\partial u^i}(u).$$

Further, if $Y = \frac{\partial\varphi}{\partial u^j}$,

$$D_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial\varphi}{\partial u^i}|_p = \frac{\partial^2\varphi}{\partial u^i u^j}(u).$$

Back to the general case, we now assume that both X and Y are tangent to S. The *covariant derivative* of Y along X at p is

$$\nabla_X Y|_p = (D_X Y|_p)^\top$$

where $(\cdot)^\top$ denotes the orthogonal projection onto T_pS. The symbol "∇" is read "nabla". In this way, the association

$$p \mapsto \nabla_X Y|_p$$

is a vector field on V tangent to S.

Lemma 3.1 (Properties of D and ∇) *Let X, $\tilde{X}$, Y, $\tilde{Y}$ be vector fields tangent to S and let f be a smooth function on S. Then:*

1. $\nabla_{fX+\tilde{X}} Y = f\nabla_X Y + \nabla_{\tilde{X}} Y$;
2. $\nabla_X(Y + \tilde{Y}) = \nabla_X Y + \nabla_X \tilde{Y}$;
3. $\nabla_X(fY) = df(X)Y + f\nabla_X Y$;
4. $X\langle Y, \tilde{Y}\rangle = \langle \nabla_X Y, \tilde{Y}\rangle + \langle Y, \nabla_X \tilde{Y}\rangle$;

and the same identities hold for ∇ replaced by D.

In this statement, fX denotes the tangent vector field $p \mapsto f(p)X(p)$, and $\langle Y, \tilde{Y}\rangle$ denotes the scalar function $p \mapsto \langle Y(p), \tilde{Y}(p)\rangle$ so that $X\langle Y, \tilde{Y}\rangle(p)$ denotes the the directional derivative in the direction of $X(p)$.

Proof. We prove (3) and (4) and leave the rest as an exercise. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth curve with $\gamma(0) = p$, $\gamma'(0) = X(p)$. Then

$$\begin{aligned} D_X(fY)|_p &= \frac{d}{dt}|_{t=0}(fY)(\gamma(t)) \\ &= \frac{d}{dt}|_{t=0} f(\gamma(t))Y(\gamma(t)) \\ &= \frac{d}{dt}|_{t=0} f(\gamma(t)) \cdot Y(\gamma(0)) + f(\gamma(0))\frac{d}{dt}|_{t=0} Y(\gamma(t)) \\ &= df_p(X(p))Y(p) + f(p)D_X Y|_p. \end{aligned}$$

Hence

$$\begin{aligned} \nabla_X(fY)|_p &= (D_X(fY))^\top \\ &= (df(X)Y + fD_X Y)^\top \\ &= df(X)Y^\top + f(\nabla_X Y)^\top \qquad \text{(orthogonal projection is linear)} \\ &= df(X)Y + f\nabla_X Y \qquad (Y \text{ is tangent}). \end{aligned}$$

Next,

$$\begin{aligned}
X\langle Y,\tilde{Y}\rangle(p) &= \frac{d}{dt}|_{t=0}\langle Y(\gamma(t)),\tilde{Y}(\gamma(t))\rangle \\
&= \langle\frac{d}{dt}|_{t=0}Y(\gamma(t)),\tilde{Y}(\gamma(0))\rangle+\langle Y(\gamma(0)),\frac{d}{dt}|_{t=0}\tilde{Y}(\gamma(t))\rangle \\
&= \langle D_XY|_p,\tilde{Y}(p)\rangle+\langle Y(p),D_X\tilde{Y}|_p\rangle \\
&= \langle(D_XY|_p)^{\top},\tilde{Y}(p)\rangle+\langle Y(p),\left(D_X\tilde{Y}|_p\right)^{\top}\rangle \qquad (Y,\tilde{Y}\text{ are tangent}) \\
&= \langle\nabla_XY|_p,\tilde{Y}(p)\rangle+\langle Y(p),\nabla_X\tilde{Y}|_p\rangle,
\end{aligned}$$

as we wished. □

Let X, Y be tangent vector fields. Of course, ν is a normal vector field, so $\langle Y,\nu\rangle=0$ and Lemma 3.1(4) says that

$$\langle D_XY,\nu\rangle+\langle Y,D_X\nu\rangle=0.$$

Le $A=-d\nu$ be the Weingarten operator. Then $A(X)=-d\nu(X)=-D_X\nu$ and we have

$$\begin{aligned}
\nabla_XY &= (D_XY)^{\top} \\
&= D_XY-\underbrace{\langle D_XY,\nu\rangle\nu}_{\text{normal component}} \\
&= D_XY+\langle Y,D_X\nu\rangle\nu \\
&= D_XY-\langle Y,AX\rangle\nu \\
&= D_XY-\Pi(X,Y)\nu.
\end{aligned}$$

Hence we arrive at the *Gauss formula*

$$D_XY=\nabla_XY+\Pi(X,Y)\nu.$$

In the remaining of this section, we show that the covariant derivative ∇ of a surface S is an intrinsic object, namely, it is completely determined by the first fundamental form I; in particular, locally isometric surfaces have the same covariant derivative. This is not an obvious assertion in view of the fact that ∇_XY was defined as the orthogonal projection onto the surface of the directional derivative D_XY, and so the ambient space $\mathbf{R}^3$ was used in this definition.

Let X, Y be vector fields on S. Since we are dealing with a local assertion, we can work in the image of a parametrization φ of S and write

$$X=\sum_{i=1}^{2}a^i\tfrac{\partial\varphi}{\partial u^i},\qquad Y=\sum_{i=1}^{2}b^j\tfrac{\partial\varphi}{\partial u^j}.$$

By using Lemma 3.1, we first obtain a local expression for $\nabla_X Y$:

$$\begin{aligned} \nabla_X Y &= \sum_{i,j} a^i \nabla_{\frac{\partial\varphi}{\partial u^i}} \left(b^j \frac{\partial \varphi}{\partial u^j} \right) \\ &= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial u^i} \frac{\partial \varphi}{\partial u^j} + b^j \nabla_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} \right). \end{aligned} \tag{3.2}$$

Since $\nabla_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j}$ is tangent to S, we can write

$$\nabla_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} = \sum_k \Gamma_{ij}^k \frac{\partial \varphi}{\partial u^k}, \tag{3.3}$$

for some smooth functions Γ_{ij}^k, the so called *Christofell symbols (of the second kind)*. Substituting into (3.2),

$$\nabla_X Y = \sum_i a^i \left(\frac{\partial b^k}{\partial u^i} + \sum_j b^j \Gamma_{ij}^k \right) \frac{\partial \varphi}{\partial u^k}, \tag{3.4}$$

which shows that ∇ depends only on $\{\Gamma_{ij}^k\}$.

We can also define the *Christofell symbols of the first kind* by putting

$$\Gamma_{ij,k} = I \left(\nabla_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j}, \frac{\partial \varphi}{\partial u^k} \right).$$

By using (3.3), we have

$$\begin{aligned} \Gamma_{ij,k} &= \sum_\ell \Gamma_{ij}^\ell I \left(\frac{\partial \varphi}{\partial u^\ell}, \frac{\partial \varphi}{\partial u^k} \right) \\ &= \sum_\ell \Gamma_{ij}^\ell g_{\ell k}. \end{aligned}$$

Multiplying through by g^{km}, where (g^{km}) denotes the inverse matrix of $I = (g_{ij})$, we get

$$\Gamma_{ij}^m = \sum_k \Gamma_{ij,k} g^{km}.$$

Hence $\{\Gamma_{ij}^k\}$ depends only on $\{\Gamma_{ij,k}\}$ and I.

In order to complete our argument, we need to show that $\{\Gamma_{ij,k}\}$ depends only on I. It is important to notice that Γ_{ij}^k (and $\Gamma_{ij,k}$) is symmetric with respect to the indices i, j, namely,

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

This is immediate from

$$\nabla_{\frac{\partial\varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} = \left(\frac{\partial^2 \varphi}{\partial u^i \partial u^j} \right)^\top.$$

Next, by using Lemma 3.1(4), we write

$$\frac{\partial g_{ij}}{\partial u^k} = \left\langle \nabla_{\frac{\partial\varphi}{\partial u^k}} \frac{\partial\varphi}{\partial u^i}, \frac{\partial\varphi}{\partial u^j} \right\rangle + \left\langle \frac{\partial\varphi}{\partial u^i} \nabla_{\frac{\partial\varphi}{\partial u^k}} \frac{\partial\varphi}{\partial u^j} \right\rangle,$$

so

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ki,j} + \Gamma_{kj,i}.$$

Doing cyclic permutations on (i, j, k), we also obtain

$$\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{ij,k} + \Gamma_{ik,j}$$

and

$$\frac{\partial g_{ki}}{\partial u^j} = \Gamma_{jk,i} + \Gamma_{ji,k}.$$

Summing the last two equations and subtracting the first one finally yields that

$$2\Gamma_{ij,k} = \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j}, \tag{3.5}$$

which completes the proof that ∇ depends only on $I = (g_{ij})$.

Example 3.6 Consider a surface of revolution parametrized as in section 2.5. Since $F = 0$, we can write

$$\nabla_{\varphi_u}\varphi_u = (D_{\varphi_u}\varphi_u)^\top = (\varphi_{uu})^\top = \langle \varphi_{uu}, \varphi_u \rangle \frac{\varphi_u}{E} + \langle \varphi_{uu}, \varphi_v \rangle \frac{\varphi_v}{G}.$$

Using the formulas from section 2.5,

$$\langle \varphi_{uu}, \varphi_u \rangle = f'f'' + g'g'' = \frac{1}{2}(f'^2 + g'^2)' = 0$$

and

$$\langle \varphi_{uu}, \varphi_v \rangle = 0.$$

Hence

$$\nabla_{\varphi_u}\varphi_u = 0.$$

Similarly, one computes that

$$\nabla_{\varphi_u}\varphi_v = \nabla_{\varphi_v}\varphi_u = \frac{f'}{f}\varphi_v$$

and

$$\nabla_{\varphi_v}\varphi_v = -ff'\varphi_u.$$

We thus obtain

$$\Gamma^1_{11} = \Gamma^2_{11} = 0, \quad \Gamma^1_{12} = 0, \quad \Gamma^2_{12} = \frac{f'}{f}, \quad \Gamma^1_{22} = -ff', \quad \Gamma^2_{22} = 0.$$

In particular, all Christofell symbols vanish along the parallels $u =$ constant corresponding to critical points of f.

3.4 The Lie bracket

If X, Y are vector fields on $\mathbf{R}^n$ or on a surface, then, in general, $D_X Y \neq D_Y X$. For instance, take $X = x^2 \cdot e_1$, $Y = e_2$. Then

$$D_X Y = D_{x^2 e_1} e_2 = x^2 D_{e_1} e_2 = 0,$$

where $D_{e_1} e_2 = 0$ because e_2 is constant, and

$$D_Y X = D_{e_2}(x^2 \cdot e_1) = dx^2(e_2) e_1 + x^2 D_{e_2} e_1 = e_1,$$

where $dx_2(e_2) = \frac{\partial x_2}{\partial x_2} = 1$. In general, the lack of comutativity is measured by the *Lie bracket*

$$[X, Y] := D_X Y - D_Y X.$$

If X, Y are tangent to S, then

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= (D_X Y - II(X,Y)\nu) - (D_Y X - II(Y,X)\nu) \\ &= D_X Y - D_Y X \\ &= [X, Y]. \end{aligned}$$

As another example, for a parametrization φ of S we have

$$[\tfrac{\partial \varphi}{\partial u^i}, \tfrac{\partial \varphi}{\partial u^j}] = D_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} - D_{\frac{\partial \varphi}{\partial u^j}} \frac{\partial \varphi}{\partial u^i} = \frac{\partial^2 \varphi}{\partial u^i \partial u^j} - \frac{\partial^2 \varphi}{\partial u^j \partial u^i} = 0.$$

This example shows that given a tangent frame $\{X_1, X_2\}$ to a surface, a necessary condition for it to be of the form $\{\frac{\partial \varphi}{\partial u^1}, \frac{\partial \varphi}{\partial u^2}\}$ for some parametrization φ is that $[X_1, X_2] = 0$.

Let X, Y be vector fields on S. We obtain a local expression for their Lie bracket. In the image of a parametrization, we can write

$$X = \sum_{i=1}^{2} a^i \tfrac{\partial \varphi}{\partial u^i}, \qquad Y = \sum_{i=1}^{2} b^j \tfrac{\partial \varphi}{\partial u^j}.$$

Using (3.4) and the symmetry of Γ_{ij}^k with respect to i, j, we have, on the image of φ:

$$\begin{aligned} [X, Y] &= \nabla_X Y - \nabla_Y X \\ &= \sum_{i,j} \left(a^i \frac{\partial b^j}{\partial u^i} - b^i \frac{\partial a^j}{\partial u^i} \right) \frac{\partial \varphi}{\partial u^j}. \end{aligned}$$

3.5 Parallel transport

Let $S \subset \mathbf{R}^3$ be a surface and let $\gamma : I \to S$ be a smooth parametrized curve (nonnecessarily regular). A *(smooth) tangent vector field on S along γ* is a (smooth) map $X : I \to \mathbf{R}^3$ such that $X(t) \in T_{\gamma(t)} S$ for all $t \in I$.

Examples 3.7 1. If Y is a tangent vector field on S and $\gamma : I \to S$ is a parametrized curve, then $X(t) = Y(\gamma(t))$ defines a tangent vector field on S along γ.

2. If $\gamma : I \to S$ is a smooth parametrized curve, then $X = \gamma'$ defines a tangent vector field on S along γ.

The *covariant derivative* of a tangent vector field X on S along γ is defined to be

$$\frac{\nabla X}{dt}\Big|_t = \left(\frac{d}{dt}X(t)\right)^{\top}.$$

In example (1) above,

$$\begin{aligned}\frac{\nabla X}{dt}\Big|_t &= \left(\frac{d}{dt}Y(\gamma(t))\right)^{\top}\\ &= D_{\gamma'(t)}Y|_{\gamma(t)}.\end{aligned}$$

In example (2) above,

$$\frac{\nabla X}{dt}\Big|_t = (\gamma''(t))^{\top}.$$

A tangent vector field X on S along γ is said to be *parallel* if $\frac{\nabla X}{dt} \equiv 0$. A smooth parametrized curve $\gamma : I \to S$ is said to be a *geodesic* if γ' is parallel along γ.

In the sequel, we write a local expression of the equation $\frac{\nabla X}{dt} \equiv 0$. In the image of a parametrization φ of S, we can write

$$\gamma(t) = \varphi(u^1(t), u^2(t)), \qquad X(t) = \sum_{i=1}^{2} a^i(t)\frac{\partial\varphi}{\partial u^i}\Big|_{(u^1(t),u^2(t))}$$

where $t \in I$. Then

$$X'(t) = \sum_{i=1}^{2}(a^i)'\frac{\partial\varphi}{\partial u^i} + \sum_{i,j=1}^{2} a^i\frac{\partial^2\varphi}{\partial u^i\partial u^j}(u^j)'$$

and

$$\begin{aligned}\frac{\nabla X}{dt} &= (X'(t))^{\top}\\ &= \sum_{i=k}^{2}(a^i)'\frac{\partial\varphi}{\partial u^i} + \sum_{i,j=1}^{2} a^i(u^j)'\nabla_{\frac{\partial\varphi}{\partial u^i}}\frac{\partial\varphi}{\partial u^j}\\ &= \sum_{i=k}^{2}(a^k)'\frac{\partial\varphi}{\partial u^k} + \sum_{i,j,k=1}^{2} a^i(u^j)'\Gamma_{ij}^k\frac{\partial\varphi}{\partial u^k}\\ &= \sum_{k=1}^{2}\left[(a^k)' + \sum_{i,j=1}^{2}\Gamma_{ij}^k(u^i)'a^j\right]\frac{\partial\varphi}{\partial u^k}.\end{aligned}$$

Hence $\frac{\nabla X}{dt} \equiv 0$ is the following system of first order linear ordinary differential equations in $a^1(t)$, $a^2(t)$:

$$(a^k)'(t) + \sum_{i,j=1}^{2} \Gamma_{ij}^k(u^1(t), u^2(t))\,(u^i)'(t)\,a^j(t) = 0 \qquad (k = 1, 2). \tag{3.8}$$

The theorem on existence and uniqueness of solution of linear ordinary differential equations says that, given the initial values $a^1(t_0) = a_0^1$, $a^2(t_0) = a_0^2$, for some $t_0 \in I$, there exists a unique solution $(a^1(t), a^2(t))$ defined on I and satisfying the given initial values. Geometrically this means that, given $v \in T_{\gamma(t_0)}S$ and taking a parametrization $\varphi : U \to S$ around $\gamma(t_0)$ with $v = a_0^1 \frac{\partial \varphi}{\partial u^1} + a_0^2 \frac{\partial \varphi}{\partial u^2}$, the vector field $X(t) = a^1(t)\frac{\partial \varphi}{\partial u^1} + a^2(t)\frac{\partial \varphi}{\partial u^2}$ is the only parallel vector field along γ such that $X(t_0) = v$, defined on an interval centered at t_0 and lying in $\gamma^{-1}(\varphi(U))$. By covering the image of γ by finitely many opens sets $V_1, \ldots, V_n$, each of which the image of a parametrization, such that $V_i \cap V_{i+1} \neq \varnothing$, and applying this result to each one of $V_1, \ldots, V_n$ in order, we deduce

Proposition 3.9 *Given a smooth parametrized curve $\gamma : [a, b] \to S$ and a tangent vector $v \in T_{\gamma(a)}S$, there exists a unique tangent vector field $X : [a, b] \to \mathbf{R}^3$ on S along γ which is parallel and satisfies $X(a) = v$.*

The vector $X(b) \in T_{\gamma(b)}S$ is called the *parallel transport* of $v = X(a)$ along γ. The parallel transport along γ defines a map $P^\gamma : T_{\gamma(a)}S \to T_{\gamma(b)}S$ which is obviously linear, since the solutions to a linear ODE depend linearly on the initial values.

Proposition 3.10 *If X, Y are parallel vector fields along γ, then $\langle X(t), Y(t)\rangle$, $||X(t)||$ and the angle between $X(t)$ and $Y(t)$ are constant functions.*

Proof. We compute

$$\begin{aligned} \frac{d}{dt}\langle X(t), Y(t)\rangle &= \langle X'(t), Y(t)\rangle + \langle X(t), Y'(t)\rangle \\ &= \langle \frac{\nabla X}{dt}, Y(t)\rangle + \langle X(t), \frac{\nabla Y}{dt}\rangle \qquad \text{(since } X, Y \text{ are tangent)} \\ &= 0. \end{aligned}$$

Hence $\langle X(t), Y(t)\rangle$ is constant, and the other assertions follow. □

Corollary 3.11 *$P^\gamma : T_{\gamma(a)}S \to T_{\gamma(b)}S$ is a linear isometry.*

Examples 3.12 1. For the plane, $\varphi(u, v) = (u, v, 0)$ is a parametrization and $I = du^2 + dv^2$. Since the coefficients of I are constant, eqn. (3.5) yields that $\Gamma_{ij}^k = 0$ for all i, j, k. The equations of parallel transport are thus $(a^k)' = 0$, $k = 1, 2$. Hence the parallel vector fields along γ are the constant vector fields. In particular, the parallel transport along γ depends on the endpoints of γ, but not on the curve itself.

2. We consider the cone C of equation $z = k\sqrt{x^2+y^2}$ for $k > 0$ and $(x, y) \neq 0$. It is obviously a graph of a smooth function, so it is a surface. We can also parametrize it as a surface of revolution by taking the generating curve to be $\gamma(s) = (f(s), 0, g(s))$, where $f(s) = \frac{1}{\sqrt{k^2+1}}s$, $g(s) = \frac{k}{\sqrt{k^2+1}}s$. Then the Gaussian curvature $K = -f''/f = 0$.

In the sequel, we show that the cone is locally isometric to the plane. Note that the angle at the vertex of the cone is $\psi = \operatorname{arccot} k \in (0, \pi/2)$. Consider the open sector of the plane V given in polar coordinates by $r > 0$, $0 < \theta < 2\pi \sin\psi$. We define a map

$$\Phi : V \to C, \qquad F(r,\theta) = \left(r\cos\left(\frac{\theta}{\sin\psi}\right)\sin\psi, r\sin\left(\frac{\theta}{\sin\psi}\right)\sin\psi, r\cos\psi \right).$$

Then Φ is smooth and its image $\Phi(V)$ is the cone minus the geratrix $y = 0$, $x > 0$, $z = kx$. The inverse map

$$\Phi^{-1} : \Phi(V) \to V, \qquad \Phi(x,y,z) = \left(\frac{1}{\sin\psi}\sqrt{x^2+y^2}, \sin\psi \operatorname{arccot}\left(\frac{x}{y}\right) \right)$$

is also smooth, since it is smooth as a function of (x, y) for $y \neq 0$. Hence $\Phi : V \to \Phi(V)$ is a diffeomorphism. We finally show that Φ is an isometry, namely, the first fundamental forms of the plane and the cone coincide on points corresponding under Φ.

The open set V is a regular surface parametrized by $\varphi(u, v) = (u\cos v, u\sin v, 0)$, where $(u, v) \in U = (0, +\infty) \times (0, 2\pi \sin\psi)$, and the corresponding coefficients of the first fundamental form are then $E = 1$, $F = 0$, $G = u^2$. Since Φ is a diffeomorphism, $\tilde{\varphi} = \Phi \circ \varphi$ can be taken as a parametrization of $\Phi(V)$ and then the corresponding coeffcients of the first fundamental form are $\tilde{E} = 1$, $\tilde{F} = 0$, $\tilde{G} = u^2$. Since $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$, Φ is an isometry.

From the local expression (3.8), we see that parallel transport is an intrinsic object. Hence the parallel transport along a curve in the cone can be read off the parallel transport along the corresponding curve in the plane. Consider a parallel curve γ in the cone given by $z = z_0$; we compute the parallel transport of its initial tangent vector v after one turn around the cone. The corresponding curve $\tilde{\gamma}$ in the plane is an arc of a circle of angle $2\pi\sin\psi$. The tangent vector to $\tilde{\gamma}$ rotates by an angle of measure $2\pi \sin\psi$ after one turn, whereas the parallel transport is the identity. It follows that the parallel transport of v along γ is rotation by $-2\pi\sin\psi$.

3. Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{R}^3$. We describe parallel transport around a small circle γ of colatitude φ. There exists a cone which is tangent to the sphere along γ; the angle at the vertex of this cone is $\psi = \frac{\pi}{2} - \varphi$. Since the tangent spaces of the sphere and the cone coincide along γ, also parallel transport along γ is the same whether we view it as a curve in the sphere or in the cone. Therefore parallel transport along the small circle of colatitude φ is rotation of angle $-2\pi\cos\varphi$ (with respect to a suitable orientation). Taking $\varphi \to 0$, by continuity we see that parallel transport along the equator (after one complete turn) is the identity.

3.6 Geodesics

As we have already mentioned, a smooth parametrized curve $\gamma : I \to S$ is a *geodesic* if γ' is parallel along γ. This means that $0 = \frac{\nabla(\gamma')}{dt} = (\gamma'')^\top$, so the acceleration γ'' in $\mathbf{R}^3$ is everywhere normal to S. In other words, γ does not accelerate viewed from S, so that "geodsics are the straightest curves in S".

If γ is geodesic, then $||\gamma'||$ is constant by Proposition 3.10. There are two cases: either $||\gamma'|| = 0$ and γ is a constant curve; or $||\gamma'||$ is a nonzero constant and γ is regular and parametrized proportionally to arc-length.

Equations for geodesics contained in the image of a parametrization φ are immediately deduced from the equations for parallel vector fields (3.8); if $\gamma(t) = \varphi(u^1(t), u^2(t))$, we take $(a^i)' = (u^i)'$ and then

$$(u^k)'' + \sum_{i,j=1}^{2} \Gamma_{ij}^k (u^i)'(u^j)' = 0, \qquad (k = 1, 2). \tag{3.13}$$

This a system of second order non-linear ordinary differential equations. Those equations show that geodesics are intrinsic objects. On the other hand, the nonlinearity implies that in general geodesics are only defined locally. Namely, the theorem of existence and uniqueness for such equations is local in nature, so we have

Proposition 3.14 *Given $p \in S$ and $v \in T_pS$, there exist $\epsilon > 0$ and a unique geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Examples 3.15 1. For the plane, $\Gamma_{ij}^k \equiv 0$, so the geodesic equations are $(u^1)'' = (u^2)'' = 0$. Hence the geodesics are the straight lines.

2. In the sphere S^2, let $p \in S^2$ and $v \in T_pS^2 = (\mathbf{R}v)^\top$, $v \neq 0$, and consider the great circle

$$\gamma(t) = \cos(t||v||) + \sin(t||v||)\frac{v}{||v||}.$$

Then $\gamma(0) = p$, $\gamma'(0) = v$ and $\gamma''(t) = -||v||^2\gamma(t) \perp T_{\gamma(t)}S^2$, so great circles are geodesics. Since there exist great circles thorugh any point with any speed, by the uniqueness part of Proposition 3.14, the great circles are all the geodesics.

3. The cilinder $x^2 + y^2 = 1$ is locally isometric to the plane. In fact, $\varphi(u, v) = (\cos v, \sin v, u)$ is a local isometry, since by restricting to $(u, v) \in (u_0, u_0+2\pi) \times \mathbf{R}$ it becomes a parametrization with $E = 1$, $F = 0$, $G = 1$. It follows that the geodesics of the cilinder are images of the geodesics of the plane under φ. In particular, the geodesics through $\varphi(0, 0)$ are of the form

$$t \mapsto (\cos(at), \sin(at), bt),$$

where $a, b \in \mathbf{R}$. Note that we get horizontal circles ($a \neq 0$, $b = 0$), vertical lines ($a = 0$, $b \neq 0$) and helices (otherwise).

4. Consider a surface of revolution parametrized as in section 2.5. For a curve v-constant, $\eta(t) = \varphi(t, v_0)$, we have $\eta' = \frac{\partial \varphi}{\partial u}$, so by the computations in example 3.6,

$$\frac{\nabla \eta'}{dt} = \nabla_{\eta'} \eta' = \nabla_{\varphi_u} \varphi_u = 0.$$

Hence meridians are always geodesics. Similarly, for the curves u-constant, $u = u_0$, we have $\nabla_{\varphi_v} \varphi_v|_{u=u_0} = -f(u_0) f'(u_0) \varphi_u$ and $f > 0$, so precisely the parallels corresponding to critical points of f are geodesics.

More generally, using the Christofell symbols computed in the quoted example, we can write the geodesic equations as

$$\begin{aligned} u'' - f(u) f'(u) (v')^2 &= 0 \\ v'' + 2\frac{f'(u)}{f(u)} u' v' &= 0 \end{aligned}$$

Note that

$$\begin{aligned} \left(f^2(u) v'\right)' &= 2 f(u) f'(u) u' v' + f^2(u) v'' \\ &= f^2(u) \left(v'' + 2\frac{f'(u)}{f(u)} u' v' \right) \\ &= 0 \end{aligned}$$

along a solution curve, so $f^2(u) v'$ is constant along a geodesic (this function is a first integral of the system).

As an application, we consider a geodesic $\gamma(t) = (u(t), v(t))$ and, for each t, the parallel $\zeta(r) = \varphi(u(t), r)$ which crosses $\gamma(t)$ at $r = v(t)$, and compute the inner product

$$\langle \gamma', \zeta' \rangle = \langle u' \tfrac{\partial \varphi}{\partial u} + v' \tfrac{\partial \varphi}{\partial v}, \tfrac{\partial \varphi}{\partial v} \rangle = u' F + v' G = f^2(u) v'$$

to be constant with respect to t. On the other hand,

$$\langle \gamma', \zeta' \rangle = ||\gamma'|| ||\zeta'|| \cos\theta = ||\gamma'|| f(u) \cos\theta,$$

where $||\gamma'||$ is constant and $\theta(t)$ is the angle between $\gamma'(t)$ and $\zeta'(v(t)) = \frac{\partial \varphi}{\partial v}(u(t), v(t))$, and $f(u(t))$ is the radius of the parallel ζ. Hence the cosine of the angle at which γ meets a parallel multiplied by the radius of that parallel is constant; this is known as *Clairaut's relation*.

In fact, the first integral allows to explicitly integrate the geodesic equations. For the sake of simplicity, assume the geodesic γ is parametrized by arc-length; together with Clairaut's relation, this gives the system

$$\begin{aligned} (u')^2 + f^2 (v')^2 &= 1 \\ f^2 v' &= c \end{aligned}$$

where c is a constant; by changing the orientation of γ if necessary we may assume that $c > 0$. If γ is not a meridian, v' is never zero and we can use v as a parameter along γ, so that $u = u(v)$. We have

$$\begin{aligned}\left(\frac{du}{dv}\right)^2 &= \frac{(u')^2}{(v')^2} \\ &= \frac{1 - f^2(v')^2}{(v')^2} \\ &= \frac{1}{(v'^2)} - f^2 \\ &= \frac{f^4}{c^2} - f^2 \\ &= \frac{f^2}{c^2}(f^2 - c^2)\end{aligned}$$

It follows that

$$f \geq c$$

and, if γ is not a parallel, $\frac{du}{dv} \neq 0$ and $f > c$ so that

$$v = c \int \frac{1}{f(u)\sqrt{f(u)^2 - c^2}}\, du.$$

3.7 The integrability equations and the Theorema Egregium

Our next goal is to investigate how true is the fact that the first and second fundamental forms locally determine a surface. We will start by looking at necessary conditions for I, II to correspond to a surface.

Suppose $\varphi : U \to S$ is a regular parametrized surface defined on an open set $U \subset \mathbf{R}^2$, and let $I = (g_{ij})$, $II = (h_{ij})$ be its fundamental forms. Recall the Gauss formula

$$D_X Y = \nabla_X Y + II(X, Y)\nu \tag{3.16}$$

and the Weingarten equation

$$D_X \nu = -A(X), \tag{3.17}$$

where

$$I(AX, Y) = II(X, Y).$$

Writing the Gauss formula in terms of the parametrization, we get

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial u^i \partial u^j} &= D_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} \\ &= \nabla_{\frac{\partial \varphi}{\partial u^i}} \frac{\partial \varphi}{\partial u^j} + II\left(\frac{\partial \varphi}{\partial u^i}, \frac{\partial \varphi}{\partial u^j}\right) \nu \\ &= \sum_k \Gamma^k_{ij} \frac{\partial \varphi}{\partial u^k} + h_{ij}\nu. \end{aligned} \tag{3.18}$$

Doing the same with the Weingarten equation,

$$\begin{aligned}\frac{\partial \nu}{\partial u^i} &= D_{\frac{\partial \varphi}{\partial u^i}} \nu \\ &= -A\left(\frac{\partial \varphi}{\partial u^i}\right) \\ &= -\sum_j h_i^j \frac{\partial \varphi}{\partial u^j} \\ &= -\sum_{j,k} h_{ik} g^{kj} \frac{\partial \varphi}{\partial u^j}. \end{aligned} \tag{3.19}$$

We would like to integrate the equations (3.18) and (3.19) in terms of φ. There are obvious neccesaary conditions for that, namely

$$\begin{aligned} 0 &= \frac{\partial^3 \varphi}{\partial u^k \partial u^i \partial u^j} - \frac{\partial^3 \varphi}{\partial u^j \partial u^i \partial u^k} \\ &= \frac{\partial}{\partial u^k}\left(\sum_s \Gamma_{ij}^s \frac{\partial \varphi}{\partial u^s} + h_{ij}\nu\right) - \frac{\partial}{\partial u^j}\left(\sum_r \Gamma_{ik}^r \frac{\partial \varphi}{\partial u^r} + h_{ik}\nu\right) \\ &= \sum_s \left(\frac{\partial \Gamma_{ij}^s}{\partial u^k}\frac{\partial \varphi}{\partial u^s} + \Gamma_{ij}^s \frac{\partial^2 \varphi}{\partial u^k \partial u^s}\right) + \frac{\partial h_{ij}}{\partial u^k}\nu + h_{ij}\frac{\partial \nu}{\partial u^k} \\ &\qquad -\sum_r \left(\frac{\partial \Gamma_{ik}^r}{\partial u^j}\frac{\partial \varphi}{\partial u^r} + \Gamma_{ik}^r \frac{\partial^2 \varphi}{\partial u^j \partial u^r}\right) - \frac{\partial h_{ik}}{\partial u^j}\nu + h_{ik}\frac{\partial \nu}{\partial u^j} \\ &= \sum_s \left(\frac{\partial \Gamma_{ij}^s}{\partial u^k} - \frac{\partial \Gamma_{ik}^s}{\partial u^j}\right)\frac{\partial \varphi}{\partial u^s} \\ &\qquad + \sum_r \Gamma_{ij}^r \left(\sum_s \Gamma_{kr}^s \frac{\partial \varphi}{\partial u^s} + h_{kr}\nu\right) - \sum_r \Gamma_{ik}^r \left(\sum_s \Gamma_{jr}^s \frac{\partial \varphi}{\partial u^s} + h_{jr}\nu\right) \\ &\qquad + \left(\frac{\partial h_{ij}}{\partial u^k} - \frac{\partial h_{ik}}{\partial u^j}\right)\nu - h_{ij}\sum_{m,s} h_{km} g^{ms}\frac{\partial \varphi}{\partial u^s} + h_{ik}\sum_{ms} h_{jm} g^{ms}\frac{\partial \varphi}{\partial u^s}. \end{aligned}$$

Considering separately the tangential and normal components, we respectively get the *Gauss equation*

$$\begin{aligned} \frac{\partial \Gamma_{ij}^s}{\partial u^k} - \frac{\partial \Gamma_{ik}^s}{\partial u^j} &+ \sum_r \left(\Gamma_{ij}^r \Gamma_{rk}^s - \Gamma_{ik}^r \Gamma_{rj}^s\right) \\ &= \sum_m (h_{ij} h_{km} - h_{ik} h_{jm}) g^{ms} \quad \text{for all } i, j, k, s \end{aligned} \tag{3.20}$$

and the *Codazzi-Mainard equation*

$$\frac{\partial h_{ij}}{\partial u^k} - \frac{\partial h_{ik}}{\partial u^j} + \sum_r \left(\Gamma_{ij}^r h_{rk} - \Gamma_{ik}^r h_{rj}\right) = 0 \quad \text{for all } i, j, k. \tag{3.21}$$

Taken together, they are known as the *integrability* (or *compatibility*) equations.

As a by-product of our computation, we derive the following consequence of the integrability equations. Put $i = j = 1$, $k = 2$ in the Gauss eqn. (3.20), multiply through by g_{s2}, and sum over s to get

$$\begin{aligned}\sum_s \left(\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^s_{12}}{\partial u^1} \right) g_{s2} + \sum_{r,s} \left(\Gamma^r_{11}\Gamma^s_{r2} - \Gamma^r_{12}\Gamma^s_{r1} \right) g_{s2} \\ = \sum_{m,s} (h_{11}h_{2m} - h_{12}h_{1m}) g^{ms} g_{s2} \\ = h_{11}h_{22} - (h_{12})^2 \\ = \det(II).\end{aligned}$$

The left-hand side of this equation involves only the g_{ij}'s and the Γ^k_{ij}'s, and we already know that the Christofell symbols are completely determined by the g_{ij}'s. Hence $\det(II)$ depends only on I. Recalling that $K = \det(II)/\det(I)$, we finally get the **Theorema Egregium**.

Theorem 3.22 (Gauss, 1826) *The Gaussian curvature of a surface is an intrinsic invariant of the surface.*

In other words, locally isometric surfaces have the same Gaussian curvature at corresponding points.

Remark 3.23 The expression

$$\frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^s_{12}}{\partial u^1} + \sum_r \left(\Gamma^r_{11}\Gamma^s_{r2} - \Gamma^r_{12}\Gamma^s_{r1} \right) =: R^s_{121}$$

is a component of the so called Riemann curvature tensor, and

$$\sum_s R^s_{121} g_{s2} = R_{1212},$$

so the Theorema Egregium can be restated in the form

$$K = \frac{R_{1212}}{g_{11}g_{22} - (g_{12})^2}.$$

3.8 A very quick digression on systems of first order partial differential equations

In constrast to the case of *ordinary* differential equations, systems of PDE's do not always have solutions. We start with a simple example. Consider a vector field $\vec{X} = P\vec{i} + Q\vec{j}$ defined on an open set $U \subset \mathbf{R}^2$, where $P, Q : U \to \mathbf{R}^2$ are smooth functions. Finding a potencial for $\vec{X}$, i.e. a smooth scalar function f on U such that $\operatorname{grad} f = \vec{X}$, is equivalent to solving the system

$$\begin{aligned} \frac{\partial f}{\partial x} &= P(x, y) \\ \frac{\partial f}{\partial y} &= Q(x, y). \end{aligned}$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, a necessary condition for the existence of solutions is that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

namely, the rotational $\operatorname{rot} \vec{X} = \vec{0}$. It is also known that if U is a rectangle, or star-shaped, or even simply-connected, then this condition is also sufficient.

A more general system has the form

$$\begin{aligned} \frac{\partial z}{\partial x} &= P(x, y, z) \\ \frac{\partial z}{\partial y} &= Q(x, y, z). \end{aligned}$$

In this case, the necessary condition is easily seen to be

$$\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} Q = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} P.$$

The general case is dealt with the following theorem, for whose proof the reader is referred to app. B in J. J. Stoker, *Differential Geometry*, Wiley Interscience, 1969.

Theorem 3.24 (Frobenius, 1877) *Consider the first-order system of partial differential equations*

$$\frac{\partial y^i}{\partial x^j} = P^i_j(x^1, \ldots, x^m; y^1, \ldots, y^n),$$

where $1 \leq i \leq n$ and $1 \leq j \leq m$ and P^i_j admit continuous derivatives of second order in all arguments. Suppose that the P^j_i satisfy the integrability conditions

$$\frac{\partial P^i_j}{\partial x^k} + \sum_\ell \frac{\partial P^i_j}{\partial y^\ell} P^k_\ell = \frac{\partial P^i_k}{\partial x^j} + \sum_\ell \frac{\partial P^i_k}{\partial y^\ell} P^j_\ell.$$

Then there exist a unique solution satisfying the initial conditions

$$y^i(x^1_0, \ldots, x^m_0) = y^i_0$$

for $1 \leq i \leq n$.

3.9 The fundamental theorem of the local theory of surfaces

The first result asserts the invariance of the fundamental forms of a surface under orientation-preserving rigid motions of Euclidean space, and contains the uniqueness part of Theorem 3.28.

Lemma 3.25 *Let $\varphi : U \to S$ be a regular parametrized surface, and let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be an orientation-preserving rigid motion, i.e. $T(x) = A(x) + b$ where $A \in SO(3)$ and $b \in \mathbf{R}^3$. Put $\tilde{\varphi} = T \circ \varphi$. Then $\tilde{\varphi} : U \to \tilde{S}$ is a regular parametrized surface and the coefficients of the fundamental forms $\tilde{g}_{ij} = g_{ij}$, $\tilde{h}_{ij} = h_{ij}$. Conversely, given two regular parametrized surfaces $\varphi : U \to S$ and $\tilde{\varphi} : U \to \tilde{S}$, where U is connected, satisfying $g_{ij} = \tilde{g}_{ij}$, $h_{ij} = \tilde{h}_{ij}$, there exists an orientation-preserving rigid motion T of $\mathbf{R}^3$ such that $\tilde{\varphi} = T \circ \varphi$.*

Proof. Differentiate $\tilde{\varphi} = A\varphi + b$; owing to the constancy of A, b, we get $\frac{\partial \tilde{\varphi}}{\partial u^i} = A\left(\frac{\partial \varphi}{\partial u^i}\right)$. Using this and the fact that A is orientation-preserving,

$$\tilde{\nu} = \frac{\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}}{||\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}||} = \frac{A\left(\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}\right)}{||A\left(\frac{\partial \tilde{\varphi}}{\partial u^1} \times \frac{\partial \tilde{\varphi}}{\partial u^2}\right)||} = A(\nu).$$

Since A is ortogonal, we immediately see that $\tilde{g}_{ij} = g_{ij}$, $\tilde{h}_{ij} = h_{ij}$.

For the second part, define $A(u) : \mathbf{R}^3 \to \mathbf{R}^3$ for $u \in U$ by setting

$$A(u)\left(\frac{\partial \varphi}{\partial u^i}\Big|_u\right) = \frac{\partial \tilde{\varphi}}{\partial u^i}\Big|_u \quad (i = 1, 2), \qquad A(u)(\nu(u)) = \tilde{\nu}(u).$$

Plainly, $A(u) \in SO(3)$; we next show that A is constant. On one hand,

$$\frac{\partial^2 \tilde{\varphi}}{\partial u^i \partial u^j} = \frac{\partial}{\partial u^i}\left(A\frac{\partial \varphi}{\partial u^j}\right) = \frac{\partial A}{\partial u^i}\frac{\partial \varphi}{\partial u^j} + A\left(\frac{\partial^2 \varphi}{\partial u^i \partial u^j}\right).$$

On the other hand, by the Gauss formula (3.18),

$$\frac{\partial^2 \tilde{\varphi}}{\partial u^i \partial u^j} = \sum_k \tilde{\Gamma}^k_{ij}\frac{\partial \tilde{\varphi}}{\partial u^k} + \tilde{h}_{ij}\tilde{\nu} = \sum_k \Gamma^k_{ij} A\left(\frac{\partial \varphi}{\partial u^k}\right) + h_{ij}A(\nu) = A\left(\frac{\partial^2 \varphi}{\partial u^i \partial u^j}\right).$$

Putting this together yields

$$\frac{\partial A}{\partial u^i}\frac{\partial \varphi}{\partial u^j} = 0. \tag{3.26}$$

Similarly,

$$\frac{\partial \tilde{\nu}}{\partial u^i} = \frac{\partial A}{\partial u^i}\nu + A\left(\frac{\partial \nu}{\partial u^i}\right),$$

and the Weingarten equation

$$\frac{\partial \tilde{\nu}}{\partial u^i} = -\sum_{j,k} \tilde{h}_{ik} \tilde{g}^{kj} \frac{\partial \tilde{\varphi}}{\partial u^j} = -\sum_{j,k} h_{ik} g^{kj} A \left(\frac{\partial \varphi}{\partial u^j} \right) = A \left(\frac{\partial \nu}{\partial u^i} \right)$$

together imply that

$$\frac{\partial A}{\partial u^i} \nu = 0. \tag{3.27}$$

From (3.26) and (3.27), we get that $\frac{\partial A}{\partial u^i} = 0$ on U and hence A is constant. Finally, also $\tilde{\varphi} - A \circ \varphi$ is a constant b, for $\frac{\partial}{\partial u^i}(\tilde{\varphi} - A \circ \varphi) = \frac{\partial \tilde{\varphi}}{\partial u^i} - A(\frac{\partial \varphi}{\partial u^i}) = 0$, which finishes the proof. □

Theorem 3.28 (Bonnet, 1867) *Let be given smooth functions g_{ij}, h_{ij} defined on an open set $U \subset \mathbf{R}^2$ ($1 \leq i, j \leq 2$) such that $g_{ij} = g_{ji}$, $h_{ij} = h_{ji}$ and the matrix (g_{ij}) is positive-definite. Suppose that g_{ij}, h_{ij} satisfy the equations of Gauss (3.20) and Codazzi-Mainardi (3.21). Then, for given initial conditions*

$$u_0 \in U,\ p_0 \in \mathbf{R}^3,\ X_{1,0},\ X_{2,0},\ \nu_0 \in \mathbf{R}^3$$

with ν_0 a unit vector and $\langle X_{1,0}, X_{2,0} \rangle = g_{ij}(u_0)$, there exists an open neighborhood V of u_0 contained in U and a unique regular parametrized surface $\varphi : V \to \mathbf{R}^3$ whose Gauss map is ν with the following properties:

1. *$\varphi(u_0) = p_0$;*
2. *$\frac{\partial \varphi}{\partial u^i}(u_0) = X_{i,0}$;*
3. *$\nu(u_0) = \nu_0$;*
4. *(g_{ij}) and (h_{ij}) are the fundamental forms of φ.*

Proof. We introduce new vector-valued variables X_1, X_2

$$\frac{\partial \varphi}{\partial u^j} = X_j \tag{3.29}$$

and write the Gauss formula and the Weingarten equation in the form of a first-order system of PDE's in X_1, X_2, ν:

$$\begin{aligned} \frac{\partial X_j}{\partial u^i} &= \sum_k \Gamma_{ij}^k X_k + h_{ij} \nu \\ \frac{\partial \nu}{\partial u^i} &= -\sum_j h_{ij} g^{jk} X_k. \end{aligned} \tag{3.30}$$

We first solve (3.30): the integrability conditions of Theorem 3.24 are exactly the equations of Gauss and Codazzi-Mainardi, which are satisfied by assumption, so there exists a unique solution (X_1, X_2, ν) defined on a neighborhood of u_0

and satisfying the initial conditions $(X_{1,0}, X_{2,0}, \nu_0)$ at u_0. Let us check that the solutions satisfy

$$\langle \nu, \nu \rangle = 1,\ \langle \nu, X_i \rangle = 0,\ \langle X_i, X_j \rangle = g_{ij} \tag{3.31}$$

on a neighborhood of u_0. We differentiate the left-hand sides of these equations to get

$$\begin{aligned}
\frac{\partial}{\partial u^i}\langle \nu, \nu \rangle &= 2\langle \frac{\partial \nu}{\partial u^i}, \nu \rangle = -2\sum_{k,l} h_{ik} g^{kl} \langle \nu, X_l \rangle \\
\frac{\partial}{\partial u^i}\langle \nu, X_j \rangle &= \langle \frac{\partial \nu}{\partial u^i}, X_j \rangle + \langle \nu, \frac{\partial X_j}{\partial u^i} \rangle \\
&= -\sum_{k,l} h_{ik} g^{kl} \langle X_l, X_j \rangle + \sum_k \Gamma_{ij}^k \langle \nu, X_k \rangle + h_{ij} \langle \nu, \nu \rangle \\
\frac{\partial}{\partial u^k}\langle X_i, X_j \rangle &= \langle \frac{\partial X_i}{\partial u^i}, X_j \rangle + \langle X_i, \frac{\partial X_j}{\partial u^i} \rangle \\
&= \sum_r \Gamma_{ik}^r \langle X_r, X_j \rangle + h_{ik} \langle \nu, X_j \rangle \\
&\qquad + \sum_s \Gamma_{jk}^s \langle X_s, X_i \rangle + h_{jk} \langle \nu, X_i \rangle.
\end{aligned}$$

These identities show that the functions $\langle \nu, \nu \rangle$, $\langle \nu, X_i \rangle$, $\langle X_i, X_j \rangle$ satisfy a system of PDE's in U. It is easy to check that the functions $1, 0, g_{ij}$ satisfy the same system of PDE's. (Do it!) Since the values of the two triples of functions coincide at the point u_0, by the uniqueness part of Theorem 3.24, the equations (3.31) are satisfied on a neighborhood of u_0.

The final step is to solve (3.29) in φ. The integrability conditions $\frac{\partial X_j}{\partial u^i} = \frac{\partial X_i}{\partial u^j}$ are satisfied because $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $h_{ij} = h_{ji}$. Therefore there exists a smaller neighborhood V of u_0 and a unique solution $\varphi : V \to \mathbf{R}^3$ with $\varphi(u_0) = p_0$. Since $\langle \nu, \nu \rangle = 1$, $\langle \nu, \frac{\partial \varphi}{\partial u^i} \rangle = \langle \nu, X_i \rangle = 0$, ν is a unit normal vector field along φ. Moreover, the fundamental forms of φ are

$$\langle \tfrac{\partial \varphi}{\partial u^i}, \tfrac{\partial \varphi}{\partial u^j} \rangle = \langle X_i, X_j \rangle = g_{ij}$$

and

$$\left\langle \frac{\partial^2 \varphi}{\partial u^i \partial u^j}, \nu \right\rangle = \left\langle \sum_k \Gamma_{ij}^k \frac{\partial^\varphi}{\partial u^k} + h_{ij}\nu, \nu \right\rangle = h_{ij},$$

as we wished. As a final remark, note that

$$\nu = \pm \frac{\frac{\partial \varphi}{\partial u^1} \times \frac{\partial \varphi}{\partial u^2}}{||\frac{\partial \varphi}{\partial u^1} \times \frac{\partial \varphi}{\partial u^2}||}$$

where the sign is "+" or "−" according to whether $\{X_{1,0}, X_{2,0}, \nu_0\}$ is a positive basis of $\mathbf{R}^3$ or not. □

3.10 Differential forms

Let U be an open set of $\mathbf{R}^n$. A *differential* 1*-form* (or a *differential form of degree* 1) on U is a map

$$\omega : p \in U \to (\mathbf{R}^n)^*;$$

ω is said to be smooth if, for all i, the function $\omega(e_i) : U \to \mathbf{R}$ given by $\omega(e_i)(p) = \omega_p(e_i)$ is smooth, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbf{R}^n$.

Suppose $\{X_1, \ldots, X_n\}$ is a smooth orthonormal frame on U, that is, the X_i are smooth vector fields and $\{X_1(p), \ldots, X_n(p)\}$ is an orthonormal basis for $p \in U$. Then we can consider the *dual coframe* $\{\omega^1, \ldots, \omega^n\}$ by specifying $\omega^i(X_j) = \delta^i_j$ (Kronecker delta). For instance, the dual coframe to the canonical basis of $\mathbf{R}^n$ is usually denoted $\{dx^1, \ldots, dx^n\}$. Of course, dx^i is just the linear projection onto the x^i-axis of $\mathbf{R}^n$.

Example 3.32 On $U = \mathbf{R}^2 \setminus \{(0,0)\}$, consider the orthonormal frame

$$\begin{aligned} X_1 &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}} e_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} e_2, \\ X_2 &= \frac{-x_2}{\sqrt{x_1^2 + x_2^2}} e_1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} e_2. \end{aligned}$$

Then the dual coframe has

$$\begin{aligned} \omega^1 &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\, dx^1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\, dx^2, \\ \omega^2 &= \frac{-x_2}{\sqrt{x_1^2 + x_2^2}}\, dx^1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\, dx^2. \end{aligned}$$

We will also consider differential forms of degree 2. A *differential* 2*-form* on $U \subset \mathbf{R}^n$ is a map Ω that takes $p \in U$ to a skew-symmetric bilinear map

$$\Omega_p : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R};$$

Ω is said to be smooth if $\Omega(e_i, e_j) : U \to \mathbf{R}$ is a smooth function for all i, j. There are two important ways of manufacturing 2-forms starting with 1-forms.

The first one is the exterior product. If ω, η are 1-forms on U, their *exterior product* is defined to be the 2-form $\omega \wedge \eta$ given by

$$(\omega \wedge \eta)_p(u, v) = \omega_p(u)\eta_p(v) - \omega_p(v)\eta_p(u),$$

where $p \in U$ and u, $v \in \mathbf{R}^n$. It is immediate to see that $(\omega \wedge \eta)_p$ is skew-symmetric and bilinear. It is also easy to see that $\omega \wedge \eta$ is smooth if ω, η are smooth. For future use, we note the following properties:

1. $\omega \wedge \eta = -\eta \wedge \omega$;

2. $\omega \wedge \omega = 0$.

The second way of constructing 2-forms from 1-forms is exterior derivation. If ω is a 1-from on U, we can write $\omega = \sum_{i=1}^n a^i\, dx_i$, where $a_i : U \to \mathbf{R}$ are smooth functions. The *exterior derivative* of ω is the 2-form

$$\begin{aligned} d\omega &= \sum_{i=1}^n da^i \wedge dx_i \\ &= \sum_{i,j=1}^n \tfrac{\partial a^i}{\partial x^j} dx^j \wedge dx^i. \end{aligned}$$

Example 3.33 Referring to Example 3.32, we have

$$d\omega^1 = 0 \quad \text{and} \quad d\omega^2 = \frac{1}{\sqrt{x_1^2 + x_2^2}}\, dx_1 \wedge dx_2.$$

The next lemma shows how to compute $d\omega$ without invoking coordinates. However, note that on left-hand side of (3.35), X and Y need to be *vector fields*, whereas $(d\omega)_p(X, Y)$ makes sense even if X, Y are just vectors.

Lemma 3.34 *If ω is a smooth 1-form on an open set U of $\mathbf{R}^n$, and X, Y are smooth vector fields on U, then*

$$d\omega(X, Y) = D_X(\omega(Y)) - D_Y(\omega(X)) - \omega([X, Y]). \tag{3.35}$$

Proof. Write $\omega = \sum_i a^i\, dx_i$. Since both hand sides of (3.35) are linear in ω, we may assume that $\omega = a\, dx_i$, where $a : U \to \mathbf{R}$ is smooth. We write $X = \sum_i X^i e_i$, $Y = \sum_i Y^i e_i$ and compute

$$\begin{aligned} &D_X(\omega(Y)) - D_Y(\omega(X)) - \omega([X, Y]) \\ &\quad= D_X(a\, dx^i(Y)) - D_Y(a\, dx^i(X)) - a\, dx^i([X, Y]) \\ &\quad= (D_X a)Y^i + aD_X Y^i - (D_Y a)X^i - aD_Y X^i - a[X, Y]^i \\ &\quad= da(X)Y^i - da(Y)X^i + a\underbrace{(D_X Y^i - D_Y X^i - [X, Y]^i)}_{=0} \\ &\quad= da \wedge dx^i(X, Y) \\ &\quad= d\omega(X, Y), \end{aligned}$$

as we wished. □

Examples 3.36 1. Every vector field X defines a 1-form via the equation $\omega(Y) = \langle X, Y\rangle$.

2. The line integral of a 1-form ω in U along a smooth curve $\gamma : [a, b] \to U$ is defined to be

$$\int_\gamma \omega = \int_a^b \omega(\dot\gamma(t))\, dt.$$

3. The differential of a smooth function $f : U \to \mathbf{R}$ is the 1-form

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

Then

$$\begin{aligned} d^2 f &= d(df) \\ &= \sum_i d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \\ &= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j \\ &= 0. \end{aligned}$$

In particular, $d(dx^i) = 0$.

4. More generally, a 1-form ω is called *exact* if there exists a function f such that $df = \omega$. A 1-form ω is called *closed* if $d\omega = 0$. We have shown that every exact 1-form is closed. The converse is true if the domain U simply-connected; this can be proven using Theorem 3.24.

3.11 Connection forms and the integrability equations

Our next goal is to express the Gauss and Codazzi-Mainardi equations in terms of differential forms. In order to express the covariant derivative of a surface in terms of differential forms, we first consider the directional derivative in the ambient space; fix fix a frame $\{X_1, X_2, X_3\}$ in $\mathbf{R}^3$ with dual coframe $\{\omega^1, \omega^2, \omega^3\}$ and define 1-forms by setting

$$\omega^i_j(Y) = \omega^i(D_Y X_j)$$

for i, $j = 1, 2, 3$. Then

$$D_Y X_j = \sum_{i=1}^{3} \omega^i_j(Y) X_i.$$

We also have that

$$\omega^i_j + \omega^j_i = 0$$

for all i, j, because

$$\begin{aligned} \omega^i_j(Y) + \omega^j_i(Y) &= \langle D_Y X_j, X_i\rangle + \langle D_Y X_i, X_j\rangle \\ &= D_Y \langle X_i, X_j\rangle \\ &= 0, \end{aligned}$$

since $\langle X_i, X_j\rangle$ is constant.

Henceforth we suppose that a surface S in $\mathbf{R}^3$ is given and we take an *adapted orthonormal frame*, namely, assume that X_1, X_2 are tangent to S and

$X_3 = \nu$ is normal to S. Let $\{\omega^1, \omega^2, \omega^3\}$ denote the dual coframe. Since the covariant derivative is the tangential component of the directional derivative, for a tangent vector Y and $i, j = 1, 2$, we have that

$$\omega^i_j(Y) = \omega^i(D_Y X_j) = \omega^i(\nabla_Y X_j).$$

The restrictions of the ω^i_j (i, j = 1, 2) to the tangent spaces of S are called *connection forms* of S. They determine (and are determined by) the covariant derivative. Since ω^i_j is skew-symmetric in i, j, there is in fact only one curvature form ω^1_2.

If Y is tangent to S and $j = 1, 2$, then

$$\nabla_Y X_j = (D_Y X_j)^\top = \sum_{i=1}^{2} \omega^i_j(Y) X_j \tag{3.37}$$

and

$$\omega^3_j(Y) = \langle D_Y X_j, X_3\rangle = -\langle X_j, D_Y \nu\rangle = \langle X_j, AY\rangle = II(X_j, Y) \tag{3.38}$$

Equations (3.37), (3.38) are the Gauss formula (3.16) and Weingarten equation (3.17) written in the language of differential forms. In order to do the same with the integhrability equations (3.20) and (3.21), let us first prove the following lemma.

Lemma 3.39 *If X, Y, Z are smooth vector fields defined on an open set U of $\mathbf{R}^n$, then*

$$D_X D_Y Z - D_Y D_X Z = D_[X, Y]Z.$$

(Equivalently, $[D_X, D_Y] = D_{[X,Y]}$ as operators)

Remark 3.40 In case $X = e_i$, $Y = e_j$, the lemma follows form the equality of mixed second partial derivatives. Indeed, $D_{e_i} D_{e_j} Z - D_{e_j} D_{e_i} Z = \frac{\partial^2 Z}{\partial x^i \partial x^j} - \frac{\partial^2 Z}{\partial x^j \partial x^i} = 0$ and $D_{[e_i, e_j]} = 0$ (since $[e_i, e_j] = 0$).

Proof of lemma 3.39. Write $X = \sum_i X^i e_i$ and $Y = \sum_j Y^j e_j$. Then

$$\begin{aligned}
D_X D_Y Z - D_Y D_X Z &= \sum_i X^i D_{e_i} \left(\sum_j Y^j e_j\right) - \sum_j Y^j D_{e_j} \left(\sum_i X^i e_i\right) \\
&= \sum_{i,j} X^i Y^j D_{e_i} D_{e_j} Z + \sum_{i,j} X^i \tfrac{\partial Y^j}{\partial x^i} D_{e_j} Z \\
&\quad - \sum_{i,j} Y^j X^i D_{e_j} D_{e_i} Z + \sum_{i,j} Y^j \tfrac{\partial X^i}{\partial x^j} D_{e_i} Z \\
&= \sum_{i,j} \left(X^i \tfrac{\partial Y^j}{\partial x^i} - Y^i \tfrac{\partial X^j}{\partial x^i} \right) D_{e_j} Z \\
&= \sum_j [X, Y]^j D_{e_j} Z \\
&= D_{[X,Y]} Z,
\end{aligned}$$

as desired. □

Proposition 3.41 (Maurer-Cartan structural equations) *For all $i,$, j, we have that*

$$d\omega^i_j + \sum_{k=1}^{3} \omega^i_k \wedge \omega^k_j = 0.$$

In particular

$$\begin{array}{rcll} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &=& 0 & \textit{(Gauss eqn.)} \\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 &=& 0 & \textit{(Codazzi-Mainardi eqn.)} \end{array}$$

Proof. If X, Y are smooth vector fields on $\mathbf{R}^3$, then

$$\begin{aligned} \left(d\omega^i_j + \sum_{k=1}^{3} \omega^i_k \wedge \omega^k_j \right)(X,Y) &= d\omega^i_j(X,Y) + \sum_{k=1}^{3} (\omega^i_k \wedge \omega^k_j)(X,Y) \\ &= D_X(\omega^i_j(Y)) - D_Y(\omega^i_j(X)) - \omega^i_j([X,Y]) \\ &\qquad + \sum_k (\omega^i_k(X)\omega^k_j(Y) - \omega^i_k(Y)\omega^k_j(X)) \\ &= \sum_k D_X(\omega^k_j(Y))\omega^i(X_k) - \sum_k D_Y(\omega^k_j(Y))\omega^i(X_k) \\ &\qquad - \sum_k \omega^k_j([X,Y])\omega^i(X_k) \\ &\qquad + \sum_k \omega^k_j(Y)\omega^i(D_X X_k) - \sum_k \omega^k_j(X)\omega^i(D_Y X_k) \\ &= \omega^i\big(D_X(\sum_k \omega^k_j(Y)X_k) - D_Y(\sum_k \omega^k_j(X)X_k) \\ &\qquad - \sum_k \omega^k_j([X,Y])X_k\big) \\ &= \omega^i(D_X D_Y X_j - D_Y D_X X_j - D_{[X,Y]} X_j) \\ &= 0, \end{aligned}$$

using Lemma 3.39, as we wished. □

The *curvature form* of S is the 2-form *defined on S* (meaning that it is restricted to the tangent spaces of S)

$$\Omega^1_2 = d\omega^1_2.$$

The relation to (Gaussian) curvature is expressed by the following proposition.

Proposition 3.42 *We have that*

$$\Omega^1_2 = K\,\omega^1 \wedge \omega^2,$$

where K is the Gaussian curvature of S.

Proof. By skew-symmetry, he only nontrivial ordered pair of vectors on which to evaluate a 2-form on S is (X_1, X_2). On one hand,

$$\begin{aligned}\Omega^1_2(X_1, X_2) &= d\omega^1_2(X_1, X_2)\\ &= -\omega^1_3 \wedge \omega^3_2(X_1, X_2) \quad \text{(by Propostion 3.41)}\\ &= -\omega^1_3(X_1)\omega^3_2(X_2) + \omega^1_3(X_2)\omega^3_2(X_1)\\ &= \omega^3_1(X_1)\omega^3_2(X_2) - \omega^3_1(X_2)\omega^3_2(X_1)\\ &= II(X_1, X_1)II(X_2, X_2) - II(X_1, X_2)II(X_2, X_1)\\ &= \det(II)\\ &= K \quad \text{(since } \{X_1, X_2\} \text{ is orthonormal).}\end{aligned}$$

On the other hand,

$$\omega^1 \wedge \omega^2(X_1, X_2) = 1.$$

Finally, $\Omega^1_2(X_1, X_2) = K\omega^1 \wedge \omega^2(X_1, X_2)$ implies that $\Omega^1_2 = K\,\omega^1 \wedge \omega^2$. □

Remark 3.43 The reasoning in the proof of Proposition 3.42 shows in fact that any smooth 2-form Ω on S can be written $\Omega = f\omega^1 \wedge \omega^2$ for some smooth function $f : S \to \mathbf{R}$. Suppose $\varphi : U \to S$ is a parametrization and $B \subset U$ is compact. The integral of the 2-form Ω on $\tilde{B} = \varphi(B)$ is defined to be the surface integral of f on $\tilde{B}$:

$$\int_{\tilde{B}} \Omega := \int\int_{\tilde{B}} f\, dA = \int\int_B f \circ \varphi \sqrt{EG - F^2}\, dudv,$$

and its value is known not to depend on the parametrization. On the other hand, by taking $\{X_1, X_2\}$ to be the result of the Gram-Schmidt orthonormalization of $\{\frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial v}\}$,

$$\begin{aligned}\omega^1 \wedge \omega^2(\tfrac{\partial\varphi}{\partial u}, \tfrac{\partial\varphi}{\partial v}) &= \langle X_1, \tfrac{\partial\varphi}{\partial u}\rangle\langle X_2, \tfrac{\partial\varphi}{\partial v}\rangle - \langle X_1, \tfrac{\partial\varphi}{\partial v}\rangle\langle X_2, \tfrac{\partial\varphi}{\partial u}\rangle\\ &= \frac{E}{\sqrt{E}}\sqrt{G - \frac{F}{E^2}} - 0\\ &= \sqrt{EG - F^2}.\end{aligned}$$

It is also easy to see that for any other positively oriented orthonormal frame $\{X'_1, X'_2\}$, the dual coframe $\{\omega^{1\prime}, \omega^{2\prime}\}$ satisfies $\omega^{1\prime} \wedge \omega^{2\prime} = \omega^1 \wedge \omega^2$. For this reason, $\omega^1 \wedge \omega^2$ is called the *area element* of S and written $\omega^1 \wedge \omega^2 = dA$.

3.12 The Gauss-Bonnet theorem

The Gauss-Bonnet is one of the most important theorems in Differential Geometry. The global version expresses the invariance the total (Gaussian) curvature of a closed orientable surface under deformations in the ambient space preserving the topology. For this reason, it is said that this theorem relates the geometry and topology of a closed surface.

We start our discussion with the notion of geodesic curvature. Let $\gamma : I \to S$ be a curve parametrized by arc-length whose image lies in a regular parametrized surface $\varphi : U \to S$. We want to consider the curvature of γ from the point of view of an observer in S. We construct a frame along γ which is adapted to S: take $e_1 = \gamma'$ and $e_2 = \pm e_1 \times \nu$, where the sign is chosen so that $\{e_1, e_2, \nu\}$ is a positive basis of $\mathbf{R}^3$. The curvature κ of γ as a curve in $\mathbf{R}^3$ is of course

$$\kappa = ||e_1'|| = ||D_{e_1} e_1||.$$

Since e_1 has constant length 1, $\langle D_{e_1} e_1, e_1 \rangle = 0$. The normal component is the *normal curvature*,

$$\kappa_\nu = \langle D_{e_1} e_1, \nu \rangle = -\langle e_1, D_{e_1} \nu \rangle = \langle e_1, A e_1 \rangle = II(e_1, e_1),$$

and the tanegential component is the *geodesic curvature*,

$$\kappa_g = \langle D_{e_1} e_1, e_2 \rangle.$$

In particular,

$$\kappa^2 = \kappa_\nu^2 + \kappa_g^2.$$

Note that $\kappa_g = 0$ if and only if $\langle \nabla_{e_1} e_1, e_2 \rangle = 0$ (since e_2 is tangent) if and only if $\nabla_{e_1} e_1 = 0$ (since $\langle \nabla_{e_1} e_1, e_1 \rangle = 0$), which is the same as $\nabla_{\gamma'} \gamma' = 0$, namely, γ is a geodesic. This shows that the geodesic curvature is a measure of how far from being a geodesic the curve is. Plainly, we also have the equations

$$\begin{aligned} \nabla_{e_1} e_1 &= \kappa_g e_2 \\ \nabla_{e_1} e_2 &= -\kappa_g e_1. \end{aligned}$$

We can now state the first theorem.

Theorem 3.44 (Gauss-Bonnet, first local version) *Let $\varphi : U \subset S$ be a regular parametrized surface, and consider a subset $B \subset U$ diffeomorphic to a closed disk, where the boundary ∂B in oriented in the counter-clockwise sense. Then*

$$\int_{\varphi(B)} K \, dA + \int_{\varphi(\partial B)} \kappa_g \, ds = 2\pi.$$

Examples 3.45 1. For a disk of radius r in the plane $\mathbf{R}^2 \subset \mathbf{R}^3$, we have $K = 0$ and $\kappa_g = \frac{1}{r}$, so $\int K \, dA + \int \kappa_g \, ds = \frac{1}{r} \int ds = \frac{1}{r} 2\pi r = 2\pi$.

2. For the closed hemisphere

$$S_+^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1, \ z \geq 0\}$$

we have $K = 1$, $\kappa_g = 0$ (the equator is a geodesic), so $\int K \, dA + \int \kappa_g \, ds = \int dA = \text{area}(S_+^2) = \frac{1}{2} 4\pi = 2\pi$.

Proof of Theorem 3.44. Recall that S is oriented by the unit normal ν coming from φ. Consider a parametrization $\gamma : [a,b] \to \partial B$ (in the counter-clockwise sense) so that $\varphi \circ \gamma : [a,b] \to \varphi(\partial B)$ is a parametrization by arc-length. To compute the geodesic curvature, we need an adapted positive orthonormal frame $\{e_1, e_2\}$ along where $e_1 = (\varphi \circ \gamma)'$. We also consider $\{X_1, X_2\}$, positive orthonormal frame on S which is the result of the Gram-Schmidt process to $\{\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v}\}$; in particular, $X_1 = \frac{\partial \varphi}{\partial u} / || \frac{\partial \varphi}{\partial u} ||$. Then there exists a smooth function $\theta : [a,b] \to \mathbf{R}$ such that

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Writing $\gamma(s) = (x(s), y(s))$, we have

$$\cos \angle(e_1, X_1) = \frac{I((\varphi \circ \gamma)', \frac{\partial \varphi}{\partial u})}{||(\varphi \circ \gamma)'|| || \frac{\partial \varphi}{\partial u} ||} = \frac{x' g_{11} + y' g_{12}}{\sqrt{g_{11}}}.$$

We define a one-parameter family of inner products

$$g_{ij}^t = (1-t) g_{ij} + t \delta_{ij},$$

continuous deformation of the first fundamental form of S into the standard inner product of $\mathbf{R}^2$. The angle $\angle(e_1, X_1)(s,t)$ is continuous as function of s and t. Since the difference $\angle(e_1, X_1)(b,t) - \angle(e_1, X_1)(a,t)$ is always an integral multiple of 2π, it must be a constant function of t. By the Umlaufsatz, $\angle(e_1, X_1)(b,1) - \angle(e_1, X_1)(a,1) = 2\pi$. Hence $\theta(b) - \theta(a) = \angle(e_1, X_1)(b,0) - \angle(e_1, X_1)(a,0) = 2\pi$.

We compute

$$
\begin{aligned}
2\pi &= \theta(b) - \theta(a) \\
&= \int_a^b \frac{d\theta}{ds}\, ds \\
&= -\int_a^b \frac{1}{\sin\theta}\left(\langle \nabla_{e_1} e_1, X_1\rangle + \langle e_1, \nabla_{e_1} X_1\rangle\right) ds \\
&= -\int_a^b \frac{1}{\sin\theta}\Big(\cos\theta\, \underbrace{\langle \nabla_{e_1} e_1, e_1\rangle}_{=0} - \sin\theta \langle \nabla_{e_1} e_1, e_2\rangle \\
&\qquad + \cos\theta\, \underbrace{\langle X_1, \nabla_{e_1} X_1\rangle}_{=0} + \sin\theta \langle X_2, \nabla_{e_1} X_1\rangle\Big) ds \\
&= \int_a^b (\kappa_g + \omega_2^1(e_1))\, ds \\
&= \int_{\varphi(\partial B)} \kappa_g\, ds + \int_{\varphi\partial B)} \omega_2^1 \\
&= \int_{\varphi(\partial B)} \kappa_g\, ds + \int_{\varphi(B)} d\omega_2^1 \qquad \text{(by Stokes theorem)} \\
&= \int_{\varphi(\partial B)} \kappa_g\, ds + \int_{\varphi(B)} K\, dA,
\end{aligned}
$$

as desired. □

In the second version of the local Gauss-Bonnet theorem, we alllow ∂B to have corners.

Theorem 3.46 (Gauss-Bonnet, second local version) *Same assumptions as in 3.44, except that now B is only homeomorphic to a closed disk and ∂B is piecewise smooth. Let α_i be the exterior angle at the ith vertex of ∂B. Then*

$$\int_{\varphi(B)} K\, dA + \int_{\varphi(\partial B)} \kappa_g\, ds + \sum_i \alpha_i = 2\pi.$$

We only make some remarks about what needs to be changed in the proof of this theorem in relation to Theorem 3.44. The first ingredient is the Umalufsatz for piecewise smooth closed curves. Suppose $\gamma : [a,b] \to \partial B$ is continuous, closed ($\gamma(a) = \gamma(b)$), and piecewise smooth in the sense that there exists a partition $a = s_0 < s_1 < \dots s_{n+1} = b$ such that $\gamma|_{[s_i, s_{i+1}]}$ is smooth for $i = 0, \dots, n$. For the sake of convenience, we also assume that $\gamma(s_0) = \gamma(s_n)$ is not a vertex. By smoothing γ near its vertices, one shows that the Umlaufsatz remains valid, te index of rotation of γ is 1. Now we can write

$$\begin{aligned} 2\pi &= \theta(b) - \theta(a) \\ &= \sum_{i=0}^{n}(\theta(s_{i+1}-) - \theta(s_i+)) + \sum_{i=1}^{n}(\theta(s_i+) - \theta(s_i-)) \\ &= \sum_{i=0}^{n}\int_{s_i}^{s_{i+1}} \frac{d\theta}{ds}\, ds + \sum_{i=1}^{n}\alpha_i, \end{aligned}$$

and the rest of the proof goes as before.

Corollary 3.47 (Geodesic n-gon) *If the sides of ∂B are geodesic segments, then*

$$\int_{\varphi(B)} K\, dA = 2\pi - \sum_{i=1}^{n}\alpha_i.$$

Corollary 3.48 (Theorema Elegantissimum, Gauss) *For a geodesic triangle ($n = 3$), we have that*

$$\int_{\varphi(B)} K\, dA = \beta_1 + \beta_2 + \beta_3 - \pi,$$

where $\beta_i = \pi - \alpha_i$ is the interior angle.

Corollary 3.49 *The sum of the interior angles of a geodesic triangle in a surface S is* $\left\{\begin{array}{l} > \pi \\ = \pi \\ < \pi \end{array}\right.$ *if* $\left\{\begin{array}{l} K > 0 \\ K = 0 \\ K < 0 \end{array}\right.$ *, resp.*

As an application of the second version of the local Gauss-Bonnt theorem, we have the following proposition.

Proposition 3.50 *If the Gaussian curvature $K \geq 0$ on a simply-connected surface S, then two geodesics that start at a point $p \in S$ cannot meet again (i.e. there is no geodesic 2-gon in S).*

Proof. Suppose the geodesics meet again. Then they bound a region $\mathcal{R}$ diffeomorphic to a disk, by simply-connectedness of S. By Theorem 3.46, $\int\int_{\mathcal{R}} K\, dA + \alpha_1 + \alpha_2 = 2\pi$. Note that α_1, $\alpha < \pi$ since two distinct geodesics cannot be tangent at a point, and the integral term is nonpositive, so we get a contradiction. □

In particular (case $\alpha_1 = \alpha = 2$):

Corollary 3.51 *There is no simple closed geodesic in a simply-connected nonpositively curved surface.*

Of course, a cilinder is not simply-connected and violates the conclusion of the preceding corollary.

Theorem 3.52 (Gauss-Bonnet, global version) *Let S be a compact orientable surface. Then*

$$\int\int_S K\,dA = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

Remark 3.53 (Digression on the Euler characteristic of a compact surface) A *triangulation* $\mathcal{T}$ of a compact surface S is a decomposition $S = \cup\Delta_i$ into finitely many triangles such that a non-empty intersection $\Delta_i \cap \Delta_j$ $(i \neq j)$ consists of one common side or one common vertex. Radó proved in 1925 that every compact surface admits a triangulation. The Euler characteristic of S with respect to $\mathcal{T}$ is defined to be $\chi(S, \mathcal{T}) = V_{\mathcal{T}} - E_{\mathcal{T}} + F_{\mathcal{T}}$, where $V_{\mathcal{T}}$, $E_{\mathcal{T}}$, $F_{\mathcal{T}}$ denote respectively the total numbers of vertices, edges, faces of triangles in $\mathcal{T}$. It is not difficult to see that $\chi(S, \mathcal{T}) = \chi(S, \mathcal{T}')$ for two triangulations $\mathcal{T}$, $\mathcal{T}'$ of S. This can be proved in two stages: first one checks that it is true in case $\mathcal{T}'$ si a refinement of $\mathcal{T}$; in the general case, one sees that $\mathcal{T}$, $\mathcal{T}'$ admit a commn refinement. This being so, one can define the *Euler characteristic* of S as $\chi(S) = \chi(S, \mathcal{T})$ for any triangulation $\mathcal{T}$ of S. In fact, Poincaré showed that $\chi(S)$ is a topological invariant of S, that is, two homemorphic surfaces have the same Euler characteristic. Theorem 3.52 then says that the total curvature $\int\int_S K\,dA$ is a topological invariant.

The sphere S^2 has $\chi(S^2) = 2$, as is easily seen. This relation is reminiscent of formulas by Descartes and Euler (Euler's relation for convex polyhedra). Similarly, one sees that the torus T^2 has $\chi(T^2) = 0$. More generally, every compact orientable surface is homeomorphic to a sphere with g handles (cylinders) attached; the number is a topological invariant of the surface called *genus*. So $g = 0$ for the sphere and $g = 1$ for the torus. Also, the relation $\chi = 2 - 2g$ is easily checked.

Proof of Theorem 3.52. It is possible to choose a triangulation $\mathcal{T}$ of S such that each triangle Δ_i is contained in the image of a parametrization compatible with the orientation of S. By Theorem 3.46,

$$\int\int_{\Delta_i} K\,dA + \int_{\partial\Delta_i} \kappa_g\,ds + \sum_{j=1}^{3} \alpha_{ij} = 2\pi,$$

where α_{ij} are the external angles of Δ_i. Summing over $i = 1, \ldots, F$, we get

$$\int\int_S K\,dA + \sum_{i=1}^{F} \int_{\partial\Delta_i} \kappa_g\,ds + \sum_{i=1}^{F}\sum_{j=1}^{3} \alpha_{ij} = 2\pi\,F.$$

The second term vanishes, because each edge is counted twice, each time with a different orientation induced by the corresponding triangle. Moreover, if

$\beta_{ij} = \pi - \alpha_{ij}$ is the internal angle,

$$\sum_{i=1}^{F}\sum_{j=1}^{3}\alpha_{ij} = \sum_{i=1}^{F}\sum_{j=1}^{3}\pi - \sum_{i=1}^{F}\sum_{j=1}^{3}\beta_{ij} = 3F\,\pi - 2\pi\,V,$$

since the sum of the internal angles around a vertex is 2π. The proof is completed by noting that $3F = 2E$ since each edge is shared by two faces. □

As an easy application, we have:

Corollary 3.54 *A compact orientable surface in* $\mathbf{R}^3$ *with constant (Gaussian) curvature is homeomorphic to the sphere.*

Proof. Indeed, a compact surface S in $\mathbf{R}^3$ admits a point p with $K(p) > 0$, so the constant curvature must be positive. By Gauss-Bonnet, $2 - 2g = \chi(S) > 0$ implying $g = 0$. □

Indeed Liebmann's theorem (1899) says that S must be a round sphere of radius $1/\sqrt{K}$.