Semigroup Ideals and Generalized Hamming Weights

Maria Bras-Amorós

CIMPA Research School
Algebraic Methods in Coding Theory
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Contents

Main reference

The results in this talk can be found in

M. Bras-Amorós, K. Lee, A. Vico-Oton:
New Lower Bounds on the Generalized Hamming Weights of AG Codes,
IEEE Transactions on Information Theory, vol. 60, n. 10, pp. 5930-5937, October 2014. ISSN: 0018-9448.

Maximum integer not in an ideal

Max integer not in a semigroup (Frobenius number)

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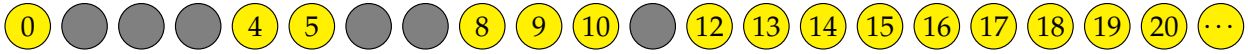
- Λ contains 0,
- Λ is closed under addition,
- Λ has a finite complement in $\mathbb{N}_0$.



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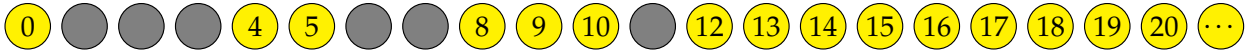

The elements in this complement are called the gaps of the semigroup and the number of gaps is the genus.

The maximum gap is the Frobenius number of the semigroup and the conductor is the Frobenius number plus one.

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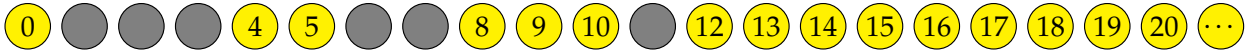

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Lemma

1. $F \leqslant 2g - 1$ *(pigeonhole principle)*
2. $F = 2g - 1 \iff \Lambda$ *symmetric (that is,* $i \in \Lambda \iff F - i \notin \Lambda$*).*

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Ideals of a numerical semigroup

A subset $I \subseteq \Lambda$ is an ideal of a numerical semigroup if and only if

$$I + \Lambda \subseteq I.$$

In particular, $\Lambda \setminus I$ is finite.

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Goal: $\max(\mathbb{N}_0 \setminus I)$.

Example

If $I = \Lambda$ then $\max(\mathbb{N}_0 \setminus I) = F$. In particular,

1. $\max(\mathbb{N}_0 \setminus I) \leqslant 2g - 1$
2. $\max(\mathbb{N}_0 \setminus I) = 2g - 1 \iff \Lambda$ symmetric.

Preliminaries: Barucci's theorem

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Theorem (Barucci)

Any ideal of a numerical semigroup is an intersection of irreducible ideals and irreducible ideals have the form $\Lambda \setminus D(i)$ for some i.

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Example

$\Lambda \setminus D(6) = \{5, 9, 10, 13, \rightarrow\}$ is an irreducible ideal of Λ.

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In $\{0, 4, 5, 8, 9, 10, 12, \rightarrow\}, G(6) = 3$ since $\lambda_6 = 12 = 1 + 11 = 6 + 6 = 11 + 1$.

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Lemma (Hoholdt, van Lint, Pellikaan)

$$\nu_i = i - g(i) + G(i) + 1$$

Bound

Difference of I: $\#(\Lambda \setminus I)$

Theorem

The maximum integer not belonging to an ideal I of a semigroup Λ of genus g with difference d is at most $d + 2g - 1$. That is, $d + 2g + i \in I$ for all $i \geqslant 0$.

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Theorem

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Proof: If I, I' satisfy the result then $I \cap I'$ also satisfies it.

By Barucci's Theorem it is then enough to prove the result for $I = \Lambda \setminus D(i)$.

In this case $\begin{cases} d = \nu_i \\ \max(\mathbb{N}_0 \setminus I) = \max\{c-1, \lambda_i\}. \end{cases}$

We need to see that $\nu_i + 2g \geqslant \max\{c, \lambda_i + 1\}$ (c the conductor).

If $c \geqslant \lambda_i + 1$ then we are done since $2g \geqslant c$.

If $c < \lambda_i + 1$ then $g(i) = g$, $\lambda_i = i + g$, and hence, by HvLP's Lemma,
$\nu_i + 2g = (i - g + G(i) + 1) + 2g = i + g + 1 + G(i) = \lambda_i + 1 + G(i) \geqslant \lambda_i + 1.$ □

Characterization of ideals attaining the bound

Lemma

If $G(i) = 0$ *then* $\lambda_i \geqslant c$.

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If $G(i) = 0$ *then* $\lambda_i \geqslant c$.

Proof: Suppose $G(i) = 0$. Then, $1, \ldots, \lambda_1 - 1$ gaps $\Longrightarrow \lambda_i - \lambda_1 + 1, \ldots, \lambda_i - 1$ non-gaps.

But $\lambda_i \in \Lambda \Longrightarrow [\lambda_i - \lambda_1 + 1, \ldots, \lambda_i] \subseteq \Lambda$.

Now, by adding multiples of λ_1 to the elements in this interval we get the whole set of integers $\lambda_i + k$ with $k \geqslant 0$.

Then $\lambda_i \geqslant c$. □

Characterization of ideals attaining the bound

Theorem

The next statements are equivalent:

1. *The maximum integer not belonging to I is exactly $d + 2g - 1$.*
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The next statements are equivalent:

1. *The maximum integer not belonging to* I *is exactly* $d + 2g - 1$.
2. $I = \Lambda \setminus D(i)$ *for some* i *with* $G(i) = 0$.

Proof: Suppose first that $I = \Lambda \setminus D(i)$ for some i with $G(i) = 0$.

Then $d = \nu_i$.

Also, $G(i) = 0 \Longrightarrow \lambda_i \geqslant c$ and so

- $g(i) = g$
- $\lambda_i = i + g$

Now, by HvLP's Lemma,
$d + 2g - 1 = (i - g(i) + G(i) + 1) + 2g - 1 = i - g + 0 + 1 + 2g - 1 = i + g = \lambda_i \notin I$.

□

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The next statements are equivalent:

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Proof:

Conversely, suppose that the maximum integer not belonging to I is $d + 2g - 1$.

If $I = I' \cap I''$, with I', I'' ideals, $d' = \#(\Lambda \setminus I')$, $d'' = \#(\Lambda \setminus I'')$, and $I', I'' \neq I$, then $d = \#(\Lambda \setminus I) > d', d''$.

If $d + 2g - 1 \notin I$ then $d + 2g - 1 \notin I'$ or $d + 2g - 1 \notin I''$, but $d + 2g - 1 > d' + 2g - 1, d'' + 2g - 1$, contradicting the previous bound.

By Barucci's Theorem, $I = \Lambda \setminus D(i)$ for some i. Also, $d = \nu_i$.

If $\lambda_i < c$, then $\nu_i + 2g - 1 \geqslant 1 + 2g - 1 = 2g \geqslant c$ and so $d + 2g - 1 \in I$, a contradiction.

Therefore $\lambda_i \geqslant c$. Then $\nu_i = i - g + G(i) + 1$ by HvLP's Lemma.
So $d + 2g - 1 = i + g + G(i) = \lambda_i + G(i)$. But $d + 2g - 1 \notin I \Longrightarrow G(i) = 0$. □

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Consider the semigroup

$$\Lambda = \{0, 4, 5, 8, 9, 10, 12, 13, 14, 15, 16, \rightarrow\}.$$

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$$d + 2g - 1 = 4 + 12 - 1 = 15 \in I.$$

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The ideal $I = \Lambda \setminus D(9) = \{4, 8, 9, 12, 13, 14, 16, \rightarrow\}$

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The ideal $I = \Lambda \setminus D(9) = \{4, 8, 9, 12, 13, 14, 16, \rightarrow\}$ has difference equal to $\nu_9 = \#\{0, 5, 10, 15\} = 4$,

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The ideal $I = \Lambda \setminus D(9) = \{4, 8, 9, 12, 13, 14, 16, \rightarrow\}$ has difference equal to $\nu_9 = \#\{0, 5, 10, 15\} = 4$, and

$$d + 2g - 1 = 4 + 12 - 1 = 15 \notin I.$$

This is because $G(9) = 0$. Indeed, $\{15 - 1 = 14, 15 - 2 = 13, 15 - 3 = 12, 15 - 6 = 9, 15 - 7 = 8, 15 - 11 = 4\} \subseteq \Lambda$.

Characterization of ideals attaining the bound

Theorem

The next statements are equivalent:

1. *The maximum integer not belonging to I is exactly $d + 2g - 1$.*
2. $I = \Lambda \setminus D(i)$ *for some* i *with* $G(i) = 0$.
3. $\Lambda \setminus I = \Lambda \cap ((d + 2g - 1) - \Lambda) = \{\lambda \in \Lambda : d + 2g - 1 - \lambda \in \Lambda\}$
4. $I = \{\lambda_i - h : h \in \mathbb{Z} \setminus \Lambda\}$ *for some* i *with* $G(i) = 0$.
5. $\{a + h : h \notin \Lambda, F - h \notin \Lambda\} \subseteq \Lambda$ *and* $I = (a + \Lambda) \cup \{a + h : h \notin \Lambda, F - h \notin \Lambda\}$ *for some* $a \in \Lambda$, $a > 0$.

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Corollary

Let Λ be a symmetric numerical semigroup of genus g. Suppose that I is an ideal of Λ with difference d. Then the largest integer not belonging to I is $d + 2g - 1$ if and only if I is principal.

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The ideal $I = \Lambda \setminus D(9) = \{4, 8, 9, 12, 13, 14, 16, \rightarrow\}$ has difference equal to $\nu_9 = \#\{0, 5, 10, 15\} = 4$, and

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Sequences of pairwise isometric one-point AG codes

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Two codes $C, D \subseteq \mathbb{F}_q^n$ are said to be x-isometric, for $x \in (\mathbb{F}_q^*)^n$ if

$$D = \{x * c = (x_1c_1, \ldots, x_nc_n) : c \in C\}.$$

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Example

Consider the double-repetition code in $\mathbb{F}^{*4}_3$
$C = \{(0,0,0,0),(0,0,1,1),(0,0,2,2),(1,1,0,0),(1,1,1,1),(1,1,2,2),(2,2,0,0),(2,2,1,1),(2,2,2,2)\}$

and the code

$D = \{(0,0,0,0),(0,0,1,2),(0,0,2,1),(1,2,0,0),(1,2,1,2),(1,2,2,1),(2,1,0,0),(2,1,1,2),(2,1,2,1)\}$

One can check that D is $(1,2,1,2)$-isometric to C.

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One can check that D is $(1,2,1,2)$-isometric to C.

A sequence of codes $(C_i)_{i=0,\ldots,n}$ is said to satisfy the isometry-dual condition if there exists $x \in (\mathbb{F}_q^*)^n$ such that C_i is x-isometric to $C_{n-i}^{\perp}$ for all $i = 0, \ldots, n$.

Sequences of pairwise isometric one-point AG codes

Let $P_1, \ldots, P_n, Q$ be different rational points of a (projective, non-singular, geometrically irreducible) curve with genus g and define

$$C_m = \{(f(P_1), \ldots, f(P_n)) : f \in L(mQ)\} \textit{(different than yesterday!)}$$

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$$\begin{aligned} W &= \{0\} \cup \{m \in \mathbb{N} : L(mQ) \neq L((m-1)Q)\} \text{(Weierstrass semigroup)}, \\ W^* &= \{0\} \cup \{m \in \mathbb{N} : C_m \neq C_{m-1}\} = \{m_1 = 0, m_2, \dots, m_n\}. \end{aligned}$$

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Theorem (Geil, Munuera, Ruano, Torres)

- *$W \setminus W^*$ is an ideal of W,*
- *$\{0\}, C_{m_1}, \ldots, C_{m_n}$ satisfies the isometry-dual condition $\Leftrightarrow \#W^* + 2g - 1 \in W^*$.*

Feng-Rao numbers and generalized Hamming weights

Generalized Hamming weights

The generalized Hamming weights of a linear code are, for each given dimension, the minimum size of the support of the linear subspaces of that dimension.

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Example

$C = \{(0,0,0,0),(0,0,1,1),(0,0,2,2),(1,1,0,0),(1,1,1,1),(1,1,2,2),(2,2,0,0),(2,2,1,1),(2,2,2,2)\}$

Subspaces of dimension 1:

- $\langle(1,1,0,0)\rangle$ supported on 2 coordinates
- $\langle(0,0,1,1)\rangle$ supported on 2 coordinates
- $\langle(1,1,1,1)\rangle$ supported on 4 coordinates

So, generalized Hamming weight of dimension 1 (= minimum distance) is 2.

Subspaces of dimension 2:

- $\langle(1,1,0,0),(0,0,1,1)\rangle$ supported on 4 coordinates

So, generalized Hamming weight of dimension 2 is 4.

Generalized Hamming weights

Generalized Hamming weights are used in

- the wire-tap channel of type II
- t-resilient functions
- network coding
- list decoding
- bounding the covering radius of linear codes
- secure secret sharing based on linear codes

Order bounds for algebraic geometry codes

Algebraic geometry codes

Let $P_1, \ldots, P_n, Q$ be different rational points of a (projective, non-singular, geometrically irreducible) curve with genus g and define

$$\begin{aligned} C_m &= \{(f(P_1), \ldots, f(P_n)) : f \in L(mQ)\} \\ W &= \{0\} \cup \{m \in \mathbb{N} : L(mQ) \neq L((m-1)Q)\} \text{(Weierstrass semigr.)} \end{aligned}$$

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Order bound on the minimum distance

The minimum distance of $C_{\lambda_m}^{\perp}$ is lower bounded by the order bound:

$$\delta(m) = \min\{\nu_i : i > m\}$$

Order bounds for algebraic geometry codes

Define $D(i)$ as before and $D(i_1, \ldots, i_r) = D(i_1) \cup \cdots \cup D(i_r)$.

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Define $D(i)$ as before and $D(i_1, \ldots, i_r) = D(i_1) \cup \cdots \cup D(i_r)$.

Order bound on generalized Hamming weights

The r-th generalized Hamming weight of $C_{\lambda_m}^{\perp}$ is lower bounded by the r-th order bound:

$$\delta_r(m) = \min\{\#D(i_1, \ldots, i_r) : i_1, \ldots, i_r > m\}.$$

Farrán-Munuera's Feng-Rao numbers

Theorem (Farrán-Munuera)

For each numerical semigroup Λ *and each integer* $r \geqslant 2$ *there exists a constant* $E_r = E(\Lambda, r)$*, called r-th Feng-Rao number, such that*

1. $\delta_r(m) = m + 2 - g + E_r$ *for all m such that* $\lambda_m \geqslant 2c - 2$*,*
2. $\delta_r(m) \geqslant m + 2 - g + E_r$ *for any m such that* $\lambda_m \geqslant c$*,*

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Furthermore,

3. $r \leqslant E_r \leqslant \lambda_{r-1}$ if $g > 0$ (and $r \geqslant 2$),
4. $E_r = \lambda_{r-1}$ if $r \geqslant c$,
5. $E_r = r - 1$ if $g = 0$.

The perspective of ideals

Recall, $\delta_r(m) = \min\{\#D(i_1, \ldots, i_r) : i_1, \ldots, i_r > m\}$.

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$\Lambda \setminus D(i_1, \dots, i_r) = \Lambda \setminus (D(i_1) \cup \cdots \cup D(i_r)) = (\Lambda \setminus D(i_1)) \cap \cdots \cap (\Lambda \setminus D(i_r))$

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So, $\lambda_{i_r} \leqslant (m + 2 - g + E_r) + 2g - 1 = m + g + 1 + E_r = \lambda_{m+1} + E_r \Longrightarrow$

$$E_r \geqslant \lambda_{i_r} - \lambda_{m+1} = i_r - i_1.$$

Bound on the Feng-Rao numbers

Theorem

Suppose that n_ℓ is the number of intervals of at least ℓ gaps of Λ. Then

$$E_r \geqslant \min\{r - 2 + \left\lceil \frac{r}{\ell - 1} \right\rceil, r - 1 + \left\lceil \frac{(\ell - 1)n_{\ell-1}}{\ell} \right\rceil\}.$$

In particular, if n is the number of intervals of Λ then

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Remark

If $r = 2$ or $n_1 \leqslant 2$ then our bound equals $E_r \geqslant r$. In any other case our bound is better.

Bound on the generalized Hamming weights

Corollary

Let m be such that $\lambda_m \geqslant c$ and let $\ell \geqslant 2$. Then

$$\delta_r(m) \geqslant m + 2 - g + \min\{r - 2 + \left\lceil \frac{r}{\ell - 1} \right\rceil, r - 1 + \left\lceil \frac{(\ell - 1)n_{\ell-1}}{\ell} \right\rceil\}.$$

Corollary

If Λ is a semigroup with conductor c and n intervals of gaps then, for any m with $\lambda_m \geqslant c$,

$$\delta_r(m) \geqslant \begin{cases} m - g + 2r & \text{if } r \leqslant \lceil n/2 \rceil + 1, \\ m - g + r + \lceil n/2 \rceil + 1 & \text{otherwise.} \end{cases}$$

Exercise

1. Prove the Lemma by Hoholdt, van Lint, and Pellikaan stating $\nu_i = i - g(i) + G(i) + 1$, where $g(i)$ is the number of gaps smaller than λ_i and $G(i)$ is the number of pairs of gaps adding up to λ_i.
2. Find W^* in the case of Hermitian codes.
   - Check that $W \setminus W^*$ is an ideal.
   - Prove that Hermitian codes satisfy the isoemtry dual property.