Classification, Characterization and Counting of Semigroups

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Basic notions

Numerical semigroups

Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus:=g:= $\#(\mathbb{N}_0 \setminus \Lambda)$)

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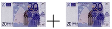
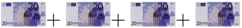
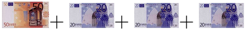
- Conductor := the unique integer c with $c - 1 \notin \Lambda$, $c + \mathbb{N}_0 \subseteq \Lambda$
- Frobenius number := the largest gap $= c - 1$
- Dominant := the non-gap previous to c.

Cash point

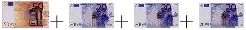
The amounts of money one can obtain from a cash point (divided by 10)

Illustration: Agnès Capella Sala

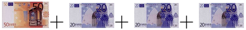
Cash point

amount		amount/10
0		0
10	*impossible!*	
20		2
30	*impossible!*	
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

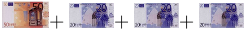
Cash point

amount		amount/10
0		0
		gap
20		2
		gap
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

Cash point

amount		amount/10
0		0
20		2
		(3)
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

Cash point

amount		amount/10
0		0
20		2
40		4
50		5
60		6
70		7
80		8
90		9
100		10
110		11
⋮	⋮	⋮

Cash point

amount		amount/10
0		0
20		2
40	 + 	4
50		5
60	 +  + 	6
70	 + 	7
80	 +  +  + 	8
90	 +  + 	9
100	 + 	10
110	 +  +  + 	11
⋮	⋮	⋮

Enumeration of a numerical semigroup

The inclusion $\Lambda \subseteq \mathbb{N}_0$ implies that there exists

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- Enumeration := the unique bijective increasing map $\lambda : \mathbb{N}_0 \to \Lambda$ $(\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 \ldots\})$

Cash point

amount		amount/10	
0		0	λ_0
20		2	λ_1
40	 + 	4	λ_2
50		5	λ_3
60	 +  + 	6	λ_4
70	 + 	7	λ_5
80	 +  +  + 	8	λ_6
90	 +  + 	9	λ_7
100	 + 	10	λ_8
110	 +  +  + 	11	λ_9
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Enumeration of a numerical semigroup

Lemma

Let Λ be a numerical semigroup with conductor c, genus g, and enumeration λ. The following are equivalent.

(i) $\lambda_i \geqslant c$

(ii) $i \geqslant c - g$

(iii) $\lambda_i = g + i$

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Lemma

Let Λ be a numerical semigroup with conductor c, genus g, and enumeration λ. The following are equivalent.

(i) $\lambda_i \geqslant c$

(ii) $i \geqslant c - g$

(iii) $\lambda_i = g + i$

Proof: Let $g(i)$ be the number of gaps smaller than λ_i. Then $\lambda_i = g(i) + i$.

(i)⇔(iii) $\lambda_i \geqslant c \iff g(i) = g \iff g(i) + i = g + i \iff \lambda_i = g + i$.

(i)⇔(ii) $c = \lambda_{c-g}$ and $\lambda_i \geqslant c = \lambda_{c-g}$ if and only if $i \geqslant c - g$. □

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

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amount		amount/10
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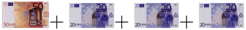
110		11
⋮	⋮	⋮

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If $a_1, \ldots, a_l$ are the generators of a semigroup Λ then

$$\Lambda = \{n_1 a_1 + \cdots + n_l a_l : n_1, \ldots, n_l \in \mathbb{N}_0\}$$

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So, $a_1, \ldots, a_l$ are necessarily coprime.

If $a_1, \ldots, a_l$ are coprime we define the semigroup generated by $a_1, \ldots, a_l$ as

$$\langle a_1, \ldots, a_n \rangle := \{n_1 a_1 + \cdots + n_l a_l : n_1, \ldots, n_l \in \mathbb{N}_0\}.$$

Apéry set

The non-gap λ_1 is always a generator. It is called the multiplicity of Λ.

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Each non-gap of Λ can be expressed as $w_i + k\lambda_1$ for some $i \in \{0, \ldots, \lambda_1 - 1\}$ and some $k \in \mathbb{N}_0$.

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So, the generators different from λ_1 must be in $\{w_1, \ldots, w_{\lambda_1 - 1}\}$.

In particular, there is always a finite number of generators.

The set $\{w_0, w_1, \ldots, w_{\lambda_1 - 1}\}$ is called the Apéry set of Λ.

Exercise

Consider the set
$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, \dots\}$.

1. Prove that H is a numerical semigroup.
2. What are its parameters?
   - conductor,
   - Frobenius number,
   - genus,
   - dominant,
   - Apéry set,
   - generators.

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1. Prove that H is a numerical semigroup.
2. What are its parameters?
   - conductor, 45
   - Frobenius number, 44
   - genus, 33
   - dominant, 43
   - Apéry set, $\{0, 49, 38, 51, 28, 53, 42, 19, 56, 45, 34, 47\}$
     $= \{0, 19, 28, 34, (38 = 19 + 19), 42, 45, (47 = 19 + 28), 49, 51, (53 = 19 + 34), (56 = 28 + 28)\}$
   - generators. $\{12, 19, 28, 34, 42, 45, 49, 51\}$

Classical problems

Frobenius' coin exchange problem

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What is the largest monetary amount that can not be obtained using only coins of specified denominations $a_1, \ldots, a_n$.

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If $a_1, \ldots, a_n$ are coprime then the set of amounts that can be obtained is the semigroup $\langle a_1, \ldots, a_n \rangle$ and the question is determining the Frobenius number.

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$n = 2$: Sylvester's formula $a_1 a_2 - a_1 - a_2$.

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$n = 2$: Sylvester's formula $a_1 a_2 - a_1 - a_2$.

$n > 2$?

Theorem (Curtis)

There is no finite set of polynomials $\{f_1, \ldots, f_n\}$ such that for each choice of $a_1, a_2, a_3 \in \mathbb{N}$, there is some i such that the Frobenius number of a_1, a_2, a_3 is $f_i(a_1, a_2, a_3)$.

Frobenius' coin exchange problem

Some refences on Frobenius' coin exchange problem:

J. L. Ramírez Alfonsín. The Diophantine Frobenius problem, volume 30 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2005.

Frank Curtis. On formulas for the Frobenius number of a numerical semi- group. Math. Scand., 67(2):190–192, 1990.

Hurwitz question

Hurwitz problems

- Determining whether there exist non-Weierstrass numerical semigroups, (Buchweitz gave a positive answer)
- Characterizing Weierstrass semigroups

Some references:

Fernando Torres. On certain N -sheeted coverings of curves and numerical semigroups which cannot be realized as Weierstrass semigroups. Comm. Algebra, 23(11):4211–4228, 1995.

Seon Jeong Kim. Semigroups which are not Weierstrass semigroups. Bull. Korean Math. Soc., 33(2):187–191, 1996.

Jiryo Komeda. Non-Weierstrass numerical semigroups. Semigroup Forum, 57(2):157–185, 1998.

N. Kaplan and L. Ye. The proportion of Weierstrass semigroups, J. Algebra 373:377–391, 2013.

Wilf's conjecture

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The number e of generators of a numerical semigroup of genus g and conductor c satisfies

$$e \geqslant \frac{c}{c-g}.$$

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The number e of generators of a numerical semigroup of genus g and conductor c satisfies
$$e \geqslant \frac{c}{c-g}.$$

Example: If $c = 2g$ (symmetric semigroups) then $\frac{c}{c-g} = \frac{2g}{g} = 2$.

Wilf's conjecture

Some references:

H. Wilf. A circle-of-lights algorithm for the money-changing problem, American Mathematical Monthly 85 (1978) 562–565.

D. E. Dobbs, G. L. Matthews. On a question of Wilf concerning numerical semigroups. International Journal of Commutative Rings, 3(2), 2003.

A. Zhai. An asymptotic result concerning a question of Wilf Alex Zhai, arXiv:1111.2779.

A. Sammartano. Numerical semigroups with large embedding dimension satisfy Wilf's conjecture, Semigroup Forum 85 (2012) 439–447.

N. Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf, J. Pure Appl. Algebra 216 (2012) 1016–1032.

A. Moscariello, A. Sammartano. On a conjecture by Wilf about the Frobenius number, Math. Z. 280 (2015) 47–53.

S. Eliahou. Wilf's conjecture and Macaulay's theorem. arXiv:1703.01761

M. Delgado, On a question of Eliahou and a conjecture of Wilf. arXiv:1608.01353

Wilf conjecture

For brute approach:

M. Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. Semigroup Forum, 76(2):379–384, 2008.

J. Fromentin, F. Hivert. Exploring the tree of numerical semigroups. Mathematics of Computation 85 (2016), no. 301, 2553–2568.

Wilf's conjecture

Exercise

Check Wilf's conjecture for
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Wilf's conjecture

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Check Wilf's conjecture for
$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, \ldots\}$.

- $e = 8$
- $\frac{c}{c-g} = \frac{45}{45-33} = \frac{45}{12} \leqslant 4$

Classification

Symmetric semigroups

Definition

A numerical semigroup with conductor c and genus g is symmetric if $c = 2g$.

Symmetric semigroups

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Example:
The Weierstrass semigroup at point P_∞ of the Hermitian curve $\mathcal{H}_4$ is symmetric.

Its conductor is $c = 12$ and its genus is $g = 6$.

i	λ_i	
0	0	
		$\leftarrow$ 3 gaps
1	4	
2	5	
		$\leftarrow$ 2 gaps
3	8	
4	9	
5	10	
		$\leftarrow$ 1 gap
6	12	$\leftarrow c = 12$
7	13	
8	14	
9	15	
10	16	
$\vdots$	$\vdots$	

Semigroups generated by two integers

Definition

Semigroups generated by two integers are the semigroups of the form

$$\Lambda = \langle a, b \rangle = \{ma + nb : a, b \in \mathbb{N}_0\}$$

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Hermitian's curve $\mathcal{H}_4$ has Weierstrass semigroup equal to $\langle 4, 5 \rangle$.

Geil's norm-trace curve over $\mathbb{F}_{q^r}$ is defined by the affine equation

$$x^{(q^r-1)/(q-1)} = y^{q^{r-1}} + y^{q^{r-2}} + \cdots + y$$

where q is a prime power.

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Geil's norm-trace curve over $\mathbb{F}_{q^r}$ is defined by the affine equation

$$x^{(q^r-1)/(q-1)} = y^{q^{r-1}} + y^{q^{r-2}} + \cdots + y$$

where q is a prime power.

It has a single rational point P_∞ at infinity and the Weierstrass semigroup at P_∞ is

$$\langle (q^r - 1)/(q - 1), q^{r-1} \rangle.$$

Semigroups generated by two integers

Lemma (Sylvester)

1. *The conductor of* $\langle a, b\rangle$ *is* $(a-1)(b-1)$
2. *The genus of* $\langle a, b\rangle$ *is* $\frac{(a-1)(b-1)}{2}$

Semigroups generated by two integers

Lemma (Sylvester)

1. *The conductor of* $\langle a, b\rangle$ *is* $(a-1)(b-1)$
2. *The genus of* $\langle a, b\rangle$ *is* $\frac{(a-1)(b-1)}{2}$

Hence, semigroups generated by two integers are symmetric.

Symmetric semigroups

Lemma

A numerical semigroup Λ is symmetric if and only if for any non-negative integer i,

$$i \notin \Lambda \iff c - 1 - i \in \Lambda.$$

i	λ_i	
0	0	
		11-10
		11-9
		11-8
1	4	
2	5	
		11-5
		11-4
3	8	
4	9	
5	10	
		11-0
6	12	
⋮	⋮	

Pseudo-symmetric semigroups

Definition

A numerical semigroup with conductor c and genus g is pseudo-symmetric if $c = 2g - 1$.

Pseudo-symmetric semigroups

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A numerical semigroup with conductor c and genus g is pseudo-symmetric if $c = 2g - 1$.

Example:
The Weierstrass semigroup at point P_0 of the Klein curve is pseudo-symmetric.

Its conductor is $c = 5$ and its genus is $g = 3$.

i	λ_i	
0	0	
		$\leftarrow$ 2 gaps
1	3	
		$\leftarrow$ 1 gaps
2	5	$\leftarrow c = 5$
3	6	
4	7	
5	8	
$\vdots$	$\vdots$	

Pseudo-symmetric semigroups

Lemma

A numerical semigroup Λ *with odd conductor* c *is pseudo-symmetric if and only if for any integer* i *different from* $(c-1)/2$*,*

$$i \notin \Lambda \iff c-1-i \in \Lambda.$$

i	λ_i	
0	0	
		4-3
		$(c-1)/2$
1	3	
		4-0
2	5	
3	6	
4	7	
5	8	
⋮	⋮	

Irreducible semigroups

Definition

Irreducible semigroups are the semigroups that can not be expressed as a proper intersection of two numerical semigroups.

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Theorem (Rosales,Branco,2003)

The set of irreducible semigroups is the union of the set of symmetric semigroups and the set of pseudo-symmetric semigroups.

Arf semigroups

Definition

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Arf semigroups

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A numerical semigroup Λ is Arf if for any $a, b, c \in \Lambda$ with $a \geqslant b \geqslant c$ we have $a + b - c \in \Lambda$.

Example
The Weierstrass semigroup at point P of the Klein quartic is Arf.

i	λ_i
0	0
1	3
2	5
3	6
4	7
5	8
6	9
7	10
$\vdots$	$\vdots$

$7 + 5 - 3 = 9 \in \Lambda$

Arf semigroups

Lemma

Suppose Λ *is Arf. If* $i, i + j \in \Lambda$ *for some* $i, j \in \mathbb{N}_0$*, then* $i + kj \in \Lambda$ *for all* $k \in \mathbb{N}_0$*. Consequently, if* Λ *is Arf and* $i, i + 1 \in \Lambda$*, then* $i \geqslant c$*.*

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Proof: Let us prove this by induction on k. It is obvious for $k = 0$ and $k = 1$. If $k > 0$ and $i, i + j, i + kj \in \Lambda$ then $(i + j) + (i + kj) - i = i + (k + 1)j \in \Lambda$. □

Arf semigroups

Lemma

Suppose Λ is Arf. If $i, i+j \in \Lambda$ for some $i, j \in \mathbb{N}_0$, then $i + kj \in \Lambda$ for all $k \in \mathbb{N}_0$. Consequently, if Λ is Arf and $i, i+1 \in \Lambda$, then $i \geqslant c$.

Proof: Let us prove this by induction on k. It is obvious for $k = 0$ and $k = 1$. If $k > 0$ and $i, i+j, i+kj \in \Lambda$ then $(i+j) + (i+kj) - i = i + (k+1)j \in \Lambda$. □

Consequently, Arf semigroups are sparse semigroups [Munuera, Torres, Villanueva, 2008], that is, there are no two consecutive non-gaps smaller than the conductor.

Hyperelliptic semigroups

Definition

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They are of the form

$$\Lambda = \{0, 2, 4, \ldots, 2k-2, 2k, 2k+1, 2k+2, 2k+3, \ldots\}$$

for some positive integer k.

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for some positive integer k.

Lemma (Campillo, Farran, Munuera, 2000)

The unique Arf symmetric semigroups are hyperelliptic semigroups.

Semigroups generated by an interval

Definition

A numerical semigroup is generated by an interval if its set of generators is $\{i, i+1, \ldots, j\}$ for some $i, j \in \mathbb{N}_0$.

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Example
The Weierstrass semigroup at point P_∞ of the Hermitian curve $\mathcal{H}_4$ is generated by the interval $\{4, 5\}$.

i	λ_i	
0	0	
1	4	
2	5	
3	8	$= 4 + 4$
4	9	$= 4 + 5$
5	10	$= 5 + 5$
6	12	$= 4 + 4 + 4$
7	13	$= 4 + 4 + 5$
8	14	$= 4 + 5 + 5$
9	15	$= 5 + 5 + 5$
10	16	$= 4 + 4 + 4 + 4$
$\vdots$	$\vdots$	$\vdots$

Exercise

Lemma

The unique numerical semigroups which are generated by an interval and Arf, are the semigroups which are equal to $\{0\} \cup \{i \in \mathbb{N}_0 : i \geqslant c\}$ *for some non-negative integer* c.

Lemma

The unique Arf pseudo-symmetric semigroups are $\{0, 3, 4, 5, 6, \dots\}$ *and* $\{0, 3, 5, 6, 7, \dots\}$ *(corresponding to the Klein quartic).*

Exercise

Lemma

The unique numerical semigroup which is pseudo-symmetric and generated by an interval is $\{0, 3, 4, 5, 6, \dots\}$.

Lemma

$\Lambda_{\{i,\dots,j\}}$ *is symmetric if and only if* $i \equiv 2 \bmod j - i$.

Acute semigroups

Definition

A numerical semigroup is ordinary if it is equal to

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Definition

A numerical semigroup is acute if it is ordinary or if its last interval of gaps is smaller than or equal to the previous one.

Acute semigroups

Example

The Weierstrass semigroup at point P_0 of the Klein quartic is acute.

i	λ_i	
0	0	
		← 2 gaps
1	3	
		← 1 gap
2	5	
3	6	
4	7	
5	8	
6	9	
7	10	
8	11	
9	12	
⋮	⋮	

Acute semigroups

Example

The Weierstrass semigroup at point P_∞ of the Hermitian curve $\mathcal{H}_4$ is acute.

i	λ_i	
0	0	
1	4	
2	5	
		$\leftarrow$ 2 gaps
3	8	
4	9	
5	10	
		$\leftarrow$ 1 gap
6	12	
7	13	
8	14	
$\vdots$	$\vdots$	

Symmetric semigroups are acute

Lemma

All symmetric semigroups are acute.

Proof: Let Λ be a non-ordinary symmetric semigroup.

Since $1 \notin \Lambda$, by the lemma on symmetric semigroups $c - 2 \in \Lambda$.

Thus, the last interval of gaps consists of one gap $(c - 1)$.

The semigroup must therefore be acute. □

Arf semigroups are acute

Lemma

All Arf semigroups are acute.

Proof: Let Λ be a non-ordinary Arf semigroup.

Consider c, c', d, d' as in the example, where $c', c'+1, \dots, d$ is the last interval of non-gaps before the conductor.

$$d \geqslant c' > d' \Longrightarrow d + c' - d' \in \Lambda.$$

i	λ_i	
0	0	$\longleftarrow$ d'
1	3	$\longleftarrow$ c', $\longleftarrow$ d
2	5	$\longleftarrow$ c
3	6	
4	7	
5	8	
6	9	
7	10	
$\vdots$	$\vdots$	

$$\left.\begin{array}{l} d + c' - d' \in \Lambda \\ d + c' - d' > d \end{array}\right\} \Longrightarrow d + c' - d' \geqslant c \Longrightarrow c - d \leqslant c' - d'.$$

□

Semigroups generated by an interval are acute

Lemma

[García-Sánchez, Rosales, 1999]
The numerical semigroup $\Lambda_{\{i,\dots,j\}}$ *generated by the interval* $\{i, i+1, \dots, j\}$ *satisfies*

$$\Lambda_{\{i,\dots,j\}} = \bigcup_{k \geqslant 0} \{ki, ki+1, ki+2, \dots, kj\}.$$

Lemma

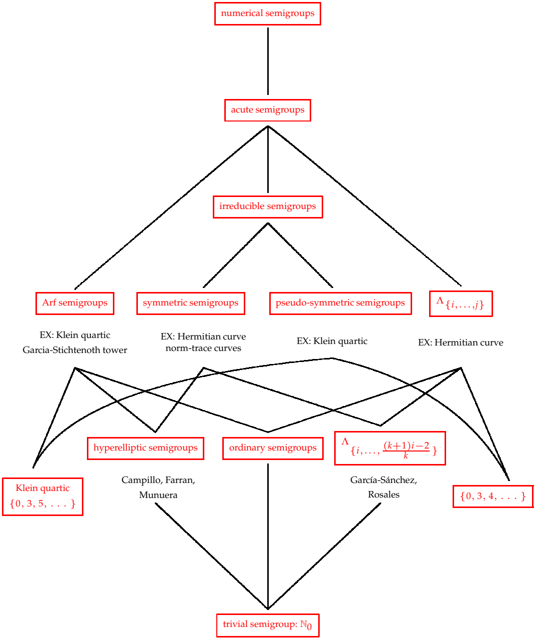
All semigroups generated by an interval are acute.

Proof: It is enough to see that the length of the gap intervals strictly decreases. □

Acute semigroups

Theorem

- *The set of acute semigroups is a proper subset of the whole set of numerical semigroups.*
- *It properly includes*
  - *Symmetric and pseudo-symmetric semigroups,*
  - *Arf semigroups,*
  - *Semigroups generated by an interval.*


numerical semigroups
acute semigroups
irreducible semigroups
Arf semigroups
symmetric semigroups
pseudo-symmetric semigroups
$\Lambda_{\{i,...,j\}}$
EX: Klein quartic
Garcia-Stichtenoth tower
EX: Hermitian curve
norm-trace curves
EX: Klein quartic
EX: Hermitian curve
hyperelliptic semigroups
ordinary semigroups
$\Lambda_{\{i,...,\frac{(k+1)i-2}{k}\}}$
Klein quartic
$\{0, 3, 5, ...\}$
Campillo, Farran,
Munuera
García-Sánchez,
Rosales
$\{0, 3, 4, ...\}$
trivial semigroup: $\mathbb{N}_0$

Characterization

Homomorphisms

Definition

Homomorphisms of numerical semigroups are the maps f such that

$$f(a+b) = f(a) + f(b).$$

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Lemma

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2. *The unique surjective homomorphism is the identity.*

Homomorphisms

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Lemma

1. *Homomorphisms of numerical semigroups are exactly the scale maps* $f(a) = ka$ *for all* a*, for some constant* k*,*
2. *The unique surjective homomorphism is the identity.*

Indeed, if f is a homomorphism then $\frac{f(a)}{a}$ is constant since

$$f(ab) = a \cdot f(b) = b \cdot f(a).$$

Furthermore, for a semigroup Λ, the set $k\Lambda$ is a numerical semigroup only if $k = 1$.

$\oplus$ operation

Definition

Given a numerical semigroup Λ define the associated $\oplus$ operation

$$\oplus_\Lambda : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$$

by

$$i \oplus_\Lambda j = \lambda^{-1}(\lambda_i + \lambda_j).$$

Equivalently,

$$\lambda_i + \lambda_j = \lambda_{i \oplus_\Lambda j}.$$

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The operation $\oplus$ is compatible with the natural order of $\mathbb{N}_0$. That is,

$$a < b \Rightarrow \begin{cases} a \oplus c < b \oplus c \\ c \oplus a < c \oplus b \end{cases} \text{ for any } c \in \mathbb{N}_0.$$

$\oplus$ operation

Example

For the numerical semigroup $\Lambda = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \dots\}$ the first values of $\oplus$ are given in the next table:

$\oplus$	0	1	2	3	4	5	6	7	$\dots$
0	0	1	2	3	4	5	6	7	$\dots$
1	1	3	4	6	7	8	10	11	$\dots$
2	2	4	5	7	8	9	11	12	$\dots$
3	3	6	7	10	11	12	14	15	$\dots$
4	4	7	8	11	12	13	15	16	$\dots$
5	5	8	9	12	13	14	16	17	$\dots$
6	6	10	11	14	15	16	18	19	$\dots$
7	7	11	12	15	16	17	19	20	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Characterization of a semigroup by $\oplus$

Lemma

The $\oplus$ operation uniquely determines a semigroup.

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Proof: Suppose that $\Lambda = \{\lambda_0 < \lambda_1 < \dots\}$ and $\Lambda' = \{\lambda'_0 < \lambda'_1 < \dots\}$ have the same associated operation $\oplus$.

Define the map

$$f(\lambda_i) = \lambda'_i.$$

It is obviously surjective and it is a homomorphism since

$$f(\lambda_i + \lambda_j) = f(\lambda_{i\oplus j}) = \lambda'_{i\oplus j} = \lambda'_i + \lambda'_j = f(\lambda_i) + f(\lambda_j).$$

So, $\Lambda = \Lambda'$. □

Characterization of a semigroup by $\oplus$

Lemma

Define $\Lambda' = d\Lambda \cup \{i \in \mathbb{N} : i \geqslant d\lambda_{a \oplus b}\}$.

Then $i \oplus_{\Lambda'} j = i \oplus_{\Lambda} j$ *for all* $i \leqslant a$ *and all* $j \leqslant b$, *and* $\Lambda' \neq \Lambda$.

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Proof:

Let λ, λ' be the enumerations of Λ, Λ'.

For all $k \leqslant a \oplus_\Lambda b$, $\lambda'_k = d\lambda_k$.

In particular, if $i \leqslant a$ and $j \leqslant b$ then $\lambda'_i = d\lambda_i$ and $\lambda'_j = d\lambda_j$.

Hence, $\lambda'_{i \oplus_{\Lambda'} j} = \lambda'_i + \lambda'_j = d\lambda_i + d\lambda_j = d\lambda_{i \oplus_\Lambda j} = \lambda'_{i \oplus_\Lambda j}$.

This implies $i \oplus_{\Lambda'} j = i \oplus_\Lambda j$. □

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This implies $i \oplus_{\Lambda'} j = i \oplus_{\Lambda} j$. □

Consequently Λ is not determined by any finite subset of $\oplus$ values.

ν sequence

Given a numerical semigroup Λ define its ν sequence as

$$\nu_i = \#\{j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda\}$$

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Example

Klein quartic

i	λ_i	ν_i	
0	0	1	$\{0\}$
1	3	2	$\{0, 3\}$
2	5	2	$\{0, 5\}$
3	6	3	$\{0, 3, 6\}$
4	7	2	$\{0, 7\}$
5	8	4	$\{0, 3, 5, 8\}$
6	9	4	$\{0, 3, 6, 9\}$
7	10	5	$\{0, 3, 5, 7, 10\}$
8	11	6	$\{0, 3, 5, 6, 8, 11\}$
9	12	7	$\{0, 3, 5, 6, 7, 9, 12\}$
10	13	8	$\{0, 3, 5, 6, 7, 8, 10, 13\}$
⋮	⋮	⋮	⋮

τ sequence

Given a numerical semigroup Λ define its τ sequence as

$$\tau_i = \max\{j \in \mathbb{N}_0 : \text{ exists } k \text{ with } j \leqslant k \text{ and } \lambda_j + \lambda_k = \lambda_i\}$$

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Example

Klein quartic

i	λ_i	τ_i	
0	0	0	$0 + 0 = 0$
1	3	0	$0 + 3 = 3$
2	5	0	$0 + 5 = 5$
3	6	1	$3 + 3 = 6$
4	7	0	$0 + 7 = 7$
5	8	1	$3 + 5 = 8$
6	9	1	$3 + 6 = 9$
7	10	2	$5 + 5 = 10$
8	11	2	$5 + 6 = 11$
9	12	3	$6 + 6 = 12$
10	13	3	$6 + 7 = 13$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Exercise

Find the ν-sequence and the τ-sequence of
$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \dots\}$.

Exercise

i	λ_i	$\{\lambda_j : \lambda_i - \lambda_j \in \Lambda\}$	ν	τ
0	0	$\{0\}$	1	0
1	12	$\{0, 12\}$	2	0
2	19	$\{0, 19\}$	2	0
3	24	$\{0, 12, 24\}$	3	1
4	28	$\{0, 28\}$	2	0
5	31	$\{0, 12, 19, 31\}$	4	1
6	34	$\{0, 34\}$	2	0
7	36	$\{0, 12, 24, 36\}$	4	1
8	38	$\{0, 19, 38\}$	3	2
9	40	$\{0, 12, 28, 40\}$	4	1
10	42	$\{0, 42\}$	2	0
11	43	$\{0, 12, 19, 24, 31, 43\}$	6	2
12	45	$\{0, 45\}$	2	0
13	46	$\{0, 12, 34, 46\}$	4	1
14	47	$\{0, 19, 28, 47\}$	4	2
15	48	$\{0, 12, 24, 36, 48\}$	5	3
16	49	$\{0, 49\}$	2	0
17	50	$\{0, 12, 19, 31, 38, 50\}$	6	2
18	51	$\{0, 51\}$	2	0
19	52	$\{0, 12, 24, 28, 40, 52\}$	6	3
20	53	$\{0, 19, 34, 53\}$	4	2
21	54	$\{0, 12, 42, 54\}$	4	1
22	55	$\{0, 12, 19, 24, 31, 36, 43, 55\}$	8	3
23	56	$\{0, 28, 56\}$	3	4
24	57	$\{0, 12, 19, 38, 45, 57\}$	6	2
25	58	$\{0, 12, 24, 34, 46, 58\}$	6	3
26	59	$\{0, 12, 19, 28, 31, 40, 47, 59\}$	8	4
27	60	$\{0, 12, 24, 36, 48, 60\}$	6	3
28	61	$\{0, 12, 19, 42, 49, 61\}$	6	2
29	62	$\{0, 12, 19, 24, 28, 31, 34, 38, 43, 50, 62\}$	11	5
30	63	$\{0, 12, 51, 63\}$	4	1
31	64	$\{0, 12, 19, 24, 28, 36, 40, 45, 52, 64\}$	10	4
32	65	$\{0, 12, 19, 31, 34, 46, 53, 65\}$	8	5
33	66	$\{0, 12, 19, 24, 28, 38, 42, 66\}$	8	4
34	67	$\{0, 12, 19, 24, 31, 36, 43, 48, 55, 67\}$	10	5
35	68	$\{0, 12, 19, 28, 34, 40, 49, 56, 68\}$	9	6
36	69	$\{0, 12, 19, 24, 31, 38, 45, 50, 57, 69\}$	10	5
37	70	$\{0, 12, 19, 24, 28, 34, 36, 42, 46, 51, 58, 70\}$	12	6
38	71	$\{0, 12, 19, 24, 28, 31, 40, 43, 47, 52, 59, 71\}$	12	5

Characterization of a semigroup by τ

Theorem

A numerical semigroup is completely determined by its τ sequence.

Proof: We can construct a numerical semigroup Λ from its τ sequence as follows:

- Let k be the minimum integer such that for all $i \in \mathbb{N}_0$,
  - $\tau_{k+2i} = \tau_{k+2i+1}$
  - $\tau_{k+2i+2} = \tau_{k+2i+1} + 1$
- Set
  - $c = k - \tau_k + 1$
  - $g = k - 2\tau_k$

  This determines λ_i for all $i \geqslant c - g$
- For $i = c - g - 1$ to 1, $\lambda_i = \frac{1}{2}\min\{\lambda_j : \tau_j = i\}$

□

Characterization of a semigroup by ν

Theorem

A numerical semigroup is completely determined by its ν sequence.

Proof: We can construct a numerical semigroup Λ from its ν sequence as follows:

- If $\nu_i = i + 1$ for all $i \in \mathbb{N}_0$ then $\Lambda = \mathbb{N}_0$
- Otherwise let $k = \max\{j : \nu_j = \nu_{j+1}\}$ (it exists and it is unique)
- Set $g = k + 2 - \nu_k$ and $c = \frac{k+g+2}{2}$
  - $0 \in \Lambda, 1, c - 1 \notin \Lambda$
  - For all $i \geqslant c, i \in \Lambda$
- For $i = c - 2$ to $i = 2$,
  - Define $\tilde{D}(i) = \{l \in \Lambda^c : c - 1 + i - l \in \Lambda^c, i < l < c - 1\}$
  - $i \in \Lambda$ if and only if $\nu_{c-1+i-g} = c + i - 2g + \#\tilde{D}(i)$

□

Semigroup characterization

Theorem

No numerical semigroup can be determined by any finite subset of

- ν *values*
- τ *values*
- $\oplus$ *values*

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- τ *values*
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Exercise

Prove the theorem.

Counting

Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

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- $n_1 = 1$, since the unique numerical semigroup of genus 1 is $\mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are

$$\{0, 3, 4, 5, \dots\},$$

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- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are

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- $n_3 = 4$
- $n_4 = 7$
- $n_5 = 12$
- $n_6 = 23$
- $n_7 = 39$
- $n_8 = 67$

$\vdots$

Counting semigroups by genus

Conjecture

[Bras-Amorós, 2008]

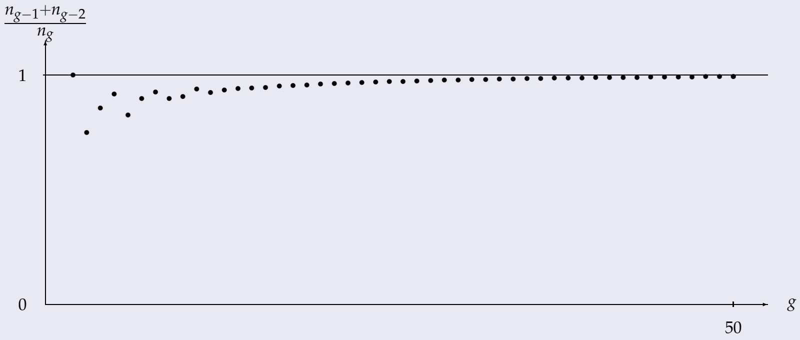
1. $n_g \geqslant n_{g-1} + n_{g-2}$
2. - $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
   - $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$

Counting semigroups by genus

g	n_g	$n_{g-1}+n_{g-2}$	$\frac{n_{g-1}+n_{g-2}}{n_g}$	$\frac{n_g}{n_{g-1}}$
0	1			
1	1			1
2	2	2	1	2
3	4	3	0.75	2
4	7	6	0.857143	1.75
5	12	11	0.916667	1.71429
6	23	19	0.826087	1.91667
7	39	35	0.897436	1.69565
8	67	62	0.925373	1.71795
9	118	106	0.898305	1.76119
10	204	185	0.906863	1.72881
11	343	322	0.938776	1.68137
12	592	547	0.923986	1.72595
13	1001	935	0.934066	1.69088
14	1693	1593	0.940933	1.69131
15	2857	2694	0.942947	1.68754
16	4806	4550	0.946733	1.68218
17	8045	7663	0.952517	1.67395
18	13467	12851	0.954259	1.67396
19	22464	21512	0.957621	1.66808
20	37396	35931	0.960825	1.66471
21	62194	59860	0.962472	1.66312
22	103246	99590	0.964589	1.66006
23	170963	165440	0.967695	1.65588
24	282828	274209	0.969526	1.65432
25	467224	453791	0.971249	1.65197
26	770832	750052	0.973042	1.64981
27	1270267	1238056	0.974642	1.64792
28	2091030	2041099	0.976121	1.64613
29	3437839	3361297	0.977735	1.64409
30	5646773	5528869	0.979120	1.64254
31	9266788	9084612	0.980341	1.64108
32	15195070	14913561	0.981474	1.63973
33	24896206	24461858	0.982554	1.63844
34	40761087	40091276	0.983567	1.63724
35	66687201	65657293	0.984556	1.63605
36	109032500	107448288	0.985470	1.63498
37	178158289	175719701	0.986312	1.63399
38	290939807	287190789	0.987114	1.63304
39	474851445	469098096	0.987884	1.63213
40	774614284	765791252	0.988610	1.63128

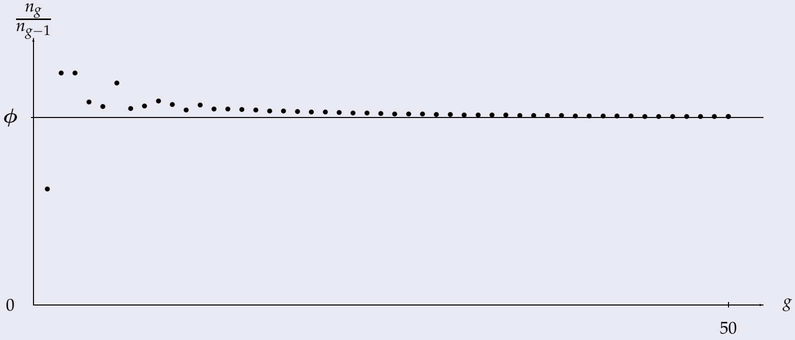
Counting semigroups by genus

Behavior of $\frac{n_{g-1}+n_{g-2}}{n_g}$



Counting semigroups by genus

Behavior of $\frac{n_g}{n_{g-1}}$



Counting semigroups by genus

What is known

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- $\lim_{g\to\infty}\frac{n_g}{n_{g-1}} = \phi$
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Weaker unsolved conjecture

- n_g is increasing

Dyck paths

Dyck paths

Definition

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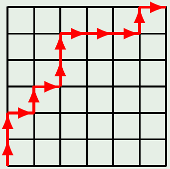

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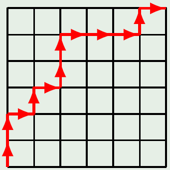

The number of Dyck paths of order n is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Dyck paths

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The square diagram of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

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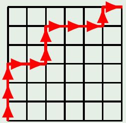
Example

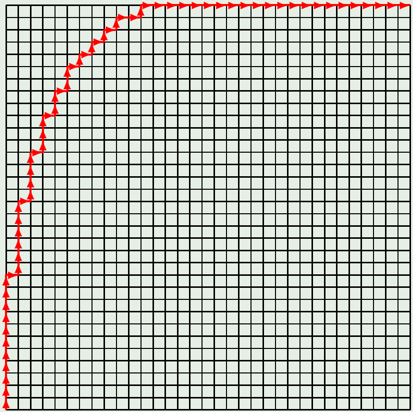
The square diagram of the numerical semigroup $\{0,4,5,8,9,10,12,\ldots\}$ is



Dyck paths

Example

The square diagram of the numerical semigroup $\{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, \ldots\}$ is



Dyck paths

Lemma

[Bras-Amorós, de Mier, 2007]
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Dyck paths

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Corollary

The number of numerical semigroups of genus g is bounded by the Catalan number $C_g = \frac{1}{g+1}\binom{2g}{g}$.

Semigroup tree and Fibonacci bounds

Tree of numerical semigroups

From genus g to genus $g-1$

A semigroup of genus g together with its Frobenius number is another semigroup of genus $g-1$.

$$\{0,2,4,5,\dots\} \mapsto \{0,2,3,4,5,\dots\}$$

Tree of numerical semigroups

From genus g to genus $g-1$

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$$\{0,2,4,5,\dots\} \mapsto \{0,2,3,4,5,\dots\}$$

A set of semigroups may give the same semigroup when adjoining their Frobenius numbers.

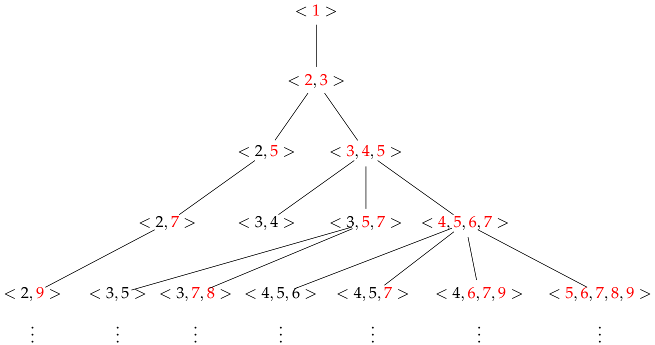
$$\begin{matrix}\{0,2,4,5,\dots\}\\ \{0,3,4,5,\dots\}\end{matrix} \mapsto \{0,2,3,4,5,\dots\}$$

Tree of numerical semigroups

From genus $g-1$ to genus g

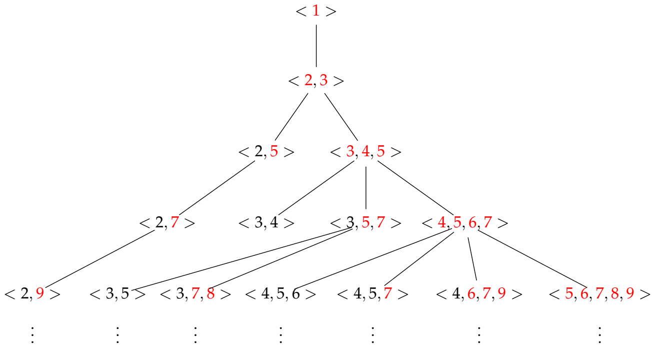
All semigroups giving Λ when adjoining to them their Frobenius number can be obtained from Λ by taking out one by one all generators of Λ larger than its Frobenius number.

Tree of numerical semigroups



The descendants of a semigroup are obtained taking away one by one all generators larger than its Frobenius number.

Tree of numerical semigroups



The descendants of a semigroup are obtained taking away one by one all generators larger than its Frobenius number.

The parent of a semigroup Λ is Λ together with its Frobenius number.
[Rosales, García-Sánchez, García-García, Jiménez-Madrid, 2003]

Tree of numerical semigroups

Lemma

The ordinary semigroup of genus g has $g+1$ descendants which in turn have $0, 1, 2, \ldots, g-2,\ g,\ g+2$ descendants.

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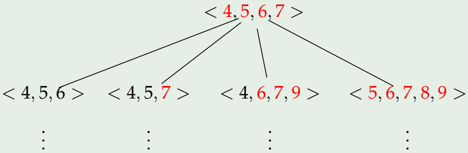

Tree of numerical semigroups

Lemma

Let $\lambda_i \in \Lambda$ be a generator of Λ (non-ordinary) larger than its Frobenius number. If $\lambda_j > \lambda_i$ satisfies

- *λ_j is not a generator of Λ*
- *λ_j is a generator of $\Lambda \setminus \{\lambda_i\}$*

then $\lambda_j = \lambda_1 + \lambda_i$.

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then $\lambda_j = \lambda_1 + \lambda_i$.

Proof: Since λ_j is not a generator of Λ, $\lambda_j = \lambda_r + \lambda_s$.
Since λ_j is a generator of $\Lambda \setminus \{\lambda_i\}$, $\lambda_j = \lambda_i + \lambda_r$.
Suppose $r > 1$. Then

$$\lambda_j = \lambda_1 + \underbrace{\lambda_i + \underbrace{\lambda_r - \lambda_1}_{>0}}_{\in \Lambda \setminus \{\lambda_i\}}, \text{ contradiction.}$$

Tree of numerical semigroups

Corollary

If the generators of Λ (non-ordinary) that are larger than its Frobenius number are $\{\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_k}\}$, then the generators of $\Lambda \setminus \{\lambda_{i_j}\}$ that are larger than its Frobenius number are

$$\{\lambda_{i_{j+1}} < \cdots < \lambda_{i_k}\},$$

or

$$\{\lambda_{i_{j+1}} < \cdots < \lambda_{i_k}\} \cup \{\lambda_1 + \lambda_{i_j}\}$$

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Corollary

If a node in the semigroup tree has k descendants, then its descendants have

- *at least* $0, \ldots, k-1$ *descendants, respectively,*
- *at most* $1, \ldots, k$ *descendants, respectively.*

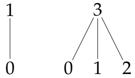
Subtree

Number of descendants of semigroups of genus 2

1	3
$\{0,2,4,5,\dots\}$	$\{0,3,4,5,\dots\}$

Subtree

Lower bound for the number of descendants of semigroups of genus 3



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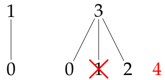
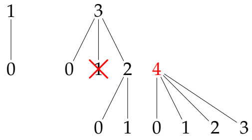


Subtree

Lower bound for the number of descendants of semigroups of genus 4



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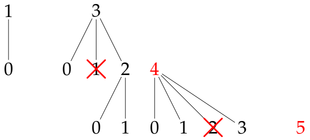
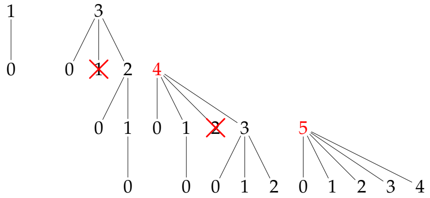


Subtree

Lower bound for the number of descendants of semigroups of genus 5



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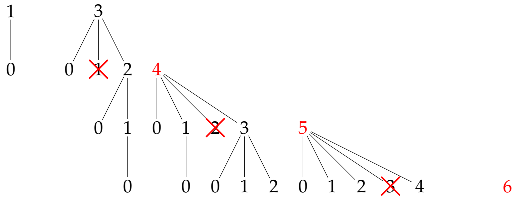
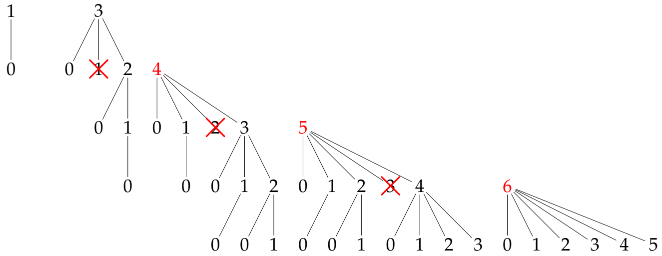


Subtree

Lower bound for the number of descendants of semigroups of genus 6



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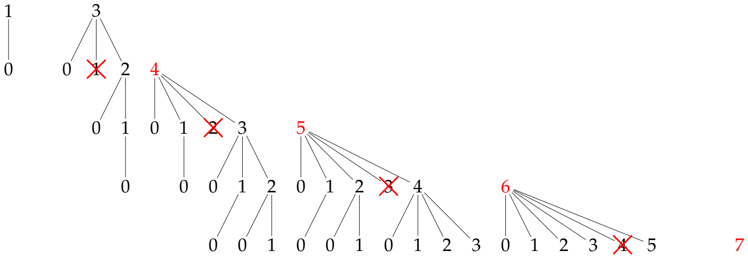

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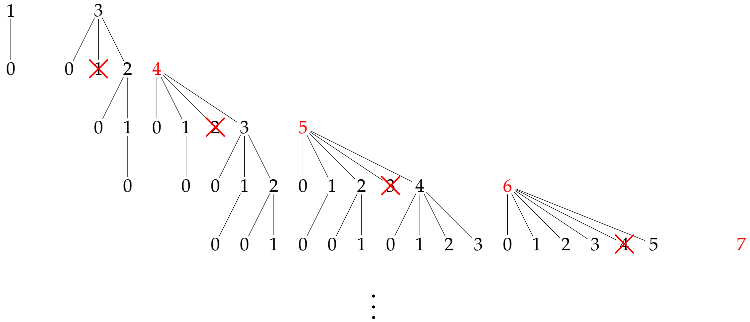
Lower bound for the number of descendants



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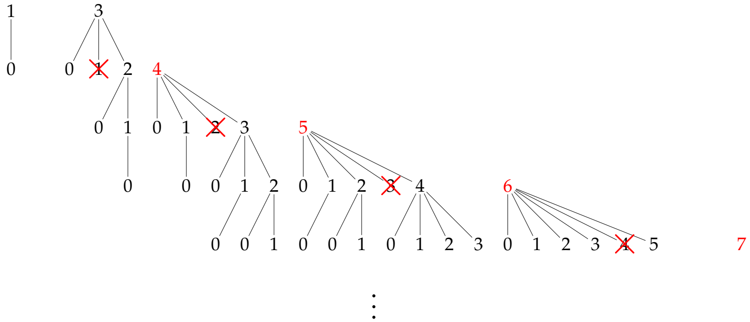

Lemma

For $g \geqslant 3$,

$$2F_g \leqslant n_g.$$

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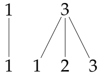

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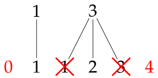

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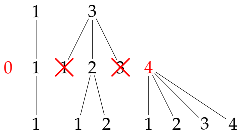

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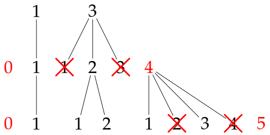

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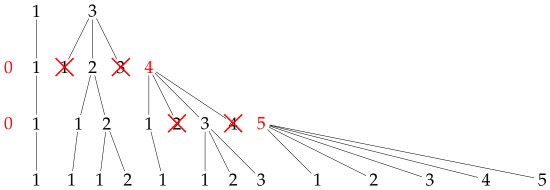

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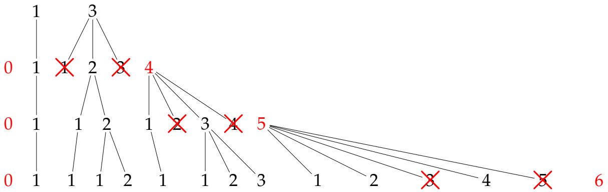

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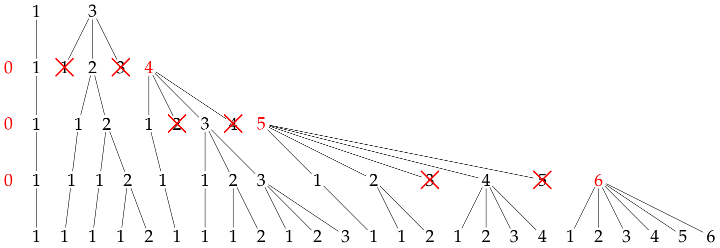

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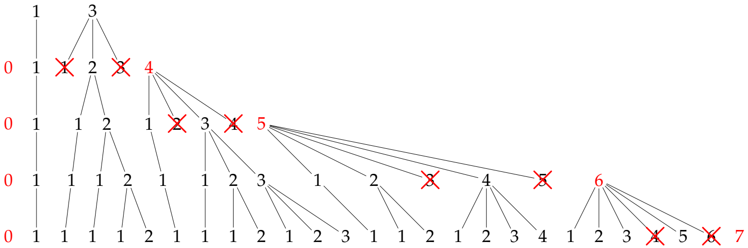

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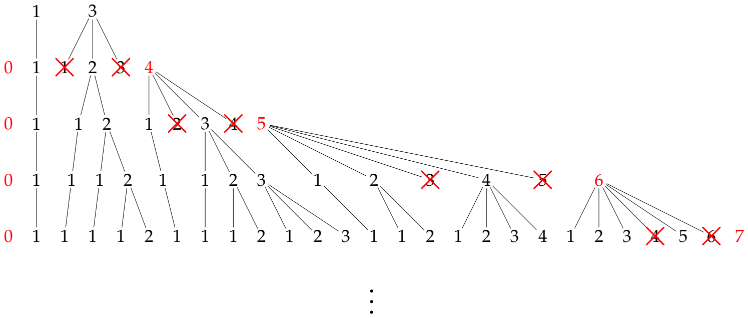

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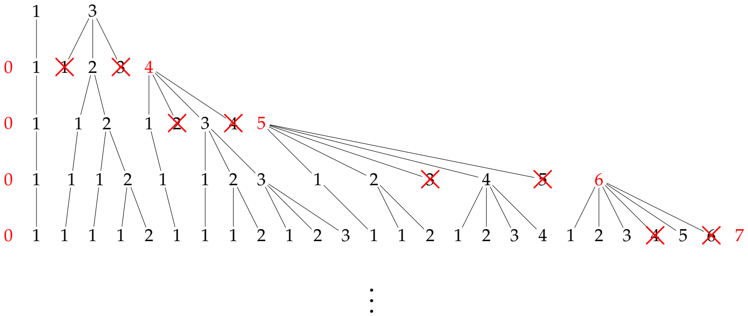


Lemma

For $g \geqslant 3$,

$$2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}.$$

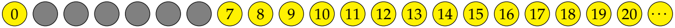
Bounds on n_g

g	$2F_g$	n_g	$1+3\cdot 2^{g-3}$	C_g
0		1		1
1		1		1
2	2	2		2
3	4	4	4	5
4	6	7	7	14
5	10	12	13	42
6	16	23	25	132
7	26	39	49	429
8	42	67	97	1430
9	68	118	193	4862
10	110	204	385	16796
11	178	343	769	58786
12	288	592	1537	208012
13	466	1001	3073	742900
14	754	1693	6145	2674440
15	1220	2857	12289	9694845
16	1974	4806	24577	35357670
17	3194	8045	49153	129644790
18	5168	13467	98305	477638700
19	8362	22464	196609	1767263190
20	13530	37396	393217	6564120420
21	21892	62194	786433	24466267020
22	35422	103246	1572865	91482563640
23	57314	170963	3145729	343059613650
24	92736	282828	6291457	1289904147324
25	150050	467224	12582913	4861946401452
26	242786	770832	25165825	18367353072152
27	392836	1270267	50331649	69533550916004
28	635622	2091030	100663297	263747951750360
29	1028458	3437839	201326593	1002242216651368
30	1664080	5646773	402653185	3814986502092304

Ordinarization transform and ordinarization tree

Ordinary numerical semigroups

A numerical semigroup is ordinary if all its gaps are consecutive.
In this case multiplicity=Frobenius number + 1.



Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity (smallest non-zero non-gap)
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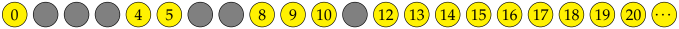

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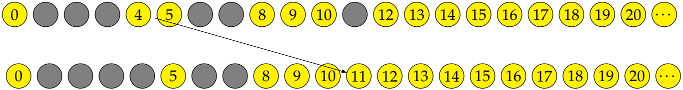

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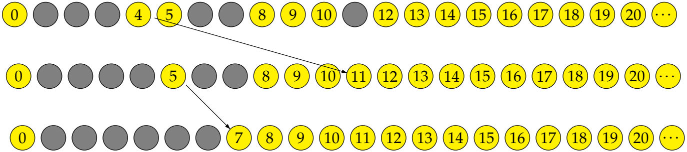

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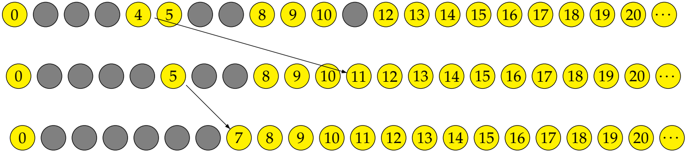

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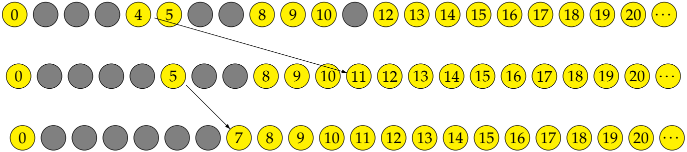

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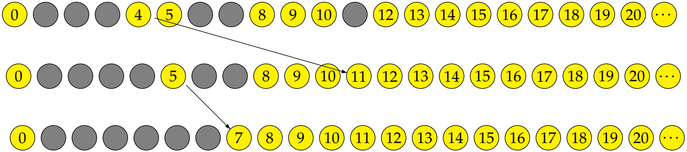

- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= ordinarization number) we obtain an ordinary semigroup.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The tree $\mathcal{T}_g$

Define a graph with

- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its ordinarization transform

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$\mathcal{T}_g$ is a tree rooted at the unique ordinary semigroup of genus g.

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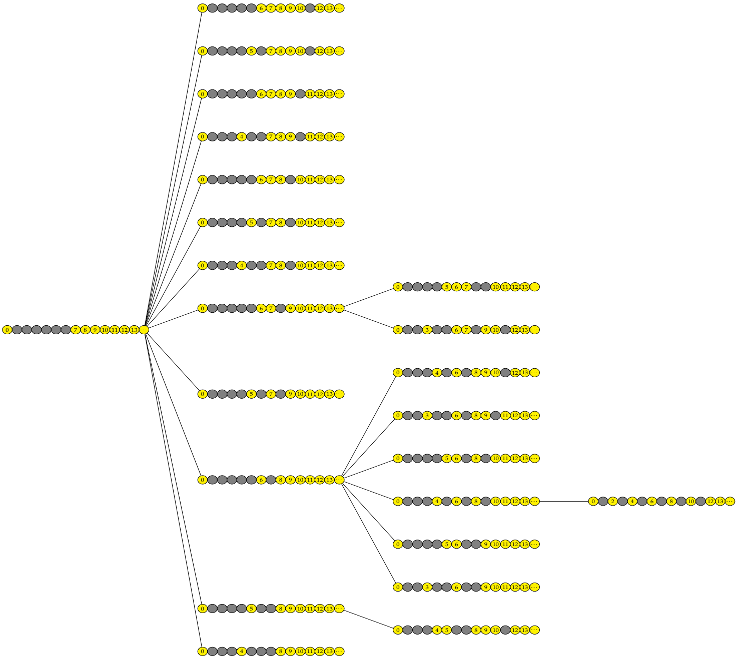
Contrary to $\mathcal{T}$, $\mathcal{T}_g$ has only a finite number of nodes (indeed, n_g).


0 7 8 9 10 11 12 13 ···
0 6 7 8 9 10 12 13 ···
0 5 7 8 9 10 12 13 ···
0 6 7 8 9 11 12 13 ···
0 4 7 8 9 11 12 13 ···
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$\mathcal{T}_g$ and $\mathcal{T}$

Lemma

If Λ_1 is a descendant of Λ_2 in $\mathcal{T}$ then Λ_1' is a descendant of Λ_2' in $\mathcal{T}$.

Lemma

If two non-ordinary semigroups Λ_1 and Λ_2 with the same genus g have the same parent in $\mathcal{T}$ then they also have the same parent in $\mathcal{T}_g$.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The depth of a semigroup of genus g in $\mathcal{T}_g$ is its ordinarization number.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The depth of a semigroup of genus g in $\mathcal{T}_g$ is its ordinarization number.

Lemma

1. *The ordinarization number of a numerical semigroup of genus g is the number of its non-zero non-gaps which are $\leqslant g$.*
2. *The maximum ordinarization number of a semigroup of genus g is $\lfloor \frac{g}{2} \rfloor$.*
3. *The unique numerical semigroup of genus g and ordinarization number $\lfloor \frac{g}{2} \rfloor$ is $\{0, 2, 4, \dots, 2g, 2g+1, 2g+2, \dots\}$.*

Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

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This conjecture would prove $n_g \leqslant n_{g+1}$.

Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

This conjecture would prove $n_g \leqslant n_{g+1}$. This result is proved for the lowest and largest depths.

Computational evidence

r\ g	g=0	g=1	g=2	g=3	g=4	g=5	g=6	g=7	g=8	g=9	g=10	g=11	g=12	g=13	g=14	g=15	g=16	g=17	g=18	g=19	g=20	g=21
r=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
r=1			1	3	5	9	12	18	22	30	35	45	51	63	70	84	92	108	117	135	145	165
r=2					1	2	9	19	39	70	118	196	281	432	586	838	1080	1490	1835	2449	2956	3804
r=3							1	1	4	16	47	97	228	442	844	1462	2447	4017	6127	9516	13693	20152
r=4									1	1	2	3	28	60	180	442	1083	2202	4611	8579	15830	27493
r=5											1	1	2	2	9	27	93	215	721	1685	4417	9633
r=6													1	1	2	2	7	9	45	89	319	889
r=7															1	1	2	2	7	7	25	47
r=8																	1	1	2	2	7	7
r=9																			1	1	2	2
r=10																					1	1

r\ g	g=22	g=23	g=24	g=25	g=26	g=27	g=28	g=29	g=30	g=31	g=32	g=33	g=34	g=35	g=36	g=37
r=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
r=1	176	198	210	234	247	273	287	315	330	360	376	408	425	459	477	513
r=2	4498	5690	6582	8162	9352	11370	12879	15480	17317	20569	22877	26812	29610	34454	37739	43538
r=3	27768	39726	52312	72494	93341	125600	157758	208370	255661	331626	401389	510031	608832	764927	899285	1114817
r=4	46615	76616	120795	189550	285103	429618	618555	905721	1255646	1790138	2418323	3354611	4425179	6031518	7767784	10392180
r=5	21378	41912	83951	153896	281388	487211	831654	1374366	2218771	3524257	5445975	8352388	12435320	18555615	26695019	38853706
r=6	2635	6446	17582	39214	90574	188007	394521	756910	1469758	2662254	4823002	8344482	14314198	23747986	38898550	62372773
r=7	142	340	1266	3483	10171	26489	69692	161111	382713	816457	1763299	3533977	7088495	13371197	25321828	45500820
r=8	23	24	96	157	553	1570	5281	14835	43790	113548	294908	701946	1652408	3632809	7973030	16368101
r=9	7	7	23	23	69	95	301	627	2457	7168	23475	68223	194677	512838	1323375	3178140
r=10	2	2	7	7	23	23	68	70	228	309	1142	2994	10901	33846	109619	318308
r=11	1	1	2	2	7	7	23	23	68	68	202	232	740	1249	4843	14332
r=12			1	1	2	2	7	7	23	23	68	68	200	201	649	759
r=13					1	1	2	2	7	7	23	23	68	68	200	200
r=14							1	1	2	2	7	7	23	23	68	68
r=15									1	1	2	2	7	7	23	23
r=16											1	1	2	2	7	7
r=17													1	1	2	2
r=18															1	1

r\ g	g=38	g=39	g=40	g=41	g=42	g=43	g=44	g=45	g=46	g=47	g=48	g=49
r=0	1	1	1	1	1	1	1	1	1	1	1	1
r=1	532	570	590	630	651	693	715	759	782	828	852	900
r=2	47510	54320	58986	67072	72419	81855	88142	98946	106170	118716	126844	141164
r=3	1299978	1590237	1836517	2226669	2545983	3059220	3477286	4134725	4669073	5518427	6185260	7256830
r=4	13180451	17322789	21616641	28040199	34458068	44142389	53663689	67788397	81530366	102094609	121404838	150477267
r=5	54507523	77486888	106094921	148091995	198378083	272201928	358476988	483240666	626315811	833944191	1063739070	1397557241
r=6	98298482	152816803	232801607	352797809	521753229	772496765	1114488292	1614321267	2277566111	3242295418	4478817624	6268430457
r=7	81612546	140878791	241699680	402445891	664483703	1072569052	1711738040	2688862529	4165828031	6388426599	9636305171	14462411903
r=8	33550240	65385970	126969443	235541563	436401532	777427260	1380117648	2375549463	4064063006	6774823275	11221522599	18200647631
r=9	7487630	16760501	36890000	77385799	160762381	319996692	631894288	1203245544	2273796763	4158339885	7567139870	13367227712
r=10	899807	2383461	6101724	14810797	34997273	79159902	175168573	373545010	782283651	1585487022	3171168252	6150909456
r=11	51663	164512	519339	1509557	4237829	11221868	28679326	70097864	166062233	379419480	845334246	1824208237
r=12	2527	5652	21994	71261	252707	803934	2492982	7226212	20114114	53281902	136131501	334153690
r=13	616	649	1925	2679	9947	27432	106780	361575	1245778	3945659	12053243	34718395
r=14	200	200	615	617	1800	1939	6144	11138	43824	140489	537134	1835716
r=15	68	68	200	200	615	615	1766	1804	5254	6320	22087	52194
r=16	23	23	68	68	200	200	615	615	1764	1765	5102	5278
r=17	7	7	23	23	68	68	200	200	615	615	1764	1764
r=18	2	2	7	7	23	23	68	68	200	200	615	615
r=19	1	1	2	2	7	7	23	23	68	68	200	200
r=20			1	1	2	2	7	7	23	23	68	68
r=21					1	1	2	2	7	7	23	23
r=22							1	1	2	2	7	7
r=23									1	1	2	2
r=24											1	1

Lemma (Bernardini and Torres (2017))

The sequence f_γ given by

$$
\begin{aligned}
f_0 &= 1, \\
f_1 &= 2, \\
f_2 &= 7, \\
f_3 &= 23, \\
f_4 &= 68, \\
f_5 &= 200, \\
f_6 &= 615, \\
f_7 &= 1764, \\
f_8 &= 5060, \\
f_9 &= 14626, \\
&\dots
\end{aligned}
$$

also counts the number of semigroups of genus 3γ and γ even gaps.

Conjecture (Bernardini,Torres)

$$f_\gamma \sim \varphi^{2\gamma}$$

Further contributions on counting

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