Survey of research, 14 December, 2005
(Complement to the "livre-docência" thesis)
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This survey of my research was written in December 2005, when I was preparing an updated version of the "livre-docência" thesis [Fis03]; it is intended as an introduction to that thesis. The livre-docência is a title unique to the University of São Paulo, though it is roughly equivalent to the French or German "Habilitation". It involves taking a special exam involving four parts, one of which consists of explaining and defending the thesis before the examining board.

Writing this thesis provided me with an opportunity to look back at the beginnings of the ideas for these projects, and to see how they had gradually come to assume the more final form of papers.

I will organize this account around a series of *Questions* and *Analogies* which provided the underlying motivations. The questions, often originally philosophical in nature, were first restated in a more mathematically precise form, as conjectures; these later became definite theorems, leading to papers. The *analogies*, between themes or objects in different settings, were percieved first at an intuitive level- as one thing reminded me of another- and then this analogy was pushed and explored until it too led to specific results.

There is a bias here, toward placing more empasis on the earlier papers; the reason for this is that we wish to bring to light the common origin of the later ideas. A more complete account of this later work is given in the survey [Fis04] as well as in the livre-docência thesis itself [Fis03].

Some questions.

A " meta-question" for me has been: "What is randomness?"

This took a specific form:

Question 1: "What does it mean to choose a point randomly (from a mathematical object: a manifold, a group, a graph?)

A first answer could be: "choose the point according to a natural distribution", that is, distributed according to a probability measure which is natural to the object concerned.

Further answers might be: choose the point according to some "randomizing mechanism"; this could be a mathematical version of throwing a dart at a dart board or watching the state of the balls on a billiard table evolve over time.

We will illustrate these by some examples:

(**1**) The circle. Here there is a natural candidate for the measure, Haar measure, since this is the unique rotation-invariant normalized Borel measure.

This reflects our intuition that a random choice of a point should be made in a rotation-invariant way. Here are some related randomizing mechanisms:

-start a Brownian motion someplace on the circle, or on the center of the disk, and stop at a large time.

-apply a large number of iterates of the doubling map $z \mapsto z^2$ to a typical point.

-make an infinite number of i.i.d. coin-tosses and choose the point from the circle with the corresponding binary expansion.

In defining these procedures, the randomness has been moved to another place (choice of a Brownian path and stopping time; choice of a point and number of iterates; making the sequence of coin tosses), and the intuition can be given concrete mathematical form: in each case, the Haar measure will appear in the limit.

(**2**) The real line. Here things get interesting. A first answer is "with respect to a translation-invariant probability measure on $\mathbb{R}$". But such a measure is now *finitely* additive, and is highly nonunique. Equivalently, one wants a linear functional λ on $L^\infty(\mathbb{R})$ which satisfies:
(i) λ is positive and normalized and translation invariant (an *invariant mean*). (That there are many essentially is due to the non-uniqueness of Hahn-Banach extension.)

I was bothered by this nonuniqueness, so I tried to imagine additional properties such a measure or mean "should" have. I came up with these:
(ii) λ is dilation invariant
(iii) λ is invariant with respect to an exponential change of scale: $\lambda(f) = \lambda(f \circ \exp)$.
(iv) λ is universally measurable and satisfies Fubini's theorem: $\lambda(\beta(\mu)) = \beta(\lambda(\mu))$ where μ is a (countably additive) Borel probability measure on L^∞ and β denotes the center of mass (the *barycenter*).
(v) λ is invariant with respect to a Brownian change of time: given $f \in L^\infty$, then for a.e. Brownian path $B(t)$, $\lambda(f) = \lambda(f(B(t))$.

The existence of such a mean was proved in [Fis87]. Some comments:

Properties (i) and (iii) imply (ii); (iv) is Mokobodzski's *measure-linearity* (see [Mey73]); properties (i), (iii) and (v) imply (iv), while (i) plus (iv) implies:
(vi) *convolution-invariance*: $\lambda(f) = \lambda(\varphi * f)$ for any $\varphi \in L^1$ with $\int \varphi = 1$. This is a property that has been studied extensively in the literature, see e.g. [Gre69].

Such a mean is still highly nonunique. But it turned out that what was really needed in the proof of the existence of a mean satisfying (i) and (iii) was the *convexity* of the exponential function. So I set up some abstract machinery to prepare the way for future pursuit of the following question:

Does there exist a natural collection of convex scale changes Co, containing exp, such that there exists a *unique* λ which satisfies the above properties and is also Co-invariant?

This problem remains very much open.

The convolution operators of (vi) together with the exponential change of scale (iii) lead to the following class of integral operators: $\varphi * f$ (what we call order-zero averaging operators) and $\log^n \circ(\varphi * (f \circ \exp^n))$ (the order-n averaging operators, for $n \geq 1$). The corresponding averaging method of order n is defined as follows: one applies the operator and then takes the limit as $t \to +\infty$.

For the particular choice $\varphi = \chi_{(0,\infty)} e^{-x}$, we have these especially nice formulas:

for $n = 0$, $f(t) \mapsto \Phi * f(t)$;
for $n = 1$, the *Césaro averaging operator*

$$f(t) \mapsto \frac{1}{t} \int_0^t f(x) \mathrm{d}x,$$

for $n = 2$, the *log averaging operator*

$$f(t) \mapsto \frac{1}{\log t} \int_1^t \frac{f(x)}{x} \mathrm{d}x$$

for $n = 3$, the *log log averaging operator*

$$f(t) \mapsto \frac{1}{\log \log t} \int_e^t \frac{f(x)}{x \log x} \mathrm{d}x$$

and so on.

The corresponding averaging methods are the *log averages* of Hardy and Riesz. So beginning with convolution-invariance (which I had learned about from Greenleaf's book, and which is a natural generalization of translation-invariance, as the convolution is an average of translates of the function) and combining this with property (iii) (exponential invariance) I had rediscovered these classical formulas, along the way. And this approach placed these classical methods in a broader context.

The relationship between the operators is this: for fixed n, by Wiener's Tauberian theorem, as long as φ has Fourier transform non-zero, the order-n operators are all equivalent. An example with nonzero Fourier transform is $\varphi = \Phi$ defined above. Moreover, as proved in [Fis87], as n increases the averaging methods are all mutually consistent and successively stronger (for any φ).

Note that by definition, an invariant mean λ which satisfies (i), (iii) and (vi) is automatically invariant with respect to the averaging operators. The proof of the existence of λ actually goes the other direction; consistency of the averaging operators leads to a mean which is exponentially invariant.

The point of view of exponential scale change led me to consider more general convex changes of scale, and to the open problem mentioned above.

(**3**)Next we try to choose a point randomly from a discrete nonamenable group G. Here the question by definition makes no sense, as there exists no translation-invariant mean on G. Nevertheless, the previous work on $\mathbb{R}$ suggests two approaches.

For simplicity we consider the free group on two generators, with μ the measure which gives equal weight $1/4$ to the generators and their inverses.

Given a bounded function f on the group, a first approach would be to take the average (using our mean λ, transferred to the integers) of the sequence given by convolution powers of μ applied to f, that is, $\lambda((\mu^{(n)} * f)(e))$. Intuitively, this defines the "mean of the function f as seen from the identity e" and is the λ-mean value of the average of f over ever-increasing disks.

The second approach is to begin a random walk at e with increments distributed like μ and then average the function along this path, again by applying the mean λ.

So we have two methods; the natural mathematical question becomes whether they are *consistent*.

The answer, given in full in a joint project with Vadim Kaimanovich, [FK98] is this:

The "average" of f is now not a single value but rather a *function.* What type of function? A *harmonic function.* So our average defines a projection operator, the *harmonic projection*, onto the μ-harmonic funtions. This is for method (i). The second method yields a funtion as well, now defined on Furstenberg's boundary at infinity of the group G.

How are these functions related? Integrating the second with respect to the hitting measure for the random walk beginning at a point, gives the value of the harmonic function: the average "as seen from" that point.

The passage between these to methods is exactly analogous to property (iv) for means on $\mathbb{R}$: via Mokobodski measure-linearity, i.e. the Fubini property.

A last remark: one knows that these are equivalent, for a group G: that G is amenable; that any bounded harmonic function is constant; that the boundary of G is a single point. Thus, in the special case of an amenable group, if we follow either averaging procedure, we get a constant, and this is a true mean value for the bounded function with which we began.

Question 2: In probability theory, there are "pathwise" theorems (the strong law of large numbers, the law of the iterated logarithm) and "ensemble" theorems (the weak law, the Central Limit

Theorem). Here is the question which intrigued me: Is it possible to come up with a pathwise version of the CLT? Specifically, is it true that given a random walk S_n in the domain of attraction of Brownian motion, that the average time spent in an interval I by $S_n/\sqrt{n}$ is the limiting expected value $\frac{1}{\sqrt{2\pi}}\int_I e^{x^2/2}\mathrm{d}x$? I first considered this for Brownian motion itself. The answer is: for the Césaro time average *no*, the limit $1/T\int_0^T \chi_I(B(t)/\sqrt{t}\mathrm{d}t$ *does not* exist, for a.e. path B; but for the log average *yes*!

The proof fell out of two insights: that the log average was appropriate, rather than the usual Césaro time average, because of the self-similarity of the Brownian paths; and that this self-similarity (the scaling property of Brownian paths, that $B(at)/\sqrt{a}$ is distributed like $B(t)$) could be given this dynamical reformulation: the map $\tau_s : B(t) \mapsto B(e^s t)/e^{s/2}$ defines a measure-preserving *flow* on path space. This is an ergodic flow, and hence we can use the Birkhoff ergodic theorem; because of the exponential time change, and because the log average is the exponential conjugate of the Césaro average, we have our proof. This became part of the paper [Fis87].

Pursuing this question for random walk paths themselves, the intuitive idea (already stated in [Fis87]) was that since random walk paths are close to Brownian paths, perhaps they converge dynamically in this flow. I searched the probability literature for relevant theorems, and discovered the following: that Strassen' s *almost-sure invariance principle* (a.s.i.p.) [Str65] provided exactly what I needed.

More precisely, Strassen had shown that for the case of i.i.d. increments X_i with mean zero and finite $(2+\delta)$ moments, one can couple a Brownian path $B(t)$ to the polygonally interpolated random walk path $S(t)$ so as to satisfy $|S(t)-B(t)| = o(t^{1/2})$ almost surely. In [Fis], we showed the equivalence between the condition $|S(t)-B(t)| = o(t^{1/2})$ and that $d(\tau_t(S), \tau_t(B)) \to 0$ as $t \to \infty$, i.e. that S and B are forward asymptotic in the scaling flow.

It follows from this second property that for any uniformly continuous bounded observable, the time averages under the flow are the same for the random walk and the Brownian paths. So by the equivalence between the two statements, this demonstrated the pathwise CLT for these random walk paths.

That left the intriguing case of finite second moments, which was much more delicate. In fact the a.s.i.p. itself then *cannot* hold, due to counterexamples of Breiman and Major; what I was able to show however was an a.s.i.p. *in log density.* The proof used a combination of ergodic theoretic and probability techniques.

This left open the case of random walk paths in the nonnormal domain of attraction (with regularly varying tails), and the related questions for all stable domains of attraction. Motivated by an early preprint version of [Fis], Berkes and Dehling attacked this problem, proving a type of a.s.i.p. (log) for stable laws.

However their statement was no longer dynamical in character, which made it unsuitable for certain applications. This situation has now been remedied in [FT06b], where we build on [BD93] to prove a dynamical a.s.i.p. in log density for all stable laws. The corresponding question for renewal processes is handled in [FT06a], and an application of these results to certain maps of the interval with an indifferent fixed point is presented in [FLT05]. See [FL01] for related work.

The idea of treating a self-similar process as a flow led me far beyond, to consider self-similar objects in general. A first step came in the form of:

Question (3) Is there an analogue of the Lebesgue density theorem for fractal sets? The Lebesgue density theorem states that for a positive measure set $A \subseteq \mathbb{R}^n$, then for a.e. pont $x \in A$, the proportion of a ball of radius ε about x occupied by A goes to 1 as $\varepsilon \to 0$. Now as is well known, this is *false* for fractal sets; the lim sup and lim inf of these quantities as $\varepsilon \to 0$ (using the Hausdorff measure of the set) are a.s. *different.* The idea was to apply the log average, and see if this converges. Together with Tim Bedford in [BF92], we managed to prove this limit (called the *order-two density*, as it used the order-two averages from [Fis87]) converged to a constant value for these sets: Sullivan's nonlinear $\mathbb{C}^{1+\alpha}$ hyperbolic "cookie cutter" Cantor sets, and the zero set of Brownian motion.

We mention that in the literature, what we called the order-two density has been rechristened the *average density*; see the books [Fal97], [Mat95a] and research papers [Fal95], [Mat95b], [Man95] among many others.

Question (4) Can one give a precise meaning to Mandelbrot's intuitive notion of the "lacunarity" of a fractal set? Mandelbrot's intriguing idea, see [Man82], was to distinguish somehow between different sets of the same dimension, as some had more "holes" than others. Part of the motivation for [BF92] was to suggest average density as a numerical measure of lacunarity. A number of authors has pursued this idea, and have calculated this value for a variety of sets; see the references mentioned above.

In [BF92] two flows appeared in the proof: the scaling flow of Brownian local time, and a special flow built over the natural extension of the hyperbolic doubling map for the cookie cutter case.

We tried to better understand the role of these flows; this led to:

Question (5) How can one model mathematically the idea of zooming down to small scales in a fractal set? Let us suppose the set is embedded in $\mathbb{R}^n$. We consider the collection of all closed subsets of the one-point compactification of $\mathbb{R}^n$, topologized by the Hausdorff metric on sets, and define a continuous flow this collection, the *magnification flow*, by $\tau^t(A) = e^t \cdot A$, just dilation of $A \subseteq \mathbb{R}^n$ by an exponential factor. Then we define the *scenery at x* of a fractal set A, given a point $x \in A$, to be the ω-limit set of $(A - x)$ under the magification flow. Thus we have shifted the location x to the origin, taken all dilations and taken all eventual limits of these pictures. The *scenery flow* of A is the magification flow acting on the union of sceneries for all $x \in A$.

In a series of papers with Tim Bedford and Mariusz Urbanski [BF96], [BFU02], we constructed explicitly the scenery flow for these examples: Brownian zero sets, hyperbolic $\mathbb{C}^{1+\alpha}$ Cantor sets, hyperbolic Julia sets. In [BF97] we studied the related "scenery process" for the hyperbolic Cantor sets, proving a rigidity theorem based on work of Sullivan [Sul87] and giving a new, dynamical interpretation to his "dual Cantor set" (it is the past of the natural extension of the hyperbolic map; this is a particularly natural construction when one has the scenery flow in mind).

A further example, explained in [Fis04], is the limit set of a Fuchsian group of second type (i.e. where the limit set is a Cantor subset of the circle) or of a geometrically finite Kleinian group. Here we showed that the scenery flow is a finite index factor of the geodesic (frame) flow of the associated surface.

Analogies (6) The analogies between the scenery flow of a fractal set and the geodesic flow on the one hand, and the pair (geodesic flow, horocycle flow) and (shift map, odometer map) on the other, are explored in [Fis92], [Fis04] and in forthcoming papers.

As explained in [Fis04], the scenery flow gives a new way of understanding our previous proofs of the existence of order-two density in [BF92]: one is taking the time average of an L^1- observable along this ergodic flow, so the limit exists, is a.s. constant and is equal to the expected value (which may, however, be hard to evaluate!) This proof works for all our examples: Brownian zero sets, hyperbolic Cantor and Julia sets, Fuchsian and Kleinian limit sets.

Question (7): Are there "fractal" sets of integers? Are there analogues of Hausdorff dimension and measure for these sets? These questions were addressed in [BF92] and [Fis92]. The example we had in mind, in addition to the random walk zero set, was a sort of integer analogue of the middle-third Cantor set: the integers $[C] = \{\Sigma_{k=0}^{n} a_i 3^k : a_i = 0 \text{ or } 2 \text{ and } n \geq 0\}$. The dimension of $A \subseteq \mathbb{Z}^+$ is defined to be the polynomial growth rate $d = \limsup \log N_n / \log n$ where $N_n = \#(A \cap [0, n])$. The d-dimensional Hausdorff measure is $\lambda(N_n/n^d)$ where λ is an invariant mean on $\mathbb{Z}^+$ which extends the log average (as described in Question (2)).

We proved in [BF92] that the dimension of the integer set A corresponding to the zeroes of a simple random walk on the integers imitates the zero set Z of a Brownian path B in that its "dimension" is $d = 1/2$, while its d-dimensional "Hausdorff measure" is equal to the average density of Z, calculated to be $\sqrt{2\pi}$. In [Fis92] we proved this for the integer Cantor set $[C]$ as well: its integer dimension is $\log 2/\log 3$, like that for the standard middle-thirds set C, while its integer Hausdorff measure is equal to the average density of C.

Question (8): Do (some) infinite measure-preserving transformations have fractal-like return sets? This was part of the insight and motivation behind [Fis92]. For a conservative (i.e. recurrent) ergodic measure preserving transformation of an infinite measure space, of course the density of return times to a subset A of finite measure os 0. The idea was that, at least in certain well-behaved cases, this return-time set might be fractal-like and so amenable to study by the techniques explored in [BF92] and [Fis92]. Some of the general theory in this direction is developed in [Fis92], and to illustrate these general ideas, the basic example of the integer Cantor set $[C]$ is studied in detail.

Question (9): Can we prove a theorem of the form "time average= space average" for some infinite measure-preserving transformations? This is, for the finite measure case, a standard restatement of the Birkhoff ergodic theorem. The problem is that the usual extension of this theorem to infinite measures, the Hopf (or Chacon-Ornstein) ratio ergodic theorem, provides no such statement. We believed that nevertheless one might be able to say more, at least in nice cases, and so began to study examples.

In [Fis92] we built explicitly an infinite measure transformation T whose return-time behavior would mimic the fractal structure of the set $[C]$ and for which such a statement does hold. We then showed that for any L^1 function φ on this measure space, the time average $\lambda(S_k(f(x))/k^d)$ where λ is our invariant mean on the integers (or, explicitly, the log average) converges to $c \int \varphi$ for a.e. x, where $S_k(f(x)) = f(x) + f(Tx) + \cdots + f(T^n x)$ and where c is the average density of the Cantor set C.

A counterexample is mentioned in [Fis92], where no such statement "time average= space average" can be possible: these are the *squashable* maps of Hajian, Ito and Kakutani. See also [Aar97].

A further example is mentioned at the end of [Fis92]. Consider a simple random walk on the integers $\mathbb{Z}$, beginning at time 0 with counting measure on $\mathbb{Z}$. This defines a stationary Markov process, and we study the shift map on paths S_n. Ignoring periodicity (we could start at even integers and consider the shift squared), we have a conservative ergodic map preserving an infinite measure. Return times to the set of paths taking on the value 0 at time 0 (which has measure 1) are exactly the zero set of the random walk. One can prove the exisience of time averages in the above sense by means of the ideas above about integer fractal sets, now using the scaling flow for Brownian local time together with an a.s.i.p. for local time of the random walk, due to Revesz.

This result was a rediscovery of a result of Chung and Erdös from 1951 [CE51] regarding Markov processes. However our new technique only covered one special case of their theorem (zeroes of the special random walk).

With Aaronson and Denker, see [ADF92], we were then able to extend the Chung-Erdös theorem, which they had proved for certain Markov shifts, to a more general setting which included e.g. some Blaschke products on the circle.

Another case where we can prove such a theorem is for horocycle flows of Fuchsian groups of second type. This is sketched in [Fis04].

We mention that in exciting new work, Ledrappier and Sarig have proved related theorems for certain abelian extensions of horocycle flows. In an intuitive sense their example extends simultaneously both the random walk and horocycle example, and this is why they use log log rather than log averages [cLS05].

Question (10): Does unique ergodicity make sense in the infinite measure setting? Now the best one can hope for is that there be an invariant nonatomic measure which is unique up to constant multiples. My first example of a transformation preserving an infinite measure which is uniquely ergodic in this sense was the shift generated by the integer Cantor set of [Fis92]. A second example we found at about the same time was the horocycle flow of a Fuchsian group of second type. It turned out that Marc Burger had discovered this second example independently and had just published a proof of unique ergodicity for the case where the limit set had dimension in $(1/2, 1)$ [Bur90]; his proof was completely different from mine, as it used harmonic analysis extensively; mine, more ergodic theoretic, had the advantage of working for all of $(0, 1)$.

We began to work together, and realized that both of our results had been anticipated by Kenny in an unpublished thesis (by a third method!) [Ken83]. We then worked to establish a stronger theorem, and based on yet another method; it remains for these results to be fully written up.

Question (11): Can we understand the transition from finite to infinite invariant measures in a parametrized model, as a sort of "phase change"? This idea was inspired by a question asked of me by Artur Lopes. We studied explicit models on interval maps, related to renewal processes [FL01]. This relates to one part of the parameter space, where we show polynomial decay of correlations and prove the Central Limit Theorem by means of a Gordin-Liverani type approach. In later work (preprint in preparation) together with Lopes and Talet we extend this study to the other parameter regimes. There are three "phases": the Gaussian, stable and Mittag-Leffler regimes; the latter corresponds to the infinite measure (and fractal-like return time) case. Much more information on these projects is given in [Fis03].

Question (12):There are two formulas for the Hausdorff dimension of a Kleinian limit set, due to Sullivan and Bowen; can they be given a unified form? Bowen's formula is valid not only for Kleinian limit sets but also e.g. for a hyperbolic Julia set J with rational map f; the statement

[Bow79], [Rue82] is, writing $P(\varphi)$ for the pressure of a function φ: there exists a unique d such that $P(-\log|Df|^d) = 0$; this is $\dim(\mathcal{J})$.

Sullivan proves that, for Γ be a geometrically finite subgroup of Möb(Δ^n), then the geodesic flow of $M = \Gamma\backslash\Delta$ is ergodic for the Patterson-Sullivan measure $\hat{\mu}$, the unique measure of maximal entropy, the entropy of which is equal to the Hausdorff dimension of the limit set $\dim(\Lambda) = d$.

We were able to unite these two formulas; the single statement is that both for $\mathcal{J}$ and for Λ:

The entropy of the marked scenery flow of the limit set is equal to the Hausdorff dimension of the limit set.

See [Fis04]. For the Julia set case, the link between scenery flow entropy and Bowen's formula is explained also in [BFU02].

Question (13): Can we prove ergodicity of the scenery flow? This would be one of the first questions one would ask about this flow, once it has been defined. The essential problem is the rotation symmetries in the space of limiting pictures; does one move ergodically through all angles? In [BFU02], we show that nonergodicity is equivalent to the following: that there exists a nontrivial Hausdorff measurable line field for the rational map f. This question was studied separately in [FU00], and the answer is that the flow is ergodic except in very specific exceptional cases. A second approach to the proof of this was then given by [May02].

Analogy (14): Understand the analogy between the scenery flow of a fractal set and the Teichmüller flow. This question came up in a discussion with P. Arnoux. The scenery flow involves zooming down toward small scales; so does the process of Rauzy induction of an interval exchange transformation, which provides a cross-section to the Teichmüller flow of a compact surface. We studied this in detail for the simplest case (the surface is the torus) in [AF01]. Essentially, we wanted to combine ideas of [Fis92] on the scenery flow and [Arn94] where a beautiful way of coding the modular flow by continued fractions is described. Our answer is to define a scenery flow on a space of *nested tilings* of the real line, related to exchanges of two intervals and nested according to the renormalization hierarchy; then we give several isomorphic models for this flow, one algebraic, one combinatorial, and one geometric: the extension to torus fiber bundles of the Teichmüller flow.

Question (15): Can the parameter space and the dynamical space be combined in one dynamical model? This was part of the motivation for [AF01]; dynamical space in this case is an irrational circle rotation, and parameter space is the collection of all rotation angles, i.e. the collection of all such maps. The rotations are returns to a circle cross-section of a vertical foliation on the torus; moving this torus along its Teichmüller flow orbit brings us to a new rotation, that given by renormalization. The scenery flow contains all of this dynamics in one picture. It is clear that this idea extends to more intervals; the intriguing question is if something similar can be carried our e.g. for quadratic maps.

Question (16): Can results and definitions for the dynamics of a single transformation be meaningfully extended to "nonstationary" dynamics? The paper [AF01] led to this question, which is to be addressed in a series of papers, beginning with [AF05]. This is an extensive project, and the reader would do best to consult the introduction of that paper!

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