September 1, 1999
Final version: to appear, Bulletin London Math. Soc.

On invariant line fields

ALBERT M. FISHER AND MARIUSZ URBAŃSKI[1]

Dept Mat IME-USP (Universidade de São Paulo),
Caixa Postal 66281, CEP 05315-970, São Paulo, Brazil
Department of Mathematics, University of North Texas, Denton, TX 76203-5118, USA,
E-mail: afisher@ime.usp.br, urbanski@dynamics.math.unt.edu
Web: http://www.math.unt.edu/~urbanski
http://www.ime.usp.br/~afisher

Abstract. We show that a rational function of degree ≥ 2 admits an invariant line field with respect to some measure μ, which is an equilibrium state of a Hölder continuous potential whose topological pressure is greater than its supremum, only in very special cases when the Julia set is either a geometric circle or an interval or it is totally disconnected and contained in a real-analytic curve.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of degree ≥ 2 and let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential defined on the Julia set such that $\mathrm{P}(\phi) > \sup(\phi)$, where $\mathrm{P}(\phi)$ is the topological pressure of ϕ with respect to the map $f : J(f) \to J(f)$. For the definition and various properties of topological pressure the reader may consult [Bo], [Ru], [Wa] or [PU] for example. It is true (see the same sources as above) that $\mathrm{P}(\phi)$ has the following metric-theoretical characterization, called the *variational principle*.

$$\mathrm{P}(\phi) = \sup\{\mathrm{h}_\mu(f) + \int \phi d\mu\},$$

where the supremum is taken over all Borel probability f-invariant measures supported on $J(f)$ and $\mathrm{h}_\mu(f)$ is the metric entropy of f with respect to the measure μ; hence one can take this as the definition of pressure. A Borel probability f-invariant measure μ on $J(f)$ is said to be an *equilibrium state* for ϕ if $\mathrm{h}_\mu(f) + \int \phi d\mu = \mathrm{P}(\phi)$. One knows that for each continuous potential on $J(f)$ there exists at least one equilibrium state; this is due to M. Lyubich in [Ly]. For Hölder continuous potentials which satisfy the condition $\mathrm{P}(\phi) > \sup(\phi)$, the equilibrium state is in fact unique, as shown in [DU] (see also [Pr]). We

[1]Supported by TARP Grant

denote this unique equilibrium state by μ_ϕ. It is ergodic; further dynamical and ergodic theoretic properties, including metric exactness and the Central Limit Theorem, have been established in [DU], [Pr], [DPU] and [Ha].

The main result of this paper is contained in the following.

Theorem 1[2]**.** Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of of degree ≥ 2 and let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential satisfying $\mathrm{P}(\phi) > \sup(\phi)$. Suppose that there exist $k \geq 1$ and a measurable function $u : J(f) \to S^1$ such that for μ_ϕ-a.e. $z \in J(f)$

$$\left(\frac{f'(z)}{|f'(z)|}\right)^k = \frac{u(f(z))}{u(z)}. \tag{1}$$

Then the map f is critically finite and either

(a) f has a superattracting fixed point with a preimage at which f has a different degree.

(b) f is critically finite with parabolic orbifold.

(c) The Julia set $J(f)$ is a geometric circle and f is biholomorphically conjugate to a finite Blaschke product.

(d) The Julia set $J(f)$ is a real-analytic closed segment and f is biholomorphically conjugate to a 2-to-1 factor of a finite Blaschke product.

(e) $J(f)$ is totally disconnected and $J(f)$ is contained in a real-analytic curve with self-intersections (if any) lying outside of the Julia set.

The reader may wish to keep in mind the following examples of Julia sets which are exceptional in the sense of (b)-(e):

(b) f is the Lattès map $z \mapsto (z^2+1)^2/4z(z^2-1)$. The Julia set is now the whole sphere and the map is covered by the conformal Anosov endomorphism of the torus $z \mapsto (1+i)z$. So, an invariant vector field is is the projection of any constant line field. This projected line field is well-defined and in fact smooth except at the orblifold points.

(c) $f(z) = z^2$ for which e.g. the vector field of unit vectors tangent to the unit circle is invariant.

(d) $f(z) = z^2 - 2$. The Julia set is the interval $[-2, 2]$ and any constant direction (line) field is invariant.

(e) $f(z) = z^2 + \sqrt{3}$. The Julia set is a Cantor subset of $\mathbb{R}$ and any constant direction (line) field is invariant.

We note that in the above examples the invariant line fields are in fact smooth except at finitely many points. Something like this holds in general and plays a key role in the proof: a main step is to show that a measurable invariant line field can be improved to one that is smooth on a large set (see Proposition 6).

[2]After this paper has been submitted for publication we received the preprint [Ma] where this theorem has been proved for the measure of maximal entropy and the case (a) was ruled out.

The present result is a strengthening of a lemma in [BFU], where we showed that in the special case of hyperbolic rational maps and Hausdorff measure, for a dense G_δ set in parameter space, there is no (Hausdorff) measurable invariant line field. That lemma was then applied in proving the ergodicity of the scenery flow of the Julia set.

The basic structure of the proof of this theorem is the same as that in the paper [Zd]. It consists of several steps. For the first one let us recall the notion of good inverse branches of iterates of f. We work with the natural extension $(\tilde{J}, \tilde{\mu}_\phi, \tilde{f})$. Recall that $\tilde{J} = \{\{x_n\}_{n=0}^\infty \subset J(f) : x_n = f(x_{n+1})\}$ and that $\tilde{\mu}_\phi$ is the only measure on $\tilde{J}$ such that $\tilde{\mu}_\phi \circ \pi^{-1} = \mu$, where $\pi : \tilde{J} \to J(f)$ is the projection onto the 0th coordinate. The map$\tilde{f} : \tilde{J} \to \tilde{J}$ is then the lift of f, defined by the formula $\tilde{f}(\{x_n\}_{n=0}^\infty) = \{f(x_n)\}_{n=0}^\infty$. Fix the numbers $L, \delta > 0$. Given a ball B centered at a point in J, an inverse branch $f_\nu^{-n} : 2B \to \overline{\mathbb{C}}$ of f^n is said to be *good* if

$$\text{diam}\big(f_\nu^{-n}(B)\big) \le L\text{e}^{-\delta n}. \tag{2}$$

Given $1 \le q \le \infty$ let

$$\text{CV}_q = \bigcup_{k=1}^{q} f^k(Crit(f))$$

be the set of all critical values of f up to order q, where $Crit(f)$ is the set of all critical points of f in $\overline{\mathbb{C}}$. The following lemma motivated by [FLM] has been essentially proved in [PUZ].

Lemma 2. As in Theorem 1 let μ_ϕ be the (unique) equilibrium state for a Hölder continuous potential $\phi : J(f) \to \mathbb{R}$ satisfying $\text{P}(\phi) > \sup(\phi)$. Then for every $\varepsilon > 0$ there exists a number $M \ge 1$ such that if B is an open ball centered at a point of the Julia set and $2B \cap \text{CV}_M = \emptyset$, then there exists a subset $\tilde{F}(\varepsilon) \subset \pi^{-1}(B)$ such that the following three properties are satisfied.

(a) $\tilde{\mu}_\phi(\tilde{F}(\varepsilon) \cap \pi^{-1}(A)) > (1-\varepsilon)\mu_\phi(A)$ for every Borel set $A \subset B$

(b) for every element $\{x_n\}_{n\ge0}$ of $\tilde{F}(\varepsilon)$ there exists $\{f_{x_n}^{-n}\}_{n\ge0}$, a compatible sequence of good inverse branches defined on $2B$ (compatible means that $f \circ f_{x_{n+1}}^{-(n+1)} = f_{x_n}^{-n}$) such that $x_n = f_{x_n}^{-n}(x_0)$ for all $n = 0, 1, 2, \ldots$.

(c) If $\{x_n\}_{n\ge0} \in \tilde{F}(\varepsilon)$, then for every $y \in B$, $\{f_{x_n}^{-n}(y)\}_{n\ge0} \in \tilde{F}(\varepsilon)$.

In the next step we will "bootstrap" the function u, showing that it can be improved. We know that it is μ_ϕ-measurable. We shall show that it has a continuous version, i.e. it can be changed on a set of measure zero so as to be continuous (while satisfying equation (1)). Here again f and ϕ satisfy the assumptions of Theorem 1.

Our first step directly concerning equation (1) aims to improve the function u. We shall prove the following.

Lemma 3. Assume the same as in Theorem 1 and Lemma 2. Then the function u has a continuous version on $J(f)\setminus\text{CV}_M$ and (1) is satisfied on $(J(f)\setminus\text{CV}_M)\cap f^{-1}(J(f)\setminus\text{CV}_M)$.

Proof. Take for U a ball B centered at z such that $2B$ contains no critical values up to order M. Consider a sequence $\{f_{z_n}^{-n}\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, produced in Lemma 2. Then it follows from (1) that for all $x,y\in B$ and every $n\geq 0$

$$\frac{u(y)}{u(x)}=\left(\frac{(f_{z_n}^{-n})'(y)}{|(f_{z_n}^{-n})'(y)|}\right)^k\cdot\left(\frac{(|f_{z_n}^{-n})'(x)|}{(f_{z_n}^{-n})'(x)}\right)^k\frac{u(f_{z_n}^{-n}(y)}{u(f_{z_n}^{-n}(x))}$$

Now, from the strong version of the Koebe distortion theorem there exist a constant $K>0$ and a function $\eta:[0,\mathrm{diam}(B)]\to[0,K]$ such that $\lim_{t\searrow 0}\eta(t)=0$ and

$$\left|\frac{(f_{z_n}^{-n})'(y)}{(f_{z_n}^{-n})'(x)}-1\right|\leq\eta(|x-y|)$$

for all $x,y\in B$. Hence

$$\left|\frac{|(f_{z_n}^{-n})'(y)|}{|(f_{z_n}^{-n})'(x)|}-1\right|\leq\eta(|x-y|)$$

and since

$$\frac{(f_{z_n}^{-n})'(y)}{|(f_{z_n}^{-n})'(y)|}\cdot\frac{|(f_{z_n}^{-n})'(x)|}{(f_{z_n}^{-n})'(x)}=1+\left(\frac{(f_{z_n}^{-n})'(y)}{(f_{z_n}^{-n})'(x)}-1\right)+\left(\frac{|(f_{z_n}^{-n})'(x)|}{|(f_{z_n}^{-n})'(y)|}-1\right)$$
$$+\left(\frac{(f_{z_n}^{-n})'(y)}{(f_{z_n}^{-n})'(x)}-1\right)\cdot\left(\frac{|(f_{z_n}^{-n})'(x)|}{|(f_{z_n}^{-n})'(y)|}-1\right),$$

these two estimates give us

$$\left|\frac{(f_{z_n}^{-n})'(y)}{(f_{z_n}^{-n})'(x)}\cdot\frac{|(f_{z_n}^{-n})'(x)|}{|(f_{z_n}^{-n})'(y)|}-1\right|\leq 2\eta(|x-y|)+\eta^2(|x-y|)\leq 3\eta(|x-y|).$$

Since $|(1+z)^k-1|\leq 2k|z|$ for every $z\in\mathbb{C}$ of sufficiently small modulus, we therefore conclude that

$$\left|\left(\frac{(f_{z_n}^{-n})'(y)}{(f_{z_n}^{-n})'(x)}\cdot\frac{|(f_{z_n}^{-n})'(x)|}{|(f_{z_n}^{-n})'(y)|}\right)^k-1\right|\leq 6k\eta(|x-y|)\tag{3}$$

provided that $|y-x|$ is sufficiently small.

We want to estimate from above of the number $\left|\frac{u(f_{z_n}^{-n}(y))}{u(f_{z_n}^{-n}(x))}-1\right|$. This turns out to be a delicate matter: we do this only for an infinite subsequence of $n's$ (which is enough!) and for almost all pairs x and y. To show this we begin with two points $x,y\in B$. By properties (a) and (c), $\pi^{-1}(x)\cap\tilde{F}(\varepsilon)\neq\emptyset$ and $\pi^{-1}(y)\cap\tilde{F}(\varepsilon)\neq\emptyset$. By property (b) and (c) there exists a bijection $H_{x,y}:\pi^{-1}(x)\cap\tilde{F}(\varepsilon)\to\pi^{-1}(y)\cap\tilde{F}(\varepsilon)$ given by the formula

$$H_{x,y}(\{x_n\}_{n\geq 0})=\{f_{x_n}^{-n}(y)\}_{n\geq 0}.$$

Let $\{\mu_x : x\in B\}$ be the system of conditional measures of the measure μ_ϕ on the leaves $\pi^{-1}(x)\cap\tilde{F}(\varepsilon)$, $x\in B$. We shall prove the following.

Sublemma. For μ_ϕ-almost all $x,y\in B$, $H_{x,y}$ maps the measure class of μ_x onto the measure class of μ_y.

Proof. First observe that there exists a constant $C > 0$ such that for all $\tilde{x} \in \tilde{F}(\varepsilon)$, all $y, z \in B$ and all $n \geq 0$

$$\left| \sum_{j=1}^{n} \phi \circ f_{x_j}^{-j}(y) - \sum_{j=1}^{n} \phi \circ f_{x_j}^{-j}(z) \right| \leq C. \tag{4}$$

Indeed, let $\alpha > 0$ be a Hölder exponent of ϕ and let H be a Hölder constant. Using (2) we therefore get

$$\begin{aligned}
\left| \sum_{j=1}^{n} \phi \circ f_{x_j}^{-j}(y) - \sum_{j=1}^{n} \phi \circ f_{x_j}^{-j}(z) \right| &\leq \sum_{j=1}^{n} \left| \phi \circ f_{x_j}^{-j}(y) - \phi \circ f_{x_j}^{-j}(z) \right| \\
&\leq \sum_{j=1}^{n} H \left| f_{x_j}^{-j}(y) - f_{x_j}^{-j}(z) \right| \leq H \sum_{j=1}^{n} (L\mathrm{e}^{-\delta j})^{\alpha} \\
&\leq HL^{\alpha} \sum_{j=1}^{\infty} \mathrm{e}^{-\delta\alpha j} = HL^{\alpha} \frac{\mathrm{e}^{-\delta\alpha}}{1 - \mathrm{e}^{-\delta\alpha}}.
\end{aligned}$$

Thus the proof of formula (4) is complete. Applying some appropriate results from [DU] we conclude that there exists a constant $Q \geq 1$ such that for all $\tilde{x} \in \tilde{F}(\varepsilon)$, all $n \geq 1$ and all Borel sets $A \subset B$

$$\begin{aligned}
Q^{-1} \exp\left(\sum_{j=1}^{n} \phi(x_j) - \mathrm{P}(\phi)n \right) \mu_{\phi}(A) \leq \mu_{\phi}(f_{x_n}^{-n}(A)) \leq \\
\leq Q \exp\left(\sum_{j=1}^{n} \phi(x_j) - \mathrm{P}(\phi)n \right) \mu_{\phi}(A).
\end{aligned} \tag{5}$$

Given now an element $\tilde{x} \in \pi^{-1}(x) \cap \tilde{F}(\varepsilon)$ and an integer $n \geq 0$ define the cylinder set

$$[x_0, x_1, \ldots, x_n] := \{\tilde{z} \in \pi^{-1}(x) \cap \tilde{F}(\varepsilon) : z_j = x_j \ \forall_{0 \leq j \leq n}\}.$$

Note that $[x_0, x_1, \ldots, x_n]$ is independent of coordinates of $\tilde{x}$ larger than n. Now, using the property (c) of Lemma 2 we see that for μ_{ϕ}-a.e. $x \in B$ and for every cylinder $[x_0, x_1, \ldots, x_n] \subset \pi^{-1}(x) \cap \tilde{F}(\varepsilon)$

$$\mu_x([x_0, x_1, \ldots, x_n]) = \lim_{r \to 0} \frac{\mu_{\phi}\big(f_{x_n}^{-n}(B(x,r)\big)}{\tilde{\mu}_{\phi}(\pi^{-1}(B(x,r)) \cap \tilde{F}(\varepsilon))}.$$

Applying property (a) of Lemma 2 and (5) we therefore get

$$Q^{-1} \leq \frac{\mu_x([x_0, x_1, \ldots, x_n])}{\exp\left(\sum_{j=1}^{n} \phi(x_j) - \mathrm{P}(\phi)n\right)} \leq \frac{Q}{1 - \varepsilon}.$$

Hence, for μ_ϕ-a.e. $x \in B$ and μ_ϕ-a.e. $y \in B$

$$\frac{1-\varepsilon}{Q^2}\exp\Big(\sum_{j=1}^n \phi(x_j)-\phi(y_j)\Big) \le \frac{\mu_x([x_0,x_1,\ldots,x_n])}{\mu_y(H_{x,y}([x_0,x_1,\ldots,x_n]))} \le \frac{Q^2}{1-\varepsilon}\exp\Big(\sum_{j=1}^n \phi(x_j)-\phi(y_j)\Big),$$

where $y_j = f_{x_j}^{-j}(y)$, $0 \le j \le n$, and, let us recall $H_{x,y}([x_0,x_1,\ldots,x_n]) = [y_0,x_y,\ldots,y_n]$. So, applying (4) we conclude that

$$\frac{1-\varepsilon}{Q^2}\mathrm{e}^{-C} \le \frac{\mu_x([x_0,x_1,\ldots,x_n])}{\mu_y(H_{x,y}([x_0,x_1,\ldots,x_n]))} \le \frac{Q^2}{1-\varepsilon}\mathrm{e}^{C}.$$

Therefore, the Sublemma is proved.

Now, in order to conclude the proof of Lemma 3 notice that by Luzin's theorem there exists a compact set $G \subset J(f)$ such that $\mu(G) > 1/2$ and $\mu|_G$ is continuous, hence uniformly continuous. In view of Birkhoff's ergodic theorem there exists a Borel set $\tilde F_1(\varepsilon) \subset \tilde F(\varepsilon)$ such that $\tilde\mu_\phi(\tilde F_1(\varepsilon)) = \tilde\mu_\phi(\tilde F(\varepsilon))$ and for all $\tilde z \in \tilde F_1(\varepsilon)$

$$(6) \qquad \lim_{n\to\infty}\frac{\#\{0 \le j \le n-1 : \tilde f^{-j}(\tilde z) \in \pi^{-1}(G)\} = \tilde\mu_\phi(\pi^{-1}(G))}{n} = \mu_\phi(G) > \frac{1}{2}.$$

Since $\tilde\mu_\phi(\tilde F_1(\varepsilon)) = \tilde\mu_\phi(\tilde F(\varepsilon))$, for μ_ϕ-a.e. $x \in B$, $\mu_x(\tilde F_1(\varepsilon) \cap \pi^{-1}(x)) = 1$. It then follows from the Sublemma that for μ_ϕ-a.e. $y \in B$,

$$\mu_y\big(\tilde F_1(\varepsilon) \cap \pi^{-1}(y) \cap H_{x,y}\big(\tilde F_1(\varepsilon) \cap \pi^{-1}(x)\big)\big) = 1.$$

So, for any such a pair $x, y \in B$ there exists at least one point $\tilde z \in \tilde F_1(\varepsilon) \cap \pi^{-1}(x)$ such that $H_{x,y}(\tilde z) \in \tilde F_1(\varepsilon) \cap \pi^{-1}(y)$. By (6) there then exists an increasing to infinity sequence $\{n_j\}_{j=1}^\infty$ such that for every $j \ge 1$, both $\tilde f^{-n}(\tilde z), \tilde f^{-n}(H_{x,y}(\tilde z)) \in \pi^{-1}(G)$. This implies that $f_{z_{n_j}}(x) = z_{n_j} = \pi(\tilde f^{-n}(\tilde z)) \in G$ and $f_{z_{n_j}}(y) = \pi((\tilde f^{-n}(H_{x,y}(\tilde z))) \in G$, where $f_{z_n} : 2B \to \overline{\mathbb{C}}$, $n \ge 0$, are good inverse branches. Therefore, using (2) and uniform continuity of $u|_G$ we conclude that $\limsup_{j\to\infty}|u(f_{z_{n_j}}^{-n_j}(y)) - u(f_{z_{n_j}}^{-n_j}(x))| = 0$. Using this, (3), and letting in (3), $n = n_j \to \infty$, we obtain

$$\left|\frac{u(y)}{u(x)} - 1\right| \le 6k\eta(|x-y|).$$

Since $\lim_{t\to 0}\eta(t) = 0$, the proof of the first part of our lemma is finished. The second part is now an immediate consequence of the first part, formula (1), and the fact that the topological support of μ_ϕ is the whole Julia set $J(f)$. ■

From now on fix

$$(7) \qquad 0 < \varepsilon < Q^{-1}\exp(\inf(\phi) - \mathrm{P}(\phi)),$$

We now want to produce extensions of u on some neighborhoods of points in $J(f) \setminus \mathrm{CV}_M$ and we want these extensions to satisfy the cohomological equation of Theorem 1. We begin with the following.

Definition 4. Let $\tilde{J}_b \subset \tilde{J}$ be the set of all sequences $\tilde{x} = \{x_n\}_{n=0}^\infty \in \tilde{J}$ such that $x_0 \notin \mathrm{CV}_\infty$, $\{x_n\}_{n=0}^\infty$ has at least one accumulation point in $J(f) \setminus \mathrm{CV}_M$ and such that there exist two open connected sets $U \subset \overline{U} \subset V$ with $x_0 \in U$ such that the inverse branches $f_{x_n}^{-n}$ of f^n sending x_0 to x_n are defined on V for all $n \geq 1$. Then by $B(\tilde{x})$ we denote the maximal open ball centered at x_0 such that all the inverse branches $f_{x_n}^{-n}$ are well-defined on $2B(\tilde{x})$.

It immediately follows from this definition that if $\tilde{x} = \{x_n\}_{n=0}^\infty \in \tilde{J}_b$, then $\tilde{f}^{-1}(\tilde{x}) = \{x_{n+1}\}_{n=0}^\infty$ also belongs to $\tilde{J}_b$. Our next step is to prove the following.

Lemma 5. Let $u : J(f) \setminus \mathrm{CV}_M \to S^1$ be a continuous function that satisfies the following cohomological equation,

$$\left(\frac{f'(z)}{|f'(z)|}\right)^k = \frac{u(f(z))}{u(z)} \quad \text{for all } z, f(z) \in J(f) \setminus \mathrm{CV}_M.$$

Let $\tilde{x} = \{x_n\}_{n=0}^\infty \in \tilde{J}_b$ and let U be one of the neighborhoods of x_0 considered in Definition 4. Then the formula

$$\begin{aligned} u_{\tilde{x}}(z) &= u(x_0) \prod_{j=1}^{\infty} \left(\frac{f'(f_{x_j}^{-j}(z))}{|f'(f_{x_j}^{-j}(z))|} \Big/ \frac{f'(x_j)}{|f'(x_j)|} \right)^k \\ &= u(x_0) \lim_{n\to\infty} \left(\frac{(f^n)'(f_{x_n}^{-n}(z))}{|(f^n)'(f_{x_n}^{-n}(z))|} \Big/ \frac{(f^n)'(x_n)}{|(f^n)'(x_n)|} \right)^k \end{aligned} \tag{8}$$

defines a real-analytic function from U to S^1 which coincides with u on $J(f) \cap U \setminus \mathrm{CV}_\infty$, is independent of U (in the sense that if U' is another set with the properties required above, then the corresponding extensions agree on the intersection $U \cap U'$), and such that if $\tilde{y} = \tilde{f}^{-1}(\tilde{x})$ then

$$\left(\frac{f'(z)}{|f'(z)|}\right)^k = \frac{u_{\tilde{x}}(f(z))}{u_{\tilde{y}}(z)} \quad \text{for all } z \in f_{x_1}^{-1}(B(\tilde{x})).$$

Proof. Let V be the set produced along with U in Definition 4. In order to check that the limit in (8) exists, notice first that the family $\{f_{x_n}^{-n} : 2B(\tilde{x}) \to \overline{\mathbb{C}}\}$ is normal by Montel's theorem (cover V by countably many open balls so small that the sets $\overline{\mathbb{C}} \setminus B$ contain at least one periodic orbit of period ≥ 3). Additionally, $\lim_{n\to\infty} \mathrm{diam}(f_{x_n}^{-n}(B)) = 0$ since $x_0 \in J(f)$. Therefore, looking at the second formula defining $u_{\tilde{x}}$ (partial products), applying the chain rule and the stronger version of Koebe's distortion theorem we conclude that the partial products of the product defining $u_{\tilde{x}}$ form a Cauchy sequence. Then independence of $u_{\tilde{x}}$ on U is obvious. The cohomological equation claimed in the lemma is an easy calculation based on (8) and we have only left to demonstrate that $u_{\tilde{x}}$ coincides with u on $J(f) \cap B(\tilde{x}) \setminus \mathrm{CV}_\infty$. Towards this end, fix $\tilde{x} \in \tilde{J}_b$ and $z \in (J(f) \cap U) \setminus \mathrm{CV}_\infty$. Then there exists a subsequence n_l such that x_{n_l} converges to a point in $J(f) \setminus \mathrm{CV}_M$. Since $\lim_{l\to\infty} \mathrm{dist}(f_{x_{n_l}}^{-n_l}(z), x_{n_l}) = 0$, in view of Lemma 3,

$$\lim_{l\to\infty} |u(f_{x_n}^{-n_l}(z)) - u(x_{n_l})| = 0.$$

Therefore, using Lemma 3, and since no points x_n, $f_{x_n}^{-n}(z)$ belong to CV_M

$$\begin{aligned} u_{\tilde{x}}(z) &= u(x_0) \lim_{l\to\infty} \left(\frac{(f^{n_l})'(f_{x_{n_l}}^{-n_l}(z))}{|(f^{n_l})'(f_{x_{n_l}}^{-n_l}(z))|} : \frac{(f^{n_l})'(x_{n_l})}{|(f^{n_l})'(x_{n_l})|} \right)^k \\ &= u(x_0) \lim_{l\to\infty} \left(\frac{u(z)}{u(f_{x_{n_l}}^{-n_l}(z))} \cdot \frac{u(x_{n_l})}{u(x_0)} \right) = \lim_{l\to\infty} u(z) \frac{u(x_{n_l})}{u(f_{x_{n_l}}^{-n_l}(z))} \\ &= u(z) \end{aligned}$$

annd the proof of Lemma 5 is complete. ■

We state now our main technical result.

Proposition 6. Let the functions $u_{\tilde{x}}$ be as defined by (8). Then either (a) or (b) holds depending on whether (i) or (ii) holds, where

(i) $u_{\tilde{x}} = u_{\tilde{y}}$ on $B(\tilde{x}) \cap B(\tilde{y})$ for all $\tilde{x}, \tilde{y} \in \tilde{J}_b$ with $x_0 = y_0 \in J(f)$.

(ii) $u_{\tilde{x}} \neq u_{\tilde{y}}$ on $B(\tilde{x}) \cap B(\tilde{y})$ for at least one pair $\tilde{x}, \tilde{y} \in \tilde{J}_b$ with $x_0 = y_0 \in J(f)$.

(a) The function $u : J(f) \setminus CV_M \to S^1$ extends in a real-analytic fashion to an open connected set G whose complement is contained in a closed countable set contained in the union of CV_∞ and the set of (at most two) exceptional points of f. In addition

$$\left(\frac{f'(z)}{|f'(z)|} \right)^k = \frac{u(f(z))}{u(z)} \quad \text{for all } z \in G \cap f^{-1}(G).$$

(b) $J(f) \supset f^p(\alpha)$ for some $p \geq 0$ and α, a real-analytic open arc such that $\alpha \cap J(f)$ is an open subset of $J(f)$. Moreover, all the self-intersections of the set $\bigcup_{n\geq 0} f^n(\alpha)$ lie outside the Julia set.

Before proving this proposition let us derive from it the dynamical and structural consequences claimed in Theorem 1. The first structural consequences of (i) are the following.

Lemma 7. If (i) holds then

(a) If $f^m(x) = f^n(y)$ for some $x, y \in G$, then $\deg_x(f^m) = \deg_y(f^n)$.

(b) If c is a critical point of f in $\overline{\mathbb{C}}$ and $f^m(c) = f(w)$ for some w, then w is either a critical point or $w \in \mathrm{CV}_\infty$.

(c) The trajectories of all critical points of f are finite.

Proof. Let $q = \deg_x(f^m)$. Then there exists an analytic function g defined on a neighborhood of x such that $g(x) \neq 0$ and on this neighborhood

$$f^m(z) = f^m(x) + (z-x)^q g(z). \tag{9}$$

Hence $(f^m)'(z) = q(z-x)^{q-1}g(z) + (z-x)^q g'(z)$ and therefore

$$\frac{(f^m)'(z)}{|(f^m)'(z)|} = \frac{q}{|q|}\left(\frac{z-x}{|z-x|}\right)^{q-1}\frac{g(z)}{|g(z)|} + O(|z-x|).$$

In particular

$$\lim_{z\to x}\left(\frac{(f^m)'(z)}{|(f^m)'(z)|}\Big/\left(\frac{z-x}{|z-x|}\right)^{q-1}\right) = \frac{q}{|q|}\frac{g(x)}{|g(x)|} \in S^1.$$

Let $u : G \to S^1$ denote the real-analytic function produced in item (a) of Proposition 6. Since $\frac{u\circ f(z)}{u(z)} = \left(\frac{f'(z)}{|f'(z)|}\right)^k$ on $G \cap f^{-1}(G)$, we get

$$\overset{(m)}{\lim_{z\to x}}\left(u(f^m(z))\Big/\left(\frac{(f^m)'(z)}{|(f^m)'(z)|}\right)^k\right) = \overset{(m)}{\lim_{z\to x}} u(z) = u(x) \in S^1,$$

where $\lim^{(m)}$ means that the limit is taken for $z \in G \cap \phi^{-1}(G) \cap \dots f^{-m}(G)$. Combining these two formulas we conclude that the limit

$$\overset{(m)}{\lim_{z\to x}}\left(u(f^m(z))\Big/\left(\frac{z-x}{|z-x|}\right)^{k(q-1)}\right)$$

exists and belongs to S^1. And since by (9)

$$\lim_{z\to x}\left(\left(\frac{z-x}{|z-x|}\right)^q\Big/\frac{f^m(z)-f^m(x)}{|f^m(z)-f^m(x)|}\right) = \frac{|g(x)|}{g(x)} \in S^1$$

we conclude that the limit

$$L_m = \overset{(m)}{\lim_{z\to x}}\left(u^q(f^m(z))\Big/\left(\frac{f^m(z)-f^m(x)}{|f^m(z)-f^m(x)|}\right)^{k(q-1)}\right)$$

exists and belongs to S^1. Since f^m maps a neighborhood of x onto a neighborhood of $f^m(x)$, we can write

$$\overset{(c)}{\lim_{w\to f^m(x)}}\left(u^q(w)\Big/\left(\frac{w-f^m(x)}{|w-f^m(x)|}\right)^{k(q-1)}\right) = L_m,$$

where $\lim^{(c)}$ means that w avoids a countable set (in fact in our case the set $\overline{\mathbb{C}}\setminus \cap \phi^{-1}(G)\cap \dots f^{-m}(G)$). Let now $s = \deg_y f^n$. Similarly we get

$$\lim_{w\to f^n(y)}^{(c)} \left(u^s(w) \Big/ \left(\frac{w - f^n(y)}{|w - f^n(y)|} \right)^{k(s-1)} \right) = L_n.$$

for some number $L_n \in S^1$. Since $f^m(x) = f^n(y)$, these last two displays imply that

$$\lim_{w\to f^m(x)}^{(c)} \left(\left(\frac{w - f^m(x)}{|w - f^m(x)|} \right)^{sk(q-1)} \Big/ \left(\frac{w - f^m(y)}{|w - f^m(y)|} \right)^{qk(s-1)} \right) = L_m^s L_n^{-q} \in S^1.$$

Let $d = sk(q-1) - qk(s-1)$. If $d \neq 0$, then there is $z \in S^1$ such that $z^d \neq L_m^s L_n^{-q}$. But we may choose $w \to f^m(x)$ such that $\frac{w-f^m(x)}{|w-f^m(x)|} \to z$, achieving a contradiction which finishes the proof of (a). Let us now demonstrate that (b) follows from (a). Indeed, if the forward orbit of c forms a periodic cycle which is backward invariant, then w belongs to this orbit, and therefore $w \in \mathrm{CV}_\infty$. Otherwise, one can find two points $x, y \in G$ and integers $k, n \geq 1$ such that $c = f^k(x)$ and $w = f^n(y)$. Then $f^{m+k}(x) = f^{1+n}(y)$, and it follows from (a) that $\deg_y(f^{1+n}) = \deg_x(f^{m+k}) \geq \deg_c f \geq 2$. Since $\deg_y(f^{1+n}) = \deg_w f \cdot \deg_y(f^n)$, we therefore conclude that either w is a critical point of f or $w \in \mathrm{CV}_n \subset \mathrm{CV}_\infty$. Since f is of degree ≥ 2 and the number of critical points of f is finite, (c) follows in turn from (b). ■

As an immediate consequence of Lemma 7 we have the following.

Corollary 8. If (i) is satisfied and if f has no superattracting periodic point with a preimage at which f has a different degree, then f is critically finite with parabolic orbifold.

Next we derive some consequences from item (ii) of Proposition 6(b), beginning with the following.

Lemma 9. If (ii) of Proposition 6 is satisfied and the Julia set $J(f)$ is not totally disconnected, then the Julia set is either a geometric circle and f is biholomorphically conjugate with a finite Blaschke product, or the Julia set $J(f)$ is a closed subarc of a geometrical circle (considered in $\overline{\mathbb{C}}$) and f is biholomorphically conjugate with a 2-to-1 factor of a finite Blaschke product.

Proof. Let $\gamma = f^p(\alpha)$ be the curve produced in Proposition 6(b). Since $J(f)$ is not totally disconnected and γ has no self-intersections in $J(f)$, at least one connected component of $\mathrm{Int}_{J(f)}(\gamma \cap J(f))$ is a non-degenerate segment of γ. Consider one such component and call it β. By the topological exactness of $f : J(f) \to J(f)$, $f^q(\beta) = J(f)$. Since $\beta \subset \gamma = f^p(\alpha)$, it therefore follows from the second part of Proposition 6(b) that $f^q(\beta)$ has no self-intersection points, that is $J(f) = f^q(\beta)$ has only points of topological order ≤ 2. So, β is either a simple closed curve or is homeomorphic with the closed segment $[0,1]$ whose endpoints, call them a and b, satisfy $f(\{a, b\}) \subset \{a, b\}$. Suppose first that $J(f) = f^q(\beta)$ is a simple closed curve (Jordan curve) and let D_1 and D_2 be the two connected components of $\overline{\mathbb{C}} \setminus f^q(\beta)$.

Since $f^q(\beta)$ is obviously rectifiable, if $f(D_i) = D_i$, $i = 1, 2$, then it follows from Theorem A of [UV] (which includes Jordan curves without parabolic points) that f is conjugate with a finite Blaschke product and the conjugating map is given by a Möbius transformation (note that $\beta = J(f)$ may contain a parabolic fixed point of f). In particular $\beta = J(f)$ is a geometric circle. If $f(D_1) = D_2$ and $f(D_2) = D_1$, then all this holds true for f^2. We therefore may assume that $\beta = \partial D_1 = S^1$ and $f(S^1) = S^1$. The classification theorem of such maps then tells us that f must be a finite Blaschke product. Suppose now that $f^q(\beta) = J(f)$ is a closed interval joining the points a and b. Without losing generality we may assume that $a = 1$, $b = -1$, and $\infty \notin f^q(\beta)$. Consider a two-sheeted cover of $\overline{\mathbb{C}}$ ramified over 1 and -1, given by the map π, where y

$$\pi(z) = \frac{z + z^{-1}}{2}.$$

Since $\pi^{-1}(\beta)$ is a closed smooth Jordan curve dissecting $\overline{\mathbb{C}}$ into two topological disks D_1 and D_2 and since $\tilde{f}$, a lift of f via π, preserves $\pi^{-1}(\beta) = J(\tilde{f})$, it follows from the previous case that $\tilde{f}$ is, up to biholomorphic conjugacy, a finite Blaschke product. So the proof of Lemma 9 is complete, and all the cases (a)–(d) of Theorem 1 are taken care of. ■

If condition (b) of Proposition 6 is still satisfied but $J(f)$ is totally disconnected, then it immediately follows from Proposition 6.b that item (a) of Theorem 1 holds. Thus the proof of Theorem 1 is complete. In view of part (e) of Theorem 1, in the case of a totally disconnected Julia set $J(f)$ it makes sense to speak of an accumulation point of $J(f)$ as being either a one-sided or two-sided accumulation point of $J(f)$ in the curve $f^p(\alpha)$. If we can describe the nature of one-sided accumulation points in this (totally) disconnected case this will help us to better understand the structure of these Julia sets; this is carried out in the next two lemmas.

Lemma 10. Each critical value (of any order) of f in $J(f)$ is a one-sided accumulation point of $J(f)$ in $f^p(\alpha)$.

Proof. Suppose that w is a critical value of f which is a two-sided accumulation point of $J(f)$. Let $z \in \overline{\mathbb{C}}$ and $n \geq 1$ be such that $f^n(z) = w$ and $(f^n)'(z) = 0$. But then $z \in J(f)$ and we would need at least two arcs passing through z in order to cover any neighborhood of z in $J(f)$. This, however, is a contradiction, since no such point exists. ■

Lemma 11. Every one-sided accumulation point of $J(f)$ in $f^p(\alpha)$ is eventually periodic.

Proof. Let x be an arbitrary one-sided accumulation point of $J(f)$ in $f^p(\alpha)$. First note that all the forward iterates of x are also one-sided accumulation points of $J(f)$. Passing to a sufficiently high iterate we may assume that $\{f^n(x) : n \geq 0\}$ contains no critical points of f. Suppose now on the contrary that x is not eventually periodic. Then the ω-limit set of x is infinite and therefore there exists $y \in J(f)$, an ω-limit point of x which is not a periodic parabolic point. We have

$$y = \lim_{k\to\infty} f^{n_k}(x) \tag{10}$$

for some sequence $\{n_k\}_{k=1}^\infty$ which is increasing to infinity. Let ω_k be the maximal arc contained in $f^p(\alpha)\cap(\overline{\mathbb{C}}\setminus J(f))$ one of whose endpoints is $f^{n_k}(x)$. It follows from (10) that

$$\lim_{k\to\infty}\operatorname{diam}(\omega_k)=0 \tag{11}$$

and that the arcs ω_k are mutually disjoint. Since y is not a rationally parabolic periodic point, the arcs ω_k do not contain critical values of any order for all k sufficiently large, say for $k\ge k_0$. Since x is not a critical point of f^{n_k} there then exist for all $k\ge k_0$ well-defined inverse branches $f_x^{-n_k}:\omega_k\cup\{f^{n_k}(x)\}\to\overline{\mathbb{C}}$ of f^{n_k} mapping $f^{n_k}(x)$ to x. Since moreover each inverse branch $f_x^{-n_k}$ extends to an open neighborhood of $f^{n_k}(x)$, $f_x^{-n_k}(\omega_k)$ is a real-analytic arc $\subset f^p(\alpha)\cap(\overline{\mathbb{C}}\setminus J(f))$ joining x and some other point of $J(f)$. Since x is a one-sided accumulation point of $J(f)$ in $f^p(\alpha)$, this arc is independent of k. Call it ω, and take an arbitrary point $z\in\omega$. By (10) and (11) $y=\lim_{k\to\infty}f^{n_k}(z)$. Since z belongs to the Fatou set $\overline{\mathbb{C}}\setminus J(f)$, it follows that y must be a rationally parabolic point. This contradiction finishes the proof. ■

Thus we have:
Corollary 12. Each critical point in $J(f)$ is eventually periodic. ■

We now turn our attention to the proof of Proposition 6. We precede it with the following.

Lemma 13. If $\tilde{x}\in\tilde{J}_b$, $\tilde{y}\in\tilde{J}$, $y_0\in U(\tilde{x})\setminus\mathrm{CV}_\infty$ and $y_n=f_{x_n}^{-n}(y_0)$ for all $n\ge0$, then $\tilde{y}\in\tilde{J}_b$ and $u_{\tilde{x}}=u_{\tilde{y}}$ on $U(\tilde{x})\cap U(\tilde{y})$, where $U(\tilde{x})$ and $U(\tilde{y})$ are the sets U described in Definition 4.

Proof. Clearly $\tilde{y}\in\tilde{J}_b$. In order to prove that $u_{\tilde{x}}=u_{\tilde{y}}$ on $U(\tilde{x})\cap U(\tilde{y})$, take $z\in U(\tilde{x})\cap U(\tilde{y})$. Since $f_{x_n}^{-n}(z)=f_{y_n}^{-n}(z)$, and by Lemma 5 $u_{\tilde{x}}(y_0)=u(y_0)$, we get

$$\begin{aligned}
\frac{u_{\tilde{x}}(z)}{u_{\tilde{y}}(z)} &= \frac{u(x_0)\prod_{j=1}^\infty\left(\frac{f'(f_{x_j}^{-j}(z))}{|f'(f_{x_j}^{-j}(z))|}\Big/\frac{f'(x_j)}{|f'(x_j)|}\right)^k}{u(y_0)\prod_{j=1}^\infty\left(\frac{f'(f_{y_j}^{-j}(z))}{|f'(f_{y_j}^{-j}(z))|}\Big/\frac{f'(y_j)}{|f'(y_j)|}\right)^k}\\
&=\frac{u(x_0)}{u(y_0)}\cdot\prod_{j=1}^\infty\left(\frac{f'(y_j)}{|f'(y_j)|}\Big/\frac{f'(x_j)}{|f'(x_j)|}\right)^k\\
&=\frac{u_{\tilde{x}_0}(y_0)}{u(y_0)}=\frac{u(y_0)}{u(y_0)}=1.
\end{aligned}$$

The proof is complete. ■

Proof of Proposition 6(a). Given $x\in J(f)\setminus\mathrm{CV}_\infty$ let $B(x)=B(x,\frac{1}{2}\mathrm{dist}(x,\mathrm{CV}_M))$. Define

$$W=\bigcup\frac12 B(x)\setminus\overline{\bigcup\frac12 B(x)\cap(\frac12 B(y)\cup Cf^{-1}(\frac12 B(z))},$$

where the first union is taken over all $x \in J(f) \setminus \mathrm{CV}_\infty$ and the second one over all pairs $x, y \in J(f) \setminus \mathrm{CV}_\infty$ such that

$$B(x) \cap B(y) \cap J(f) = \emptyset,$$

for all pairs $x, z \in J(f)$ and all connected components $Cf^{-1}(\frac{1}{2}B(z))$ of $f^{-1}(\frac{1}{2}B(z))$ for which

$$B(x) \cap Cf^{-1}(\frac{1}{2}B(z)) \cap J(f) = \emptyset.$$

We want to demonstrate first that

$$J(f) \setminus (\mathrm{CV}_\infty \cap f^{-1}(\mathrm{CV}_\infty)) \subset W. \quad (??)$$

Indeed, take $w \in J(f) \setminus (\mathrm{CV}_\infty \cap f^{-1}(\mathrm{CV}_\infty))$ and suppose on the contrary that $w \notin W$. This means that either there exist sequences $x_n \in J(f) \setminus \mathrm{CV}_\infty$, $y_n \in J(f) \setminus \mathrm{CV}_\infty$ and $s_n \in \frac{1}{2}B(x_n) \cap \frac{1}{2}B(y_n)$ such that $B(x_n) \cap B(y_n) \cap J(f) = \emptyset$ and $\lim_{n\to\infty} s_n = w$, or there exist sequences $x_n \in J(f) \setminus \mathrm{CV}_\infty$, $z_n \in J(f) \setminus \mathrm{CV}_\infty$ and $t_n \in \frac{1}{2}B(x_n) \cap f^{-1}(\frac{1}{2}B(z_n)$ such that $B(x_n) \cap Cf^{-1}(B(z_n)) \cap J(f) = \emptyset$ and $\lim_{n\to\infty} t_n = w$, where $Cf^{-1}(B(z_n))$ is the connected component of $f^{-1}(B(z_n))$ containing t_n. If the first possibility occurs, then passing to a subsequence we may assume that $x_n \to x \in J(f)$ and $y_n \to y \in J(f)$. In fact $x, y \in J(f) \setminus \mathrm{CV}_M$ since otherwise, if say $x \in \mathrm{CV}_M$, then $\lim_{n\to\infty} \mathrm{diam}(B(x_n)) = 0$ which implies that $w = \lim_{n\to\infty} s_n = x \in \mathrm{CV}_M \subset \mathrm{CV}_\infty$. Hence $\liminf_{n\to\infty} \mathrm{diam}(B(x_n)) > 0$, $\liminf_{n\to\infty} \mathrm{diam}(B(y_n)) > 0$, and as $s_n \in \frac{1}{2}B(x_n) \cap \frac{1}{2}B(y_n)$ converges to w, we therefore conclude that $w \in B(x_n) \cap B(y_n)$ for all n large enough. This is in contradiction with the facts that $w \in J(f)$ and $B(x_n) \cap B(y_n) \cap J(f) = \emptyset$. If the second possibility happens, then passing to a sequence we may assume that $x_n \to x \in J(f)$ and $z_n \to y \in J(f)$. Similarly as in the previous case $x, z \in J(f) \setminus \mathrm{CV}_M$. Indeed, if $x \in \mathrm{CV}_M$, then as above, $\lim_{n\to\infty} \mathrm{diam}(B(x_n)) = 0$ which implies that $w = \lim_{n\to\infty} t_n = \lim_{n\to\infty} x_n = x \in \mathrm{CV}_M \subset \mathrm{CV}_\infty$, a contradiction. If, on the other hand, $z \in \mathrm{CV}_M$, then $\lim_{n\to\infty} \mathrm{diam}(B(z_n)) = 0$, and therefore, $w = \lim_{n\to\infty} t_n \in f^{-1}(z) \in f^{-1}(\mathrm{CV}_M) \subset f^{-1}(\mathrm{CV}_\infty)$, again a contradiction. Hence $\liminf_{n\to\infty} \mathrm{diam}(B(x_n)) > 0$, $\liminf_{n\to\infty} \mathrm{diam}(B(z_n)) > 0$, and the latter formula implies that $\liminf_{n\to\infty} \mathrm{diam}\big(Cf^{-1}(B(z_n))\big) > 0$. Since $t_n \in \frac{1}{2}B(x_n) \cap Cf^{-1}(\frac{1}{2}B(z_n)$ it therefore follows that $w \in B(x_n) \cap Cf^{-1}(B(z_n))$ for all n large enough. This contradicts the fact that $w \in J(f)$ and $B(x_n) \cap Cf^{-1}(B(z_n)) \cap J(f) = \emptyset$. Thus (12) is proved.

We now extend u to W as follows: Fix $w \in W$ and choose $x \in J(f) \setminus \mathrm{CV}_\infty$ such that $w \in B(x)$. In view of Lemma 2, $B(x) \subset B(\tilde{x})$ for some $\tilde{x} \in \tilde{J}_b$ with $x_0 = x$. Then define

$$u(x) = u_{\tilde{x}}(w).$$

In order to check that u is well-defined, choose another point $y \in J(f) \setminus \mathrm{CV}_\infty$ and $\tilde{y} \in \pi^{-1}(y) \cap \tilde{J}_b$ such that $w \in B(x)$ and $B(y) \subset B(\tilde{y})$. Since $w \in W$, $B(x) \cap B(y) \cap (J(f) \setminus \mathrm{CV}_\infty)$, there exists $s \in B(x) \cap B(y) \cap (J(f) \setminus \mathrm{CV}_\infty)$. Define $\tilde{a} = \{f_{x_n}^{-n}(s)\}_{n\geq 0}$ and $\tilde{b} = \{f_{y_n}^{-n}(s)\}_{n\geq 0}$. Then by Lemma 13, $u_{\tilde{x}} = u_{\tilde{a}}$ on $B(\tilde{x}) \cap B(\tilde{a})$, $u_{\tilde{b}} = u_{\tilde{y}}$ on $B(\tilde{y}) \cap B(\tilde{b})$, and, by (i), $u_{\tilde{a}} = u_{\tilde{b}}$ on $B(\tilde{a}) \cap B(\tilde{b})$. Thus $u_{\tilde{x}} = u_{\tilde{y}}$ on $B(\tilde{x}) \cap B(\tilde{y}) \cap B(\tilde{a}) \cap B(\tilde{b})$, and since this intersection is a non-empty (containing s) open subset of $B(x)\cap B(y)$, we conclude that $u_{\tilde{x}}=u_{\tilde{y}}$ on $B(\tilde{x})\cap B(\tilde{y})$. In particular $u_{\tilde{x}}(w)=u_{\tilde{y}}(w)$.

Our next step is to check that (1) holds on $W\cap f^{-1}(W)$. So, fix $w\in W\cap f^{-1}(W)$. Then there exist $x,y\in J(f)\setminus \mathrm{CV}_\infty$ and $\tilde{x}\in\pi^{-1}(x)\cap\tilde{J}_b$, $\tilde{y}\in\pi^{-1}(y)\cap\tilde{J}_b$ such that $w\in\frac{1}{2}B(x)\subset\frac{1}{2}B(\tilde{x})$ and $f(w)\in\frac{1}{2}B(y)\subset\frac{1}{2}B(\tilde{y})$. By the definition of $B(y)$, there exists a holomorphic inverse branch $f_\omega^{-1}:B(y)\to\overline{\mathbb{C}}$ of f sending $f(w)$ to w. In view of (7) that is the choice of ε, Lemma 2, and (i) there exists $\tilde{v}\in\pi^{-1}(y)\cap\tilde{J}_b$ such that $v_1=f_w^{-1}(y)$ and $B(y)\subset B(\tilde{v})$. We now proceed in a similar way as before. Since $w\in W$ and $w\in\frac{1}{2}B(x)\cap f_{v_1}^{-1}(\frac{1}{2}B(y))$ (where $f_{v_1}^{-1}=f_w^{-1}$), $B(x)\cap f_{v_1}^{-1}(B(y))\cap J(f)\neq\emptyset$. We can therefore fix $\rho\in B(x)\cap f_{v_1}^{-1}(B(y))\cap J(f)\setminus\mathrm{CV}_\infty$. Define now $\tilde{\alpha}=\{f_{x_n}^{-n}(\rho)\}_{n\geq 0}$ and $\tilde{\beta}=\{f_{v_{n+1}}^{-(n+1)}(f(\rho))\}_{n\geq 0}$. Both $\tilde{\alpha},\tilde{\beta}\in\tilde{J}_b$. In view of Lemma 13, $u_{\tilde{x}}=u_{\tilde{\alpha}}$ on $B(\tilde{x})\cap B(\tilde{\alpha})$. Since $f_{v_1}^{-1}(f(\rho))=\rho$, it follows from property (i) of Proposition 6 that $u_{\tilde{\alpha}}=u_{\tilde{\beta}}$ on $B(\tilde{\alpha})\cap B(\tilde{\beta})$. Finally, using Lemma 13 with $U(\tilde{\beta})=B(\tilde{b})$ and $U(\tilde{f}^{-1}(\tilde{v}))=f_{v_1}^{-1}(B(\tilde{v}))$, we see that $u(\tilde{b})=u_{\tilde{f}^{-1}(\tilde{v})}$ on $B(\tilde{b})\cap f_{v_1}^{-1}(B(\tilde{v}))$. Thus $u_{\tilde{x}}=u_{\tilde{f}^{-1}(\tilde{v})}$ on $B(\tilde{x})\cap B(\tilde{\alpha})\cap B(\tilde{b})\cap f_{v_1}^{-1}(B(\tilde{v}))$, and since this intersection is an open subset of $B(\tilde{x})\cap f_{v_1}^{-1}(B(\tilde{v}))$ containing ρ, we conclude that $u_{\tilde{x}}=u_{\tilde{f}^{-1}(\tilde{v})}$ on $B(\tilde{x})\cap f_{v_1}^{-1}(B(\tilde{v}))$. Therefore, it follows from Lemma 5 and the definition of u that

$$\left(\frac{f'(w)}{|f'(w)|}\right)^k=\frac{u_{\tilde{v}}(f(w))}{u_{\tilde{f}^{-1}(\tilde{v})(w)}}=\frac{u(f(w))}{u(w)}.$$

Hence, (1) holds on $W\cap f^{-1}(W)$.

Now, by (12), $J(f)\setminus W$ is a compact set contained in the countable set $\mathrm{CV}_\infty\cup f^{-1}(\mathrm{CV}_\infty)$. Hence $W\cap J(f)\neq\emptyset$ and we can find an open ball $B\subset W$ and an integer $q\geq 1$ such that $B\cap J(f)\neq\emptyset$ and $f^q(B)\supset B$. We want to extend u in a real-analytic fashion from B to $\overline{\mathbb{C}}\setminus\big(\mathrm{CV}_l\cup\overline{\mathrm{CV}_\infty\setminus J(f)}\big)$, where $l=qp$ and $p\geq 0$ is so large that $f^{qp}(B)\supset J(f)$. The first step is to define real-analytic functions $u_n:B_n:=f^{qn}(B)\setminus\mathrm{CV}_{qn}\to S^1$, $n\geq 0$ such that

$$u_n|_{B_n\cap B}=u|_{B_n\cap B}.$$

So, let $U\subset B_n$ be the set of all points $w\in B_n$ for which there exists an open simply connected set $S_w\subset B_n$ such that $w\in S_w$, $S_w\cap J(f)\neq\emptyset$, and if $z\in B\cap f^{-qn}(\{w\})$, then $f_z^{-qn}(S_w)\subset B$, where $f_z^{-qn}:S_w\to\overline{\mathbb{C}}$ is the analytic inverse branch of f^{qn} sending w to z. Notice first that $U\supset(f^{qn}(B)\cap J(f))\setminus\mathrm{CV}_{qn}$. Hence $U\neq\emptyset$. We shall now prove that U is an open subset of $\overline{\mathbb{C}}$. We fix $w\in U$. Since all the analytic inverse branches $\{f_\nu^{-qn}\}$ of f^{qn} are well-defined on S_w and are (obviously) continuous, there exists $r>0$ so small that if $v\in B(w,r)$ and $f_\nu^{-qn}(v)\in B$, then $f_\nu^{-qn}(w)\in B$. Therefore, putting $S_v=S_w$, we have concluded the proof of the openness of U.

A little bit harder is to demonstrate that U is a closed subset of B_n (in the topology relative to B_n). Let us consider a converging sequence $\{w_k\}_{k\geq 1}$ of points in U such that $w:=\lim_{k\to\infty}w_k\in B_n$. We need to show that $w\in U$. First, since $w\in B_n\subset\overline{\mathbb{C}}\setminus\mathrm{CV}_{qn}$, there exists $R>0$ so small that all the analytic inverse branches $\{f_i^{-qn}\}$of f^{qn} are well-defined on $B(w,r)$ and $f_i^{-qn}(B(w,r))\subset B$ whenever $f_i^{-qn}(w)\in B$. Take now $k\geq 1$ so large that $w_k\in B(w,r)$. Without loss of generality we may assume that S_{w_k} is the δ-neighborhood of an analytic closed arc and $\delta>$ is so small that the union

$B(w,r) \cup S_{w_k}$ is simply connected. The set $B(w,r) \cup S_{w_k}$ is our candidate for S_w. And indeed, we just stated that $B(w,r) \cup S_{w_k}$ is simply connected. Since $B_{w_k} \cap J(f) \neq \emptyset$, we get $(B(w,r) \cup S_{w_k}) \cap J(f) \neq \emptyset$. Suppose now that $f_i^{-qn}(w) \in B$. Then $f_i^{-qn}(B(w,r)) \subset B$ and f_i^{-qn} extends uniquely in an analytic fashion to $B(w,r) \cup S_{w_k}$ since this union is simply connected. Since $f_i^{-qn}(w_k) \in f_i^{-qn}(B(w,r)) \subset B$, we conclude that $f_i^{-qn}(S_{w_k}) \subset B$. Thus $w \in U$, completing the argument that U is a closed subset of B_n. Since B_n is connected we conclude that $U = B_n$. In order to simplify notation, for every $j \geq 1$ and every $z \notin \mathrm{Crit}(f^j)$ set

$$\Pi_j(z) = \left(\frac{(f^j)'(z)}{|(f^j)'(z)|} \right)^k .$$

Fix now $w \in B_n$ and $x \in B \cap f^{-qn}(\{w\})$. Define then the function $u_x : S_w \to S^1$ by setting

$$u_x(z) = u(f_x^{-qn}(z)) \Pi_{qn}(f_x^{-qn}(z)).$$

We shall demonstrate that if y is another point in $B \cap f^{-qn}(\{w\})$, then

$$u_x = u_y.$$

Indeed, since S_w is an open set, since $J(f) \cap S_w \neq \emptyset$, since $J(f)$ is completely invariant, and since $J(f) \setminus W$ is a compact countable set, there exists a non-empty open set $H \subset W$ such that

$$f_x^{-qn}(H) \cup f_y^{-qn}(H) \subset W \cap f^{-1}(W) \cap \ldots \cap f^{-qn}(W).$$

Multiplying formula (1) by qn we conclude that for every $z \in H$

$$u_x(z) = u(f_x^{-qn}(z)) \Pi_{qn}(f_x^{-qn}(z)) = u(f^{qn}(f_x^{-qn}(z)) = u(z)$$

and that similarly $u_y(z) = u(z)$. Thus $u_x|_H = u|_H$. Since H is a non-empty open subset of an open connected set S_w and since both functions u_x and u_y real-analytic we therefore conclude that $u_x = u_y$. Hence we can speak about the function $u_w = u_x$ independent of the preimage x. We now choose a sequence $\{w_j\}_{j=1}^{\infty}$ of points in B_n such that $B_n = \bigcup_{j \geq 1} S_{w_j}$ and we define the function $u_n : B_n \to S^1$ by setting

$$u_n(z) = u_{w_j}(z)$$

if $z \in S_{w_j}$. In order to check that this procedure defines a real-analytic function on B_n, we need only verify that if $z \in S_{w_j} \cap S_{w_k}$, then $u_{w_j}(z) = u_{w_k}(z)$. Choosing two points $x \in B \cap f^{qn}(w_j)$ and $y \in B \cap f^{qn}(w_k)$, let $f_x^{-qn} : S_{w_j} \to \overline{\mathbb{C}}$ and $f_y^{-qn} : S_{w_k} \to \overline{\mathbb{C}}$ be the two corresponding inverse branches. Since $f_x^{-qn}(z) \in f_x^{-qn}(S_{w_j}) \subset B$ and $f_y^{-qn}(z) \in f_y^{-qn}(S_{w_k}) \subset B$, we conclude that there exist $g_x : S_z \to \overline{\mathbb{C}}$ and $g_y : S_z \to \overline{\mathbb{C}}$, two analytic inverse branches of f^{qn} determined by the conditions $g_x(z) = f_x^{-qn}(z)$ and $g_y(z) = f_y^{-qn}(z)$. Making use of what we have already proved, we write

$$\begin{aligned} u_{w_j}(z) &= u(f_x^{-qn}(z)) \Pi_{qn}(f_x^{-qn}(z)) = u(g_x(z)) \Pi_{qn}(g_x(z)) = u(g_y(z)) \Pi_{qn}(g_y(z)) \\ &= u(f_y^{-qn}(z)) \Pi_{qn}(f_y^{-qn}(z)) = u_{w_k}(z). \end{aligned}$$

Therefore, the real-analytic function $u_n : B_n \to S^1$ is well-defined.

Now for every $n \geq 0$ let,

$$W_n = (B \cap W \cap f^{-1}(W) \cap \ldots \cap f^{-qn}(W)) \setminus \mathrm{CV}_{qn} \neq \emptyset$$

Then W_n has non-empty intersection with the Julia set. It follows from our construction that $u|_{W_n} = u|_{W_n}$. Since additionally, $W_n \subset B \cap B_n$ is an open set and $B \cap B_n$ is connected, we conclude that

$$u_n|_{B\cap B_n} = u|_{B\cap B_n}. \tag{13}$$

Therefore $u_{n+1}|_{B\cap B_n \cap B_{n+1}} = u|_{B\cap B_n\cap B_{n+1}} = u_n|_{B\cap B_n \cap B_{n+1}}$. Since $B \cap B_n \cap B_{n+1} = f^{qn}(B) \setminus \mathrm{CV}_{q(n+1)}$ is a non-empty open set contained in the open connected set $B_n \cap B_{n+1}$, we thus get

$$u_{n+1}|_{B_n\cap B_{n+1}} = u_n|_{B_n\cap B_{n+1}}. \tag{14}$$

Suppose now that $m \leq n$. Since $B_m \cap B_n = f^{qm}(B) \setminus \mathrm{CV}_{qn} \subset f^{qk}(B)\mathrm{CV}_{qk} = B_k$, for every $q \leq k \leq n$, we conclude by induction from (13) that the formula

$$\overline{u}(z) = u_n(z),\ z \in B_n$$

defines a function from $G = \bigcup_{n\geq 0} B_n$ to S^1. Since each set B_n is open, so is G. Since each set B_n is connected and $\bigcap_{n\geq 0} \subset B \setminus \mathrm{CV}_\infty \neq \emptyset$, G is connected. The function $\overline{u} : G \to S_1$ is real-analytic since all the functions u_n are and all the sets B_n are open. Suppose now that $z \in \overline{\mathbb{C}} \setminus G$ and z is not an exceptional point of f. Since $\bigcup_{n\geq 0} f^{qn}(B)$ is equal to $\overline{\mathbb{C}}$ minus the set of exceptional points of f, $z \in f^{qn}(B)$ for some $n \geq 0$. Since $z \notin B_n$, this implies that $z \in \mathrm{CV}_{qn}$. Thus the complement of G is a closed countable set contained in the union of CV_∞ with the set (at most two) exceptional points of f. We are left to show that $\overline{u} \circ f = u \cdot \Pi$ on $G \cap f^{-1}(G)$. Using (13) we conclude that

$$\overline{u}|_{G\cap B} = u. \tag{15}$$

Hence, we have

$$\overline{u} \circ f^q = u \circ f^q = u \cdot \Pi_q = \overline{u} \cdot \Pi_q$$

on $G \cap f^{-q}(G) \cap B \cap f^{-q}(B) \cap f^{-1}(W) \cap f^{-2}(W) \cap \ldots f^{-q}(W)$. Since $f^q(B) \subset B$ and since $J(f) \setminus W$ is a compact countable set, this last intersection is not empty. Since it is obviously open, we conclude that

$$\overline{u} \circ f^q = \overline{u} \cdot \Pi_q \tag{16}$$

on the connected set $G \cap f^{-q}(G)$. Our aim is to replace q by 1 in (16). In order to accomplish this, let W_c be the union of all connected components of W which intersect $J(f)$. We shall first show that

$$\overline{u}|_{G\cap W_c} = u|_{G\cap W_c}. \tag{17}$$

We fix a connected component V of W_c. Then V is open and $J(f) \cap V \neq \emptyset$. Since $f^q : J(f) \to J(f)$is topologically exact, there exists a transitive point $w \in J(f) \cap V$ for f^q. In particular, there exists $n \geq 1$ such that $f^{qn} \in G \cap B$ and, by continuity of f^{qn} there exists an open ball $D(w) \subset V$ centered at w such that $f^{qn}(D(w)) \subset G \cap B$. Take now an arbitrary point $z \in D_1(w) = D(w) \cap W_c \cap f^{-1}(W_c) \cap \ldots \cap f^{-qn}(W_c) \cap G \cap f^{-1}(G) \cap \ldots \cap f^{-qn}(G)$. Using (1) qn times, (16) n times, and (15), we then get that:

$$u(z)\Pi_{qn}(z) = u(f^{qn}(z)) = \overline{u}(f^{qn}(z)) = \overline{u}(z)\Pi_{qn}(z).$$

Hence, $\overline{u}(z) = u(z)$. Since $D_1(w)$ is a non-empty open subset of $G \cap V$, this implies that $\overline{u}|_{G \cap V} = u|_{G \cap V}$ (notice that the intersection $G \cap V$ is connected). In conclusion, (17) is satisfied. Applying now (17) and (1) we have that for every $z \in G \cap f^1(G) \cap W_c \cap f^{-1}(W_c)$

$$\overline{u}(f(z)) = u(f(z)) = u(z)B(z) = \overline{u}(z)B(z).$$

Since this last intersection is a non-empty subset of the connected set $G \cap f^{-1}(G)$, we finally conclude that

$$\overline{u} \circ f = \overline{u} \cdot B \text{ on } G \cap f^{-1}(G).$$

The proof of Proposition 6(a) is complete. ■

Proof of Proposition 6(b). First notice that since $J(f)$ is a perfect set and since, by Lemma 5, $u_{\tilde{x}} = u_{\tilde{y}}$ on $B(\tilde{x}) \cap B(\tilde{y}) \cap J(f) \setminus CV_M$, the set $\Gamma = \{z : B(\tilde{x}) \cap B(\tilde{y}) : u_{\tilde{x}}(z) = u_{\tilde{y}}(z)\}$ consists of finitely many real-analytic curves and isolated points whose union contains $B(\tilde{x}) \cap B(\tilde{y}) \cap (J(f) \setminus CV_M)$. Let γ be a connected component of Γ which meets $J(f)$. By the structure of Γ, $\mathrm{Int}_{J(f)}(\gamma \cap J(f)) \neq \emptyset$, and therefore there exists an open real-analytic arc $\alpha \subset \gamma$ such that $\alpha \cap J(f)$ is a non-empty open subset of $J(f)$. By the topological exactness of $f : J(f) \to J(f)$ there exists $p \geq 0$ such that $f^p(\alpha) \supset J(f)$. This proves the first part of Proposition 6(b). For proving the second part, let us suppose that $z \in J(f)$ is a self-intersection point of the set $\bigcup_{n \geq 0} f^n(\alpha)$. Then there would exist two open arcs $\alpha_1, \alpha_2 \subset \alpha$ and two numbers $m, n \geq 1$ such that $f^m(\alpha_1) \cap f^n(\alpha_2) = \{z\}$. Since the map $f : J(f) \to J(f)$ is open and $J(f)$ is totally invariant this would imply that $\{z\} = f^m(J(f) \cap \alpha_1) \cap f^n(J(f) \cap \alpha_2)$ is an open subset of $J(f)$. This however gives a contradiction, since $J(f)$ is perfect. With this observation the proof of Proposition 6(b) and hence of the whole of Proposition 6 is complete.

Acknowledgment. We are deeply indebted to the anonymous referee whose valuable comments and suggestions influenced and improved our paper considerably.

REFERENCES

[BFU] T. Bedford, A. Fisher, M. Urbański, The scenery flow for hyperbolic Julia sets, Preprint.

[Bo] R. Bowen: Equilibrium states and the ergodic theory for Anosov diffeomorphisms. Lect. Notes in Math. **470**, (1975), Springer.

[DPU] M. Denker, F. Przytycki, M. Urbański, On the transfer operator for rational functions on the Riemann sphere, Ergod. Th. and Dynam. Sys. 16 (1996), 255-266

[DU] M. Denker, M. Urbański, Ergodic theory of Equilibrium states for rational maps, Nonlinearity 4 (1991), 103-134

[FLM] A. Freire, A. Lopes, R. Mané, An invariant measure for rational maps, Bol. Soc. Bras. Mat. 131 (1983), 45-62

[Ha] N. Haydn, Convergence of the transfer operator for rational functions, Preprint.

[Ly] V. Ljubich: Entropy properties of rational endomorphisms of the Riemann sphere. Ergod. Th. Dynam. Sys. **3**, (1983), 351-386.

[Ma] V. Mayer, Comparing measures and invariant line fields, Preprint 1999.

[Pr] F. Przytycki, On the Perron-Frobenius-Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions. Bol. Soc. Bras. Mat. 20 (1990), 95-125.

[PU] F. Przytycki, M. Urbański, Fractals in the Plane - the Ergodic Theory Methods, available on the web:http://www.math.unt.edu/~urbanski, to appear in Cambridge Univ. Press.

[PUZ] F. Przytycki, M. Urbański, A. Zdunik, Gibbs and Hausdorff measures on repellers for holomorphic maps, I, Ann. of Math. 130 (1989), 439-455.

[Ru] D. Ruelle: Thermodynamic formalism. Encycl. Math. Appl. **5**, (1978), Addison–Wesley.

[UV] M.Urbański, A. Volberg, A rigidity theorem in complex dynamics, Progress in Probability, 37 (1995), 179-187

[Wa] P.Walters, A variational principle for the pressure of continuous transformations, Amer. J. Math. 97 (1975), 937-971.

[Zd] A. Zdunik, Parabolic orbifolds and the dimension of maximal measure for rational maps, Inv. Math. 34 (1990), 627-649.