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Ratio geometry, rigidity and the scenery process for hyperbolic Cantor sets.

ABSTRACT.
Given a $\mathcal{C}^{1+\alpha}$ hyperbolic Cantor set C, we study the sequence $C_{n,x}$ of Cantor subsets which nest down toward a point x in C. We show that $C_{n,x}$ is asymptotically equal to an ergodic Cantor set valued process. The values of this process, called limit sets, are indexed by a Hölder continuous set-valued function defined on Sullivan's dual Cantor set. We show the limit sets are themselves $\mathcal{C}^{k+\alpha}$, $\mathcal{C}^{\infty}$ or $\mathcal{C}^{\omega}$ hyperbolic Cantor sets, with the highest degree of smoothness which occurs in the $\mathcal{C}^{1+\gamma}$ conjugacy class of C. The proof of this leads to the following rigidity theorem: if two $\mathcal{C}^{k+\alpha}$, $\mathcal{C}^{\infty}$ or $\mathcal{C}^{\omega}$ hyperbolic Cantor sets are $\mathcal{C}^{1}$ conjugate, then the conjugacy (with a different extension) is in fact already $\mathcal{C}^{k+\alpha}$, $\mathcal{C}^{\infty}$ or $\mathcal{C}^{\omega}$. Within one $\mathcal{C}^{1+\gamma}$ conjugacy class, each smoothness class is a Banach manifold, which is acted on by the semigroup given by rescaling subintervals. Smoothness classes nest down, and contained in the intersection of them all is a compact set which is the attractor for the semigroup: the collection of limit sets. Convergence is exponentially fast, in the $\mathcal{C}^{1}$ norm.