Equivariant Basic Cohomology and Applications

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Let G be a connected Lie group, M a G-manifold.

Borel construction:

$$M_G := EG \times_G M$$

where EG is a contractible space on which G acts freely.

Equivariant cohomology of (M, G):

$$H_G(M) := H(M_G)$$

The projection $\pi : M_G \to EG/G =: BG$ induces a module structure $H(BG) \times H_G(M) \to H_G(M)$ by $f \cdot \omega := \pi^*(f) \cup \omega$.

Borel Localization: $\widehat{H}_T(M) = \widehat{H}_T(M^T)$.

Two deRham models for smooth actions: Weil and Cartan model.

Weil model: $(\bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \otimes \Omega(M))_{\text{bas}\,\mathfrak{g}}$

Cartan model:
Smooth torus action $T \curvearrowright M$.
Infinitesimal action $\mathfrak{t} \to \mathfrak{X}(M)$; $X \mapsto X^*$, where
$X^*(p) = \frac{d}{dt} \exp(tX)p$
$\rightsquigarrow$ operators $i_X := i_{X^*}, L_X := L_{X^*}, d$.
$\Omega(M)$ is a $\mathfrak{t}$-differential graded algebra (dga).

Define the *Cartan complex* $\Omega_{\mathfrak{t}}(M) := S(\mathfrak{t}^*) \otimes \Omega(M)^T$ and the equivariant differential $d_{\mathfrak{t}}$:
Let $X_1, \dots, X_n$ be a basis of $\mathfrak{t}$, $\theta_1, \dots, \theta_n$ be a dual basis of $\mathfrak{t}^*$.
Cartan complex $\Omega_{\mathfrak{t}}(M) = \mathbb{R}[\theta_1, \dots, \theta_n] \otimes \Omega(M)^T$ with

$$d_{\mathfrak{t}}(\theta_k) = 0 \qquad d_{\mathfrak{t}}(\omega) = d\omega + \sum_k \theta_k \otimes i_{X_k}\omega$$

T-manifold M	Riemannian foliation $(M, \mathcal{F})$
infinitesimal action $\mathfrak{t} \to \mathfrak{X}(M)$	transverse action $\mathfrak{a} \to l(M, \mathcal{F})$
DeRham complex $\Omega(M)$ $\mathfrak{t}$-dga	basic subcomplex $\Omega(M, F)$ $\mathfrak{a}$-dga
equivariant cohomology $H_{\mathfrak{t}}(M)$	equivariant basic cohomology $H_{\mathfrak{a}}(M, \mathcal{F})$
$\mathfrak{t}$-orbits	leaf closures
T-fixed points	closed leaves

Let (M, g) be a complete Riemannian manifold. A *Riemannian foliation* is a foliation, whose leaves are locally equidistant. More precisely:

Definition

Let $T\mathcal{F} = \bigcup_{p \in M} T_p L_p$ be the tangent bundle of the foliation and $\nu\mathcal{F} = T\mathcal{F}^{\perp}$ its geometric normal bundle. Consider the *transverse metric* $g_T = g|(\nu\mathcal{F} \times \nu\mathcal{F})$. If $L_X g_T = 0$ for every tangential vector field X, then $\mathcal{F}$ is called a *Riemannian foliation*.

Example (Homogeneous Foliations)

The (connected components of) orbits of a locally free isometric action define a Riemannian foliation.

Example

Consider the T^2-action on $S^3 \subset \mathbb{C}^2$ by

$$\begin{aligned} T^2 \times S^3 &\to S^3 \\ ((c_1, c_2), (z_1, z_2)) &\mapsto (c_1 z_1, c_2 z_2) \end{aligned}$$

For $r \in \mathbb{R} \backslash \{0\}$ consider $\mathbb{R} \to T^2; t \mapsto (e^{2\pi it}, e^{2\pi irt})$. The action

$$\mathbb{R} \to T^2 \curvearrowright S^3$$

is locally free and defines a Riemannian foliation $\mathcal{F}_r$.

$\mathcal{F}_r$ is closed $\iff$ $r \in \mathbb{Q}$. $M/\mathcal{F}_{p/q}$ is a spherical orbifold.
If $r \in \mathbb{R} \backslash \mathbb{Q}$, then the leaf closures are the T^2-orbits,
$M/\overline{\mathcal{F}}_r = M/T^2$=[0,1].

$(M, \mathcal{F})$: foliation of codimension q.

$$\Omega^*(M, \mathcal{F}) := \{\omega \in \Omega^*(M) \mid i_X\omega = 0, L_X\omega = 0 \ \forall X \in C^\infty(T\mathcal{F})\}.$$

is a subcomplex of $\Omega^*(M)$, i.e.

$$d(\Omega^*(M, \mathcal{F})) \subset \Omega^{*+1}(M, \mathcal{F}).$$

$$H^*(M, \mathcal{F}) := H(\Omega^*(M, \mathcal{F}), d)$$

is the *basic cohomology* of $(M, \mathcal{F})$.

Objective: Determine $b_i := \dim H^i(M, \mathcal{F})$, or equivalently, the Poincaré-polynomial

$$P_t(M, \mathcal{F}) := \sum_{i=0}^{q} b_i t^i.$$

Example (Closed Riemannian Foliation)

Let $\mathcal{F}$ be a *closed* Riemannian foliation (i.e. all leaves are closed).
$\Longrightarrow M/\mathcal{F}$ is a Riemannian orbifold. Then

$$H^*(M,\mathcal{F}) \cong H^*(M/\mathcal{F})$$

If $\mathcal{F}$ is not closed, then $M/\mathcal{F}$ is not even Hausdorff.

Question: What can we say about $H^*(M,\mathcal{F})$?

Let $I(M, \mathcal{F})$ be the space of *transverse fields*, i.e. global sections of the normal bundle $\nu\mathcal{F}$ that are holonomy-invariant. Then $\Omega(M, \mathcal{F})$ is a $I(M, \mathcal{F})$-dga.

Consider Killing foliations. Examples: Homogeneous Riemannian foliations, and Riemannian foliations on simply-connected manifolds.

For a Killing foliation $\mathcal{F}$ there are commuting transverse fields $X_1, \ldots, X_k \in I(M, \mathcal{F})$ such that

$$T_p\overline{L}_p = T_pL_p \oplus \langle X_1(p), \ldots, X_k(p)\rangle$$

for all $p \in M$. [Molino, Mozgawa]
$X_1, \ldots, X_k$ form an abelian Lie-subalgebra of $I(M, \mathcal{F})$. Thus $\Omega(M, \mathcal{F})$ is a $\mathfrak{a}$-dga.

T-manifold M	Killing foliation $(M, \mathcal{F})$
infinitesimal action $\mathfrak{t} \to \mathfrak{X}(M)$	transverse action $\mathfrak{a} \to l(M, \mathcal{F})$
DeRham complex $\Omega(M)$ $\mathfrak{k}$-dga	basic subcomplex $\Omega(M, F)$ $\mathfrak{a}$-dga
equivariant cohomology $H_{\mathfrak{t}}(M)$	equivariant basic cohomology $H_{\mathfrak{a}}(M, \mathcal{F})$
$\mathfrak{t}$-orbits	leaf closures
T-fixed points	closed leaves

M: complete
$\mathcal{F}$: Killing foliation (e.g. $\mathcal{F}$ Riemannian and M 1-connected)
transversely orientable
$M/\overline{\mathcal{F}}$ compact (e.g. M compact).
C: the union of closed leaves.

Theorem (Goertsches-T: Borel-type Localization)

$$\dim H^*(C/\mathcal{F}) = \dim H^*(C,\mathcal{F}) \leq \dim H^*(M,\mathcal{F}) = \sum_i b_i.$$

In particular

$$\#\text{components of } C \leq \dim H^*(M,\mathcal{F}).$$

Theorem (Caramello-T)

$$\chi_B(M,\mathcal{F}) = \chi_B(C,\mathcal{F}|_C) = \chi(C/\mathcal{F}).$$

Let $f : M \to \mathbb{R}$ be a basic Morse-Bott function, whose critical manifolds are isolated leaf closures. We denote the index of f at the critical manifold N by λ_N.

Theorem (Alvarez López)

If M is compact, then

$$P_t(M, \mathcal{F}) \leq \sum_N t^{\lambda_N} P_t(N, \mathcal{F}),$$

where N runs over the critical leaf closures.

Theorem (Goertsches-T)

A basic Morse-Bott $f : M \to \mathbb{R}$, whose critical set is equal to C, is perfect. That means

$$P_t(M, \mathcal{F}) = \sum_N t^{\lambda_N} P_t(N/\mathcal{F}),$$

where N runs over the connected components of C and λ_N is the index of f at N.

Application to *K-contact manifolds* (e.g. Sasakian manifolds):

(M^{2n+1}, α, g): compact K-contact manifold.
α: contact form, i.e. $\alpha \wedge (d\alpha)^n \neq 0$ everywhere,
g: adapted Riemannian metric.
R: *Reeb field* defined by $\alpha(R) = 1$ and $i_R d\alpha = 0$. It is a nonvanishing Killing field with respect to g.

$\rightsquigarrow$ *Reeb orbit foliation* $\mathcal{F}$. It is a 1-dimensional homogeneous Riemannian foliation, therefore a Killing foliation.

$R :=$ Reeb field of α .
$T :=$ closure of the Reeb flow in $\mathrm{Isom}(M, g)$. Then T is a torus whose Lie algebra $\mathfrak{t}$ contains R. T-orbits are the closures of the Reeb orbits.
$C :=$ union of the closed Reeb orbits = union of all 1-dimensional T-orbits.
$\mathfrak{a} = \mathfrak{t}/\mathbb{R}R$.

Definition (Contact moment map)

For each $X \in \mathfrak{t}$, we define $\Phi^X : M \to \mathbb{R}$ by

$$\Phi^X(p) = \alpha(X^*_p).$$

Note that Φ^X is T-invariant.

Theorem (Goertsches-Nozawa-T)

For generic $X \in \mathfrak{t}$, the function Φ^X is a perfect basic Morse-Bott function whose critical set is C:

$$P_t(M, \mathcal{F}) = \sum_N t^{\lambda_N} P_t(N/\mathcal{F}).$$

Corollary

Assume that C consists of isolated closed Reeb orbits. Then we get $H^{\mathrm{odd}}(M, \mathcal{F}) = 0$.

Proof.

The indices of the critical leaves, the isolated Reeb orbits, are even, because the negative spaces are complex. □

Theorem (Goertsches-Nozawa-T)

We have

$$\sum_j \dim H^j(C/\mathcal{F}) = \sum_j \dim H^j(M,\mathcal{F}).$$

In particular, in case the closed Reeb orbits are isolated, their number is given by $\dim H^*(M,\mathcal{F})$.

$0 \neq [(d\alpha)]^k \in H^{2k}(M, \mathcal{F})$ for all $k = 0, \dots, n$.

$$\mathbb{R}[z]/(z^{n+1}) \subset H^*(M, \mathcal{F}).$$

Corollary (Rukimbira)

The Reeb flow has at least $n+1$ closed orbits.

Corollary

If the Reeb flow has exactly $n+1$ closed orbits, then $H^*(M, \mathcal{F}) \cong \mathbb{R}[z]/(z^{n+1})$ as graded rings.

Theorem (Goertsches-Nozawa-T)

If (M, α, g) is a compact K-contact $(2n+1)$-manifold whose closed Reeb orbits are isolated, then their number is exactly $n+1$ if and only if M is a real cohomology sphere (i.e. $H^*(M) = H^*(S^{2n+1})$).

Proof.

The Gysin sequence relates $H^*(M, \mathcal{F})$ to $H^*(M)$. It can be used to show

$$H^*(M, \mathcal{F}) = \mathbb{R}[z]/(z^{n+1}) \iff H^*(M) = H^*(S^{2n+1}).$$

$$0 \to H^{2k+1}(M) \to H^{2k}(M, \mathcal{F}) \xrightarrow{\delta} H^{2k+2}(M, \mathcal{F}) \to H^{2k+2}(M) \to 0,$$

□

Theorem (GNT: Duistermaat-Heckman-type theorem)

Let (M, α, g) be a $(2n+1)$-dimensional compact K-contact manifold with only finitely many closed Reeb orbits $L_1, \ldots, L_N$. Then the volume of (M, g) is given by

$$\frac{1}{2^n n!}\int_M \alpha \wedge (d\alpha)^n = (-1)^n \frac{\pi^n}{n!}\sum_{k=1}^{N} l_k \cdot \frac{\alpha|_{L_k}(X^*)^n}{\prod_j \beta_j^k(X + \mathbb{R}R)},$$

where $l_k = \int_{L_k}\alpha$ is the length of the closed Reeb orbit L_k and $\{\beta_j^k\}_{j=1}^n \subset \mathfrak{a}^*$ are the weights of the transverse isotropy $\mathfrak{a}$-representation at L_k.

Applications: Calculation of Volume

Deformations of standard Saskian structure on S^{2n+1}
Toric Sasakian manifolds
Homogeneous Sasakian manifolds

M: compact manifold
$\mathcal{F}$: orientable taut transversely Kähler foliation of dimension one and complex codimension m with only finitely many closed leaves $L_1, \ldots, L_N$.
Assume that $\wedge^{m,0}\nu^*\mathcal{F}$ is trivial as a topological line bundle.

$$\int_M u_1 c = \sum_{k=1}^{N} \left(\int_{L_k} u_1 \right) \frac{c_{\mathfrak{a}}|L_k}{c_{m,\mathfrak{a}}(\nu\mathcal{F}, \mathcal{F})|L_k},$$

where c is a basic Chern class of the normal bundle $\nu\mathcal{F}$ of degree $2m$ and $c_{\mathfrak{a}}$ its the corresponding equivariant Chern class. In particular, in the case where $c = c_m$, we obtain

$$\int_M u_1 c_m = \sum_k \int_{L_k} u_1 .$$