Structure Equations for G-Structures and G-Structure Algebroids

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Joint work with Rui Loja Fernandes

Purpose

- Explain the background geometry underlying a type of classification problem in differential geometry.
- Show examples of how to solve the classification problems

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- Show examples of how to solve the classification problems

This talk is based on:

- R.L. Fernandes & I.S., The Classifying Algebroid of a Geometric Structure I, (Trans. of the A.M.S.).
- R.L Fernandes & I.S., The Global Solutions to Cartan's Realization Problem (Arxiv)
- R. Bryant, Bochner-Kähler metrics. *J. of Amer. Math. Soc.*, **14** (2001), 623–715.

Type of Classification Problems

The Classification Problems that we consider are of Finite Type:

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The Classification Problems that we consider are of Finite Type:

- Finite Type problemas are those for which the local isomorphism class of the geometric structure being considered are determined by a finite amount of invariants.
- These are classes of geometric structures which can be described as solutions of an Exterior Differential System of Frobenius Type.

Examples

Example (Surfaces of Constant Curvature)

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Example (Surfaces of Hessian type $\frac{1}{2}(1-k^2)$)

- (M^2, σ) such that $\mathrm{Hess}_\sigma(k) = \frac{1}{2}(1-k^2)\sigma$;
- Complete set of local invariants: k, k_1, k_2, where

$$k_1 = \frac{\partial k}{\partial \theta_1}, \quad k_2 = \frac{\partial k}{\partial \theta_2},$$

$(\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2})$ - Local O.N. Frame of M.

Example: Extremal Kähler Surfaces

- An Extremal Kähler Surface is a Kähler Surface (M, σ, Ω, J) such that the Hamiltonian vector field ξ_k associated to the Gaussian curvature of σ is an infinitesimal symmetry of the Kähler structure:

$$\mathcal{L}_{\xi_k}\sigma = 0, \quad \mathcal{L}_{\xi_k}\Omega = 0, \quad \mathcal{L}_{\xi_k}J = 0.$$

- If M is compact these correspond to critical points of the Calabi functional.
- 2-dimensional Böchner-Kähler manifolds.
- There are 2 $\mathbb{R}$-valued functions and one $\mathbb{C}$-valued function that provide a complete set of invariants (to be described soon).

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- $\eta \in \Omega^1(F_G(M), \mathfrak{g})$: a principal bundle **connection**.

Basics of G-structures

A diffeomorphism $\phi : M_1 \to M_2$ lifts to an isomorphism:

$$\phi_* : F(M_1) \to F(M_2).$$

Definition

Given G-structures $F_G(M_1)$ and $F_G(M_2)$, a G-**equivalence** is a diffeomorphism $\phi : M_1 \to M_2$ such that:

$$\phi_*(F_G(M_1)) = F_G(M_2).$$

Classical problem:

- When are two G-structures (locally) equivalent?

This encodes the equivalence of many geometric problems.

Basics of G-structures : Examples

Examples:

- Riemannian structures $\iff \mathrm{O}_n$-structures;
- Almost complex structures $\iff \mathrm{GL}_n(\mathbb{C})$-structures;
- Almost symplectic structures $\iff \mathrm{Sp}_n$-structures;
- Almost hermitian structures $\iff \mathrm{U}_n$-structures.

Basics of G-structures : Tautological Form

The tautological form $\theta \in \Omega^1(F_G(M), \mathbb{R}^n)$, $\xi \mapsto p^{-1}(\mathrm{d}_p\pi(\xi))$ controls the equivalence problem and characterises G-structures among G-principal bundles:

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Proposition

If $P \to M$ is a G-principal bundle, and $\tau \in \Omega^1(P, \mathbb{R}^n)$ is a tensorial 1-form, then there exists a unique embedding of principal bundles $\varphi : P \to F(M)$ such that $\varphi^*\theta = \tau$.

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Conclusion: Category of G-Structures with equivalences $\simeq$ Category of principal G-bundles with tensorial forms.

Connections

Recall that a connection is a 1-form $\omega \in \Omega^1(F_G(M), \mathfrak{g})$ such that

$$R_g^*\omega = \mathrm{Ad}_g^{-1}\omega, \quad \omega(\tilde{\alpha}_p) = \alpha, \quad \forall \alpha \in \mathfrak{g}.$$

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- Torsion of ω: $c : F_G(M) \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathbb{R}^n)$

$$c(p)(u, v) = \mathrm{d}\theta(\xi_u, \xi_v), \quad \xi_u, \xi_v \in \mathrm{Ker}\omega_p, \quad \theta(\xi_u) = u, \quad \theta(\xi_v) = v.$$

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- Curvature of ω: $R : F_G(M) \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathfrak{g})$

  $$R(p)(u, v) = \mathrm{d}\omega(\xi_u, \xi_v), \quad \xi_u, \xi_v \in \mathrm{Ker}\omega_p, \quad \theta(\xi_u) = u, \quad \theta(\xi_v) = v.$$

Torsion Free Connections

The existence of Torsion free connections on a G-structure is the first (and many times only) obstruction to integrability of the underlying geometric structures:

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Examples:

- Almost complex structures is complex $\iff F_{\mathrm{GL}_n(\mathbb{C})}(M)$ admits a torsion free connection;
- Almost symplectic structures is symplectic $\iff F_{\mathrm{Sp}_n}(M)$ admits a torsion free connection;
- Almost hermitian structures is Kähler $\iff F_{\mathrm{U}_n}(M)$ admits a torsion free connection.

Structure Equations

Key Remark: $(\theta, \omega)_p : T_p F_G(M) \to \mathbb{R}^n \oplus \mathfrak{g}$ is an isomorphism.
We can interpret (θ, ω) as a coframe on $F_G(M)$.

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Equivalence of G-structures with connections is controlled by the structure equations:

$$\begin{cases} \mathrm{d}\theta = c \circ \theta \wedge \theta - \omega \wedge \theta \\ \mathrm{d}\omega = R \circ \theta \wedge \theta - \omega \wedge \omega \\ \text{Higher order consequences of these equations} \end{cases}$$

Structure Equations: Example 1

Example (Constant Curvature Surfaces: $G = \mathrm{SO}_2$)

- Connection $\omega \in \Omega^1(F_{\mathrm{SO}_2}(M), \mathfrak{so}_2)$ – Levi-Civita connection
- Structure equations:

$$\begin{cases} \mathrm{d}\theta^1 = -\theta^2 \wedge \omega \\ \mathrm{d}\theta^2 = \theta^1 \wedge \omega \\ \mathrm{d}\eta = -k\theta^1 \wedge \theta^2 \\ \mathrm{d}k = 0 \end{cases}$$

- $\theta = (\theta^1, \theta^2) \in \Omega^1(F_{\mathrm{SO}_2}(M), \mathbb{R}^2)$ is the tautological form of the orthogonal frame bundle
- $k : F_{\mathrm{SO}_2}(M) \longrightarrow \mathbb{R}$ is the Gaussian curvature.

Structure Equations: Example 2

Example ((M^2, σ) such that $\mathrm{Hess}_g k = \frac{1}{2}(1-k^2)\sigma$: $G = \mathrm{SO}_2$)

- Structure equations:

$$\begin{cases} \mathrm{d}\theta^1 = -\theta^2 \wedge \omega \\ \mathrm{d}\theta^2 = \theta^1 \wedge \omega \\ \mathrm{d}\eta = -k\theta^1 \wedge \theta^2 \\ \mathrm{d}k = k_1\theta^1 + k_2\theta^2 \\ \mathrm{d}k_1 = \frac{1}{2}(1-k^2)\theta_1 - k_2\omega \\ \mathrm{d}k_2 = \frac{1}{2}(1-k^2)\theta_2 + k_1\omega \end{cases}$$

- ω - Levi-Civita; $\theta = (\theta^1, \theta^2)$ - tautological form; $(k, k_1, k_2) : F_{\mathrm{SO}_2}(M) \to \mathbb{R}^3$.

Structure Equations: Example 3

Example (Extremal Kähler Surfaces (M, σ, Ω, J): $G = U_1$)

- Structure equations:

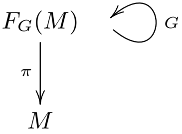
$$\begin{cases} d\theta = -\omega \wedge \theta \\ d\omega = \frac{K}{2}\,\theta \wedge \bar{\theta} \\ dK = -(\bar{T}\theta + T\bar{\theta}) \\ dT = U\theta - T\omega \\ dU = -\frac{K}{2}(\bar{T}\theta + T\bar{\theta}) \end{cases}$$

- $\omega \in \Omega^1(F_{U_1}(M), \mathbb{C})$ tautological form; $\omega \in \Omega^1(F_{U_1}(M), i\mathbb{R})$ Levi-Civita connection
- $(K, T, U) : F_{U_1}(M) \to \mathbb{R} \times \mathbb{C} \times \mathbb{R}$.

Classification Problem

Given a structure group $G \subset \mathrm{GL}_n$ and a set of structure equations of a finite type problem, an integration (or realization) is:

- A manifold M of dimension n
- A G-structure



  with tautological form $\theta \in \Omega^1(F_G(M), \mathbb{R}^n)$
- A connection $\omega \in \Omega^1(F_G(M), \mathfrak{g})$

Such that the structure equations are satisfied.

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Such that the structure equations are satisfied.

PROBLEM: Classify all realizations of a finite type problem up to local/global equivalence; Construct examples; Describe the local/global symmetry groups of realizations; etc...

Example: Surfaces of Constant Curvature

- If we "dualize" the structure equations for constant curvature surfaces:

$$\begin{cases} \mathrm{d}\theta^1 = -\theta^2 \wedge \eta \\ \mathrm{d}\theta^2 = \theta^1 \wedge \eta \\ \mathrm{d}\eta = -\kappa\theta^1 \wedge \theta^2 \\ \mathrm{d}\kappa = 0 \end{cases} \implies \begin{cases} e_1 = [e_2, e_3] \\ e_2 = [e_3, e_1] \\ e_3 = \kappa[e_1, e_2] \\ \kappa \text{ is constant.} \end{cases}$$

  we obtain a bundle of Lie algebras $A \to \mathbb{R}$.

- We look for an "SO_2 - integrations", i.e., Lie group H integrating A_κ with free and proper SO_2-action:

$\kappa < 0$	$A_\kappa = \mathfrak{sl}_2$	SL_2	$\mathrm{SL}_2/\mathrm{SO}_2 \simeq \mathbb{H}(\kappa)$
$\kappa = 0$	$A_\kappa = \mathfrak{euc}_2$	$\mathbb{R}^2 \ltimes \mathrm{SO}_2$	$\mathbb{R}^2 \rtimes \mathrm{SO}_2/\mathrm{SO}_2 \simeq \mathbb{R}^2$
$\kappa > 0$	$A_\kappa = \mathfrak{so}_3$	SO_3	$\mathrm{SO}_3/\mathrm{SO}_2 \simeq S^2(\frac{1}{\kappa})$

Lie Algebroids

In general we do not get (a bundle of) Lie algebras, but a Lie algebroid:

Definition

A Lie Algebroid is a vector bundle $A \to X$ with

- a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$;
- a bundle map $\rho : A \to TX$ called the anchor of A

satisfying the Leibniz identity

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta$$

for all $\alpha, \beta \in \Gamma(A)$, and $f \in C^\infty(X)$.

Lie Algebroids

- $\mathrm{Im}(\rho) \subset TX$ is a singular integrable distribution $\implies$ Leaves of A in X
- $\mathrm{Ker}\rho_x \subset A_x$ is a Lie algebra: Isotropy Lie algebra

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Proposition: (Consequence of Koszul's Formula for d)

Let $A \to X$ be a vector bundle. There is a one to one correspondence between Lie algebroid structures on A and derivations $\mathrm{d} : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ such that $\mathrm{d}^2 = 0$.

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Conclusion

Necessary conditions to for existence of G-realizations:

$$d^2 = 0 \implies \text{Lie algebroid!}$$

G-Structure Algebroids

The Lie algebroids appearing in classification problems have extra structure. They come equipped with:

- A **principal G-action**;
- A **tensorial** 1-**form** $\theta \in \Gamma(A^*) \otimes \mathbb{R}^n$;
- A **connection** 1-**form** $\omega \in \Gamma(A^*) \otimes \mathfrak{g}$;

G-Structure Algebroids in Normal Form

- As a vector bundle $A \to X$ is always trivial with fiber $\mathbb{R}^n \oplus \mathfrak{g}$;
- X comes equipped with an action of G;
- The natural inclusion

$$i : X \ltimes \mathfrak{g} \longrightarrow A = X \times (\mathbb{R}^n \oplus \mathfrak{g})$$

  is a Lie algebroid morphism. It determines an action of G on A by inner automorphisms.
- The bracket is given on constant sections by

$$[(u,\alpha),(v,\beta)](x) = (\alpha\cdot v - \beta\cdot u - c(x)(u,v), [\alpha,\beta]_{\mathfrak{g}} - R(x)(u,v))$$

  where......

G-Structure Algebroids in Normal Form II

- $c : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathbb{R}^n)$ is a G-equivariant map called the **torsion of** (A, θ, ω);
- $R : X \to \mathrm{Hom}(\wedge^2\mathbb{R}^n, \mathfrak{g})$ is a G-equivariant map called the **curvature of** (A, θ, ω);
- The anchor of A takes the form

$$\rho_x(u, \alpha) = F(x, u) + \psi(x, \alpha),$$

where $F : X \times \mathbb{R}^n \to TX$ is a G-equivariant bundle map and $\psi : X \times \mathfrak{g} \to TX$ is the infinitesimal action map associated to the G action on X.

Example: Hessian Curvature – (M^2, g) such that $\text{Hess}_g \kappa = \frac{1}{2}(1-\kappa^2)g$

$$\begin{cases} d\theta^1 = -\theta^2 \wedge \eta \\ d\theta^2 = \theta^1 \wedge \eta \\ d\eta = -\kappa\theta^1 \wedge \theta^2 \\ d\kappa = \kappa_1\theta^1 + \kappa_2\theta^2 \\ d\kappa_1 = \frac{1}{2}(1-\kappa^2)\theta_1 - \kappa_2\eta \\ d\kappa_2 = \frac{1}{2}(1-\kappa^2)\theta_2 + \kappa_1\eta \end{cases} \Longrightarrow \begin{cases} [\alpha_2, \beta] = \alpha_1 \\ [\beta, \alpha_1] = \alpha_2 \\ [\alpha_1, \alpha_2] = \kappa\beta \\ \rho(\alpha_1) = \kappa_1\partial_\kappa + \frac{1}{2}(1-\kappa^2)\partial_{\kappa_1} \\ \rho(\alpha_2) = \kappa_2\partial_\kappa + \frac{1}{2}(1-\kappa^2)\partial_{\kappa_2} \\ \rho(\beta) = -\kappa_2\partial_{\kappa_1} + \kappa_1\partial_{\kappa_2} \end{cases}$$

Where $X = \mathbb{R}^3$ with coordinates $(\kappa, \kappa_1, \kappa_2)$;
$A = X \times (\mathbb{R}^2 \oplus \mathfrak{so}_2) = X \times \mathbb{R}^3$ with basis of sections $\alpha_1, \alpha_2, \beta$;
The SO_2 action on X is induced by $\rho(\beta)$: rotation around the κ axis.

G-Structure Algebroid for EK-Surfaces

- $X = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$;
- $A = X \times (\mathbb{C} \oplus \mathfrak{u}(1))$;
- $U(1)$-action on X:

$$(K, T, U)g = (K, g^{-1}T, U), \quad g \in U(1),$$

associated to the infinitesimal action $\psi : X \times \mathfrak{u}(1) \to TX$:

$$\psi(\alpha)|_{(K,T,U)} = (0, -\alpha T, 0)\,, \quad \alpha \in \mathfrak{u}(1).$$

- Bracket:

$$[(z, \alpha), (w, \beta)]|_{(K,T,U)} := (\alpha w - \beta z, -\frac{K}{2}(z\bar{w} - \bar{z}w)),$$

- Anchor:

$$\rho(z, \alpha)|_{(K,T,U)} := \left(-T\bar{z} - \bar{T}z, Uz - \alpha T, -\frac{K}{2}T\bar{z} - \frac{K}{2}\bar{T}z\right).$$

Classification Problem Revisited

If (P, θ, ω) is a G-structure with connection, then $TP \to P$ is a G-structure algebroid with torsion $c = c_\omega$ and curvature $R = R_\omega$.

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If $(A, \theta, \omega) \to X$ is the G-structure algebroid corresponding to a finite type classification problem for G-structures with connections the there is a 1-1 correspondence

$$\{\text{Solutions of the problem}\} \longleftrightarrow \{G\text{-structure algebroid morphisms}\}$$

$$\begin{array}{ccc} TP & \xrightarrow{\ \phi\ } & A \\ \downarrow & & \downarrow \\ P & \xrightarrow[\ h\ ]{} & X \end{array}$$

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Example

- The Maurer-Cartan form $\omega_{\mathrm{MC}} \in \Omega^1(G, \mathfrak{g})$ is a Lie algebroid morphism $\omega_{\mathrm{MC}} : TG \to \mathfrak{g}$;
- A map $\phi : TP \to \mathfrak{g}$ is a morphism iff it satisfies the M-C equation;
- Every morphism ϕ is locally the pull-back of ω_{MC} (universal property).

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We must consider M-C forms on Lie groupoids!

G-Structure Groupoids

Let $G \subset \mathrm{GL}_n$.
A G-**Structure Groupoid** is a Lie groupoid $\varGamma \rightrightarrows X$ with a (right) locally free and proper G-action such that $s(\gamma \cdot g) = s(\gamma)$,

$$(\gamma_1\gamma_2) \cdot g = (\gamma_1 \cdot g)\gamma_2,$$

and a **tautological** (s-foliated) **1-form** $\Theta \in \Omega^1_s(\varGamma, \mathbb{R}^n)$, where Θ is

- Right invariant: $R^*_\gamma \Theta = \Theta$;
- G-equivariant: $\Psi^*_g \Theta = g^{-1} \cdot \Theta$;
- Strongly Horizontal:
  $\Theta_\gamma(\xi) = 0 \quad \text{iff} \quad \xi = (\alpha_\varGamma)|_\gamma$, for some $\alpha \in \mathfrak{g}$.

G-Structure Groupoids with Connections

A **Connection** on a G-structure groupoid $\Gamma \rightrightarrows X$ is a (s-foliated) 1-form $\Omega \in \Omega^1_s(\Gamma, \mathbb{R}^n)$ which satisfies:

- Right invariant: $R^*_\gamma \Omega = \Omega$;
- G-equivariant: $\Psi^*_g \Omega = \mathrm{Ad}_{g^{-1}} \cdot \Omega$;
- Vertical: $\Omega_\gamma(\alpha_\Gamma) = \alpha$ for all $\alpha \in \mathfrak{g}$.

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G-structure groupoids with connections give rise to families of G-structures with connection:

$$\begin{array}{c} s^{-1}(x) \quad \circlearrowleft G \\ \Big\downarrow \pi \\ s^{-1}(x)/G \end{array}$$

Solutions to Realization Problem

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$$\begin{array}{ccc} Ts^{-1}(x) & \xrightarrow{\omega_{\mathrm{MC}}} & A \\ \downarrow & & \downarrow \\ s^{-1}(x) & \xrightarrow[t]{} & X \end{array}$$

is a morphism of G-structure algebroids, where

$$(\omega_{\mathrm{MC}})_\gamma(\xi) = d_\gamma R_{\gamma^{-1}}(\xi) \in A_{t(\gamma)}$$

Solutions to Realization Problem

The Lie algebroid of a G-structure groupoid $(\Gamma, \Theta, \Omega) \rightrightarrows X$ is a G-structure algebroid;

$$\begin{array}{ccc} Ts^{-1}(x) & \xrightarrow{\omega_{\mathrm{MC}}} & A \\ \downarrow & & \downarrow \\ s^{-1}(x) & \xrightarrow[t]{} & X \end{array}$$

is a morphism of G-structure algebroids, where

$$(\omega_{\mathrm{MC}})_\gamma(\xi) = d_\gamma R_{\gamma^{-1}}(\xi) \in A_{t(\gamma)}$$

These solutions are universal

Integrability of G-Structure Algebroids I

- Not every Lie algebroid is isomorphic to the Lie algebroid of a Lie groupoid. If $A = \mathrm{Lie}(\mathcal{G})$ we say that A is **integrable**;

- Not every G-structure algebroid is isomorphic to the Lie algebroid of a G-structure groupoid (even when A is integrable). If $(A, \theta) = \mathrm{Lie}(\Gamma, \Theta)$ we say that (A, θ) is G-**integrable**;

Integrability of G-Structure Algebroids II

- (A, θ) is G-integrable if and only if it is integrable and there exists Γ integrating A such that the action map $i : \mathfrak{g} \ltimes X \to A$ integrates to a groupoid morphism $\iota : G \ltimes X \to \Gamma$. **there are explicit (and computable!) obstructions for this.**

- If (A, θ) is G-integrable then there exists a **canonical G-structure groupoid** $\Sigma_G(A) \rightrightarrows X$ which integrates A and is characterised by $\pi_1(s^{-1}(x)/G) = \{1\}$. This groupoid covers any other G-integration of A.

Main Results: Local Existence of Solutions

Theorem (R. Fernandes, I.S.)

Let $(A, \theta) \to X$ be a G-structure algebroid and $x \in X$. Then there exists a G-invariant open neighbourhood $U \subset L_x$ such that $A|_U$ is G-integrable.

Main Results: Local Existence of Solutions

Consequence: If $A \to X$ is the G-structure algebroid of a finite type classification problem, then for each $x \in X$ there exists a realization

$$\begin{array}{ccc} TF_G(M) & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ F_G(M) & \xrightarrow[h]{} & X \end{array}$$

such that $x \in \mathrm{Im}(h)$.

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such that $x \in \mathrm{Im}(h)$.

The moduli space (stack) of germs of solutions to the classification problem up to isomorphism is represented by $G \ltimes X \rightrightarrows X$.

Global Solutions

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Assume $G \subset \mathrm{O}_n$ so that completeness is metric (there is a more general definition). If

$$\begin{array}{ccc} TF_G(M) & \xrightarrow{\ \phi\ } & A \\ \downarrow & & \downarrow \\ F_G(M) & \xrightarrow[\ h\ ]{} & X \end{array}$$

is a complete realization then $\mathrm{Im}(h) = L$ is a leaf of A.

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Global Moduli Space: If the G-structure algebroid $A \to X$ of a finite type classification problem for G-structures with connections is G-integrable, then the canonical G-integration $\Sigma_G(M) \rightrightarrows X$ represents the moduli space (stack) of simply connected and complete solutions of the classification problem.

Back to Examples: Classification of EK-Surfaces

Conditions	$U(1)$-**frame bundle**: $\mathrm{s}^{-1}(x)$	**Solutions**: $\mathrm{s}^{-1}(x)/U(1)$
$K = 0$	$\mathrm{SO}(2) \ltimes \mathbb{R}^2$	$\mathbb{R}^2$
$K = c > 0$	$\mathbb{S}^3$	$\mathbb{S}^2$
$K = c < 0$	$\mathrm{SO}(2,1)$	$\mathbb{H}^2$
$\Delta = 0,\ c_1 = c_2 = 0$	$(\mathbb{R}^2 \times \mathbb{R})/\mathbb{Z}$	$\mathbb{R}^2$
$\Delta = 0,\ c_2 < 0$	$\mathbb{R}^2 \times \mathbb{S}^1$	$\mathbb{R}^2$
$\Delta = 0,\ c_2 > 0$	$(\mathbb{R}^2 \times \mathbb{R})/\mathbb{Z}$ or $(\mathbb{R}^2 \times \mathbb{S}^1)$	$\mathbb{R}^2$
$\Delta < 0$	$\mathbb{R}^2 \times \mathbb{S}^1$	$\mathbb{R}^2$
$\Delta > 0$	$\mathbb{R}^2 \times \mathbb{S}^1$	$\mathbb{R}^2$

Hessian Type - Reading Geometry from the Leaves

Surfaces (M, σ) such that $\mathrm{Hess}_\sigma(k) = \frac{1}{2}(1-k^2)\sigma$. The associated

classifying Lie algebroid is $A = \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, with Lie bracket and anchor:

$$[\alpha_1, \alpha_2] = -k\beta \quad [\alpha_1, \beta] = \alpha_2 \quad [\alpha_2, \beta] = -\alpha_1$$

$$\rho(\alpha_1) = k_1 \frac{\partial}{\partial k} + \frac{1}{2}(1-k^2)\frac{\partial}{\partial k_1}$$

$$\rho(\alpha_2) = k_2 \frac{\partial}{\partial k} + \frac{1}{2}(1-k^2)\frac{\partial}{\partial k_2}$$

$$\rho(\beta) = -k_2 \frac{\partial}{\partial k_1} + k_1 \frac{\partial}{\partial k_2}.$$

Metrics of Hessian Curvature - Geometry from Leaves

Computing the obstructions (infinitesimal G-monodromy):

Orbit foliation of A: level sets of

$$F(k_1, k_2, k) := k_1^2 + k_2^2 + \frac{1}{3}k^3 - k$$

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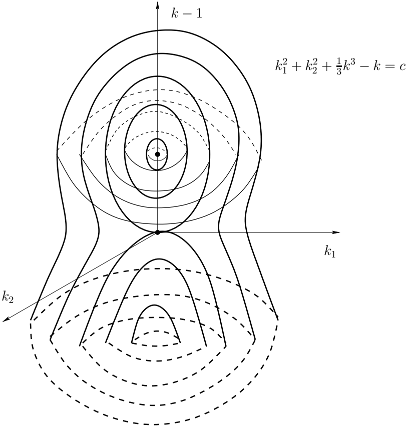
- At the two fixed points $(0, 0, 1)$ and $(0, 0, -1)$, there are solutions (constant curvature metrics);
- In the region filled by spheres there does not exist a G-integration for almost every leaf (but there exists G-integrations on some spheres);
- Over every other leaf in the other regions there exist G-integrations.



Thank you!