Semigroups in Semi-simple Lie Groups and Eigenvalues of Second Order Differential Operators on Flag Manifolds

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  Controllability problem.
- Group generation is almost trivial: if and only if Γ generates $\mathfrak{g}$. (G connected).
- Special set $\Gamma = \{X, \pm Y_1, \ldots, \pm Y_k\}$. Coming from

$$\frac{dg}{dt} = X(g) + u_1(t) Y_1(g) + \cdots + u_k(t) Y_k(g)$$

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  There is a recent proof by SM-Ariane Santos, applying topology of flag manifolds.
- The method for complex groups work for some real ones. E.g. $\mathfrak{sl}(n, \mathbb{H})$.

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- Example of conjecture: $\{X, \pm Y\} \subset \mathfrak{sl}(n, \mathbb{R})$ is not controllable if X, Y are symmetric matrices.

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  Contains $\mathrm{supp}\mu^n \subset (\mathrm{supp}\mu)^n$
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  Contains $\mathrm{supp}\mu^n \subset (\mathrm{supp}\mu)^n$
  $\mu^n = n$th convolution power of μ.
- Not originated from control theory. Can be applied to the controllability problem.

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- Representations: U on a vector space by operators $U(g)$. Form the operator

$$U(\mu)v = \int_G (U(g)v)\mu(dg)$$

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- Here will focus on the representations.

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  $F_\lambda = \{f : G \to \mathbb{C} : f(gmhn) = e^{\lambda(\log h)} f(g)$. $\lambda \in \mathbb{C}$.
  $\lambda \in \mathfrak{a}^*$.
  (Special case of $f(gmhn) = \theta(m) e^{\lambda(\log h)} f(g)$ with λ complex and $\theta : M \to \mathbb{C}_\times$ homomorphism.)

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  $U_\lambda(g) = U(g)$ restricted to F_λ

Compact picture

- Each F_{λ} is in bijection with the function space $F_K = \{f : K \to \mathbb{C}\}$ by $f \in F_K \mapsto \widetilde{f} \in F_{\lambda}$, $\widetilde{f}(kan) = f(k)$.

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- Equivalent representations **compact picture** : $F = F_K$ or $F = F_\mathbb{F}$
  $U_\lambda(g) f(x) = \rho_\lambda(g, x) f(gx)$, $g \in G$, $x \in K$
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  $K = G/AN$ viewed as homogeneous space of G.
- Cocycle: $\rho_\lambda(g, x) = e^{\lambda(\log h)}$ where $gu = khn$ and $x = ux_0$.
  $x_0 = 1 \cdot AN =$ origin of K

Example: $\mathrm{Sl}(2,\mathbb{R})$ or $\mathbb{C}$

- $G = \mathrm{Sl}(2,\mathbb{R})$,
  $K = S^1 = \mathrm{SO}(2)$, $\mathbb{F} = \mathbb{P}^1$
  $\rho_\lambda(g, x) = \|gx\|^p$
  $U_p(g) f(x) = \|gx\|^p f(gx), g \in \mathrm{Sl}(2,\mathbb{R})$, $x \in S^1$
- Other realization: Homogeneous functions
  $F_p = \{f : \mathbb{R}^2 \to \mathbb{C} : f(cx) = c^p f(x)$, $c > 0$.
  $U_p(g) f(y) = f(gy)$, $g \in \mathrm{Sl}(2,\mathbb{R})$, $y \in \mathbb{R}^2$.

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Back to probabilities

- μ has exponential moments if $\int \rho_\lambda(g, x)\, \mu(dg) < \infty$ all x and λ
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- μ is exposed (*étalée*) if $\mathrm{int} S_\mu \neq \emptyset$.
  $U_\lambda(\mu)$ is compact on $\mathcal{C}(K)$.
  discrete spectra with finite dimensional spectral spaces
  $r_\lambda =$ spectral radius of $U_\lambda(\mu)$
  is an eigenvalue

Result

- $S_{\mu} = G$ if and only if the map $\lambda \mapsto r_{\lambda}$ is analytic.

Result

- When $S_\mu \neq G$ points of nonanalyticity are obtained from the structure of S_μ (flag type).

Continuous time version

- Application to controllability of $\Gamma = \{X, \pm Y_1, \ldots, \pm Y_k\}$.
  S_Γ = semigroup generated by e^{tX}, $X \in \Gamma$, $t \geq 0$

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  S_Γ = semigroup generated by e^{tX}, $X \in \Gamma$, $t \geq 0$
- Related to

$$\frac{dg}{dt} = X(g) + u_1(t) Y_1(g) + \cdots + u_k(t) Y_k(g)$$

- Associated Itô stochastic differential equation

$$dg = X(g)\, dt + \sum_{j=1}^{k} Y_j(g) \circ dW_j.$$

Continuous time: Solutions and semigroups

- One-parameter semigroup of measures (under convolution): $\mu_t = P_t(1, \cdot) =$ transition probability of the solution starting at 1.

  $\mu_{t+s} = \mu_t * \mu_s$

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- By the support theorem (Strook-Varadhan-Kunita)

$$\mathrm{cl} S_\Gamma = \mathrm{cl} \bigcup_{t \geq 0} \mathrm{supp} \mu_t$$

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- $U_\lambda(L_\lambda) = U_\lambda(X) + \frac{1}{2}\sum_{i=1}^{k} U_\lambda(Y_i)^2$
  infinitesimal representation of the universal envelopping algebra

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- $r_Y(x) = \widetilde{Y} q_Y(x) = Y^2\mathrm{a}(1, x)$
- $\widetilde{X}$ = vector field induced by $X \in \mathfrak{g}$
  $\mathrm{a}(g, x) = \log \rho(g, x)$

Controllability: Preliminaires

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- $S_\Gamma = G$ if and only if $S_{\mu_t} = G$, $t > 0$.
- $r_\lambda(t)$ = spectral radius of $U_\lambda(\mu_t)$
  L_λ has a largest eigenvalue γ_λ
  $r_\lambda(t) = e^{t\gamma_\lambda}$

Controllability: Theorem

- Under the Lie algebra rank condition $S_\Gamma = G$ if and only if $\lambda \mapsto \gamma_\lambda$ is everywhere analytic.

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- Spectra L_λ (infinitesimal data) $\longleftrightarrow$ Controllability

Semigroups in $\mathrm{Sl}(2, \mathbb{R})$

Facts:

- Let $S \subset \mathrm{Sl}(2, \mathbb{R})$ be a semigroup with $\mathrm{int} S \neq \emptyset$. Then $S = \mathrm{Sl}(2, \mathbb{R})$ if and only if S acts transitively on the projective line $\mathbb{P}^1$.

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- When $S \neq \mathrm{Sl}(2,\mathbb{R})$ $(\mathrm{int} S \neq \emptyset)$ there exists a unique proper closed subset $C \subset \mathbb{P}^1$ such that $\mathrm{cl} Sx = C$ for all $x \in C$. (Invariant control set.)

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- There exists $c > 0$ such that

$$\frac{\|gx\|}{\|x\|} > c \qquad [x] \in C.$$

Operators in the invariant control set

- Assume $S_{\mu} \neq \mathrm{Sl}(2, \mathbb{R})$ and let $C \subset \mathbb{P}^1$ be its invariant control set.

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- Define

$$U_p^C(\mu) f(x) = \int_G \rho_p(g, x) f(gx) \mu(dg)$$

$$= \int_G \frac{\|gx\|^p}{\|x\|^p} f(gx) \mu(dg)$$

for the operator restricted to the Banach space of continuous functions $\mathcal{C}(C)$.

Facts about the operators

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  Because there is a strictly positive eigenfunction by irreducibility: $\mathrm{cl} S_\mu x = C$ all $x \in C$.
- $p \mapsto r_p^C$ is analytic in the real line.
  By pertubation theory of compact operators: multiplicity $1 \Longrightarrow$ analyticity.

Facts about the operators

- $\gamma_C(p) = \log r_p^C$ is a convex function:

$$\begin{aligned}\gamma_C(p) &= \lim \frac{1}{n} \log \int \frac{\|gx\|^p}{\|x\|^p} \mu^n(dg) \qquad \text{any } x \in C \\ &= \lim \frac{1}{n} \log \left\| \left(U_p^C \right)^n \right\|.\end{aligned}$$

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Moment Lyapunov Exponent

- By Gelfand formula $r(T) = \lim_n \|T^n\|^{1/n}$ and $\|T\| = \sup_x |T1(x)|$ if T is a positive operator.

Shape of $\gamma_C(p)$

- $\gamma'_C(0) > 0$:

$$\gamma'_C(0) = \lim \frac{1}{n} \log \frac{\|g_n x\|^p}{\|x\|^p}$$

Top Lyapunov exponent (g_n = random product)

Shape of $\gamma_C(p)$

- $\gamma_C'(0) > 0$:

$$\gamma_C'(0) = \lim \frac{1}{n} \log \frac{\|g_n x\|^p}{\|x\|^p}$$

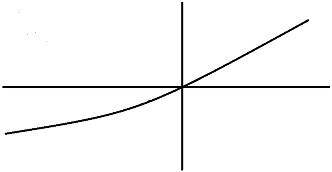
Top Lyapunov exponent (g_n = random product)

- $$\lim_{p \to -\infty} \gamma_C(p) < 0$$

Property of the semigroup: $\frac{\|gx\|}{\|x\|} > c$ if $g \in S_\mu$ and $[x] \in C$.

Shape of $\gamma_C(p)$

- .



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- $U_p(\mu)$, $r_p =$ spectral radius, $\gamma(p) = \log r_p$
- If $S_\mu \neq G$ there is no irreducibility. Existence of strictly positive eigenfunction and multiplicity 1 of r_p is not immediate.
- If $p \in (-1, +\infty)$ then there exists an eigenfunction f_p, $U_p(\mu) = r_p f_p$ with $f > 0$ in $\mathbb{P}^1$:

$$f_p(x) = \int_{\mathbb{P}^1} |\cos\theta(x,y)|^p \nu_p(dy)$$

where ν_p is an eigenmeasure. Integrability is ensured only at $p > -1$.

Shape of $\gamma(p)$

- If $p \in (-1, +\infty)$ then r_p has multiplicity one and

$$\gamma(p) = \lim \frac{1}{n} \log \int \frac{\|gx\|^p}{\|x\|^p} \mu^n(dg) = \gamma_C(p).$$

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- The adjoint of $U_p(\mu)$ in $L^2(\mathbb{P}^1)$ is

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- The shape of $\gamma(p)$ in the interval $(-\infty, -1)$ is symmetric-like to the shape in $(-1, +\infty)$.
  Applied to μ^{-1}

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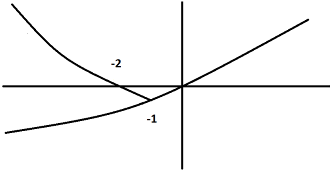

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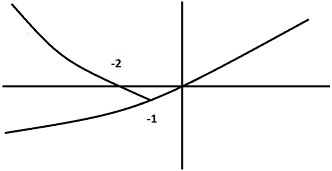

- Analyticity fails at -1.
  And multiplicity is bigger thant 1.

Semi-simple groups in general

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- There are flag manifolds where C_Θ is contractible.
  $h^n C$ shrinks to a point.
- The maximal one with this property is the flag type $\mathbb{F}_{\Theta(S)}$ of S ($\mathrm{int} S \neq \emptyset$ and $S \neq G$).

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- In $\mathbb{F}_{\Theta(S)}$ there is the cocycle $\rho_{\omega_{\Theta(S)}}(g,x)$ defined by $g_* m = \rho_{\omega_{\Theta(S)}}(g^{-1},x)\, m$ where m is the unique K-invariant measure.

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- Hence $\lambda \mapsto r_\lambda$ fails to be analytic at $\lambda = -\omega_{\Theta(S)}$.
- Lack of analyticity is read by the flag type of $S_{\mu|}$.

Comments

- This work was started with the objective of developing measure theoretic (probabilistic) tools to study semigroups in semi-simple Lie groups. The methods to study semigroups S with $\mathrm{int} S$ are mainly topological. Having a measure theoretic approach may open the possibility to study more general classes of semigroups and eventually get the concept of **flag type** of a semigroup in a more general context. For example Zariski dense semigroups in algebraic groups and eventually

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- The results obtained relating controllability (flag type) to spectral radii suggest applications of differential operator theory to controllability. Up to now only applications in the other direction.

Examples of operators dim 2

- $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\qquad \theta \in \mathbb{P}^1 = S^1$
  $\lambda = p\lambda_1$

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$$\begin{aligned} L_p &= \mathrm{sen}\theta \frac{d}{d\theta} + \cos^2\theta \frac{d^2}{d\theta^2} - \frac{\mathrm{sen}2\theta}{2}\frac{d}{d\theta} \\ &\quad + p\,\mathrm{sen}\theta \frac{d}{d\theta} + p\left(\cos\theta + \cos^2\theta\right) + p^2\mathrm{sen}^2\theta. \end{aligned}$$

Examples of operators dim 2

- $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  coordinate system $t \mapsto [(\cosh t, \sinh t)]$:

$$\frac{d^2}{dt^2} + \left(p \frac{2\sinh 2t}{\cosh 2t} - 2\sinh 2t \right) \frac{d}{dt}$$
$$+ p \frac{1}{\cosh 2t} + p \frac{4}{\cosh^2 t} + p^2 \frac{2\sinh^2 2t}{\cosh^2 2t}$$