Prequatization, differential cohomology and the genus integration

Rui Loja Fernandes

Department of Mathematics
University of Illinois at Urbana-Champaign, USA

2nd Workshop São Paulo J. of Math. Sci.
USP, November 2019

This talk is an *exercise* based on:

- Ivan Contreras & RLF, “Genus Integration, Abelianization and Extended Monodromy”, `arXiv:1805.12043`.
- Discussions with Alejandro Cabrera on obstructions to strict deformation quantization.

This talk is an *exercise* based on:

- Ivan Contreras & RLF, “Genus Integration, Abelianization and Extended Monodromy”, `arXiv:1805.12043`.
- Discussions with Alejandro Cabrera on obstructions to strict deformation quantization.

Aim:

- relationship between the classical prequantization condition and integration of Lie algebroids

This talk is an *exercise* based on:

- Ivan Contreras & RLF, "Genus Integration, Abelianization and Extended Monodromy", `arXiv:1805.12043`.
- Discussions with Alejandro Cabrera on obstructions to strict deformation quantization.

Aim:

- relationship between the classical prequantization condition and integration of Lie algebroids

Wait!!! Wasn't this solved a long time ago?!

This talk is an *exercise* based on:

- Ivan Contreras & RLF, "Genus Integration, Abelianization and Extended Monodromy", `arXiv:1805.12043`.
- Discussions with Alejandro Cabrera on obstructions to strict deformation quantization.

Aim:

- relationship between the classical prequantization condition and integration of Lie algebroids

Wait!!! Wasn't this solved a long time ago?!

- M. Crainic, Prequantization and Integrability, *J. Sympl. Geom.* (2004)

This talk is an *exercise* based on:

- Ivan Contreras & RLF, "Genus Integration, Abelianization and Extended Monodromy", `arXiv:1805.12043`.
- Discussions with Alejandro Cabrera on obstructions to strict deformation quantization.

Aim:

- relationship between the classical prequantization condition and integration of Lie algebroids

Wait!!! Wasn't this solved a long time ago?!

• M. Crainic, Prequantization and Integrability, *J. Sympl. Geom.* (2004)

... but this paper assumes manifold is **1-connected**.

The prequantization condition

- $\omega \in \Omega^2(M)$ – closed 2-form
  - **Group of periods of** ω:

$$\mathrm{Per}(\omega) := \left\{ \int_\sigma \omega : \sigma \in H_2(M, \mathbb{Z}) \right\} \subset (\mathbb{R}, +)$$

  - **Group of spherical periods of** ω:

$$\mathrm{SPer}(\omega) := \left\{ \int_\sigma \omega : \sigma \in \pi_2(M) \right\} \subset \mathrm{Per}(\omega)$$

The prequantization condition

- $\omega \in \Omega^2(M)$ – closed 2-form
  - **Group of periods of** ω:

$$\mathrm{Per}(\omega) := \left\{ \int_\sigma \omega : \sigma \in H_2(M, \mathbb{Z}) \right\} \subset (\mathbb{R}, +)$$

  - **Group of spherical periods of** ω:

$$\mathrm{SPer}(\omega) := \left\{ \int_\sigma \omega : \sigma \in \pi_2(M) \right\} \subset \mathrm{Per}(\omega)$$

Definition

(M, ω) satisfies the **prequantization condition** if $\mathrm{Per}(\omega) \subset \mathbb{R}$ is a discrete subgroup, i.e., if there exists $a \in \mathbb{R}$ such that

$$\mathrm{Per}(\omega) = a\mathbb{Z} \subset \mathbb{R}.$$

One can also consider the weaker requirement that $\mathrm{SPer}(\omega) \subset \mathbb{R}$ is a discrete subgroup. One of our aims is to understand the differences...

The prequantization condition

Notation:

$$\mathbb{S}^1_a := \mathbb{R}/a\mathbb{Z}$$

Note that one can have $a = 0$ in which case $\mathbb{S}^1_0 = \mathbb{R}$.

The prequantization condition

Notation:

$$\mathbb{S}^1_a := \mathbb{R}/a\mathbb{Z}$$

Note that one can have $a = 0$ in which case $\mathbb{S}^1_0 = \mathbb{R}$.

Theorem (Souriau 1967, Kostant 1970)

Let $\omega \in \Omega^2_{\mathrm{cl}}(M)$. There exists a principal $\mathbb{S}^1_a$-bundle $\pi : P \to M$ with connection $\theta \in \Omega^1(P, \mathbb{R})$ satisfying $\pi^*\omega = \mathrm{d}\theta$ if and only if $\mathrm{Per}(\omega) \subset a\mathbb{Z}$.

The prequantization condition

Notation:

$$\mathbb{S}^1_a := \mathbb{R}/a\mathbb{Z}$$

Note that one can have $a = 0$ in which case $\mathbb{S}^1_0 = \mathbb{R}$.

Theorem (Souriau 1967, Kostant 1970)

Let $\omega \in \Omega^2_{\mathrm{cl}}(M)$. There exists a principal $\mathbb{S}^1_a$-bundle $\pi : P \to M$ with connection $\theta \in \Omega^1(P, \mathbb{R})$ satisfying $\pi^*\omega = \mathrm{d}\theta$ if and only if $\mathrm{Per}(\omega) \subset a\mathbb{Z}$.

- What are the possible such principal $\mathbb{S}^1_a$-bundle $\pi : P \to M$ with connection θ?

The prequantization condition

Notation:

$$\mathbb{S}^1_a := \mathbb{R}/a\mathbb{Z}$$

Note that one can have $a = 0$ in which case $\mathbb{S}^1_0 = \mathbb{R}$.

Theorem (Souriau 1967, Kostant 1970)

Let $\omega \in \Omega^2_{\mathrm{cl}}(M)$. There exists a principal $\mathbb{S}^1_a$-bundle $\pi : P \to M$ with connection $\theta \in \Omega^1(P, \mathbb{R})$ satisfying $\pi^*\omega = \mathrm{d}\theta$ if and only if $\mathrm{Per}(\omega) \subset a\mathbb{Z}$.

- What are the possible such principal $\mathbb{S}^1_a$-bundle $\pi : P \to M$ with connection θ?

The answer is provided by *differential cohomology*.

Differential cohomology (Cheeger & Simons)

Definition

A **differential character** of degree k on M relative to $a\mathbb{Z}$ is a group homomorphism $\chi : Z_k(M) \to \mathbb{S}^1_a$ for which there exists a closed form $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)$ such that:

$$\chi(\partial\sigma) = \int_\sigma \omega \pmod{a\mathbb{Z}}, \quad \forall \sigma \in C_{k+1}(M).$$

$\hat{H}^k(M, \mathbb{S}^1_a) = \{\text{differential characters of degree } k\}$

Differential cohomology (Cheeger & Simons)

Definition

A **differential character** of degree k on M relative to $a\mathbb{Z}$ is a group homomorphism $\chi : Z_k(M) \to \mathbb{S}^1_a$ for which there exists a closed form $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)$ such that:

$$\chi(\partial\sigma) = \int_\sigma \omega \pmod{a\mathbb{Z}}, \quad \forall \sigma \in C_{k+1}(M).$$

$\hat{H}^k(M, \mathbb{S}^1_a) = \{\text{differential characters of degree } k\}$

- ω is uniquely determined by the differential character χ and $\mathrm{Per}(\omega) \subset a\mathbb{Z}$:

$$\delta_1 : \hat{H}^k(M, \mathbb{S}^1_a) \to \Omega^{k+1}_{a\mathbb{Z}}(M), \quad \chi \mapsto \omega.$$

Differential cohomology (Cheeger & Simons)

Definition

A **differential character** of degree k on M relative to $a\mathbb{Z}$ is a group homomorphism $\chi : Z_k(M) \to \mathbb{S}^1_a$ for which there exists a closed form $\omega \in \Omega^{k+1}_{\mathrm{cl}}(M)$ such that:

$$\chi(\partial\sigma) = \int_\sigma \omega \pmod{a\mathbb{Z}}, \quad \forall \sigma \in C_{k+1}(M).$$

$\hat{H}^k(M, \mathbb{S}^1_a) = \{\text{differential characters of degree } k\}$

- ω is uniquely determined by the differential character χ and $\mathrm{Per}(\omega) \subset a\mathbb{Z}$:
$$\delta_1 : \hat{H}^k(M, \mathbb{S}^1_a) \to \Omega^{k+1}_{a\mathbb{Z}}(M), \quad \chi \mapsto \omega.$$
- Choose lift $\tilde{\chi} : C_k(M) \to \mathbb{R}$ and define $c : C_{k+1}(M) \to \mathbb{R}$ by:
$$c(\sigma) := \int_\sigma \omega - \tilde{\chi}(\partial\sigma).$$
Then $c \in Z^{k+1}(M, a\mathbb{Z})$ and $[c] \in H^{k+1}(M, a\mathbb{Z})$ does not depend on $\tilde{\chi}$:
$$\delta_2 : \hat{H}^k(M, \mathbb{S}^1_a) \to H^{k+1}(M, a\mathbb{Z}), \quad \chi \mapsto [c].$$

Differential cohomology

If $r : H^{k+1}(M, a\mathbb{Z}) \to H^{k+1}(M, \mathbb{R})$ is the natural map, then: $r([c]) = [\omega]$.

Differential cohomology

If $r : H^{k+1}(M, a\mathbb{Z}) \to H^{k+1}(M, \mathbb{R})$ is the natural map, then: $r([c]) = [\omega]$.

Theorem (Cheeger & Simons, 1985)

There is a short exact sequence:

$$H^k(M, \mathbb{R})/r(H^k(M, a\mathbb{Z})) \longrightarrow \hat{H}^k(M, \mathbb{S}^1_a) \xrightarrow{(\delta_1, \delta_2)} R^{k+1}(M, a\mathbb{Z})$$

where:

$$R^\bullet(M, a\mathbb{Z}) = \{(\omega, u) \in \Omega^\bullet_{a\mathbb{Z}}(M) \times H^\bullet(M, a\mathbb{Z}) : [\omega] = r(u)\}.$$

- Differential cohomology provides a refinement of integral cohomology and differential forms with $a\mathbb{Z}$-periods.
- Differential cohomology has a graded ring structure:

$$* : \hat{H}^k(M, \mathbb{S}^1_a) \times \hat{H}^l(M, \mathbb{S}^1_a) \to \hat{H}^{k+l+1}(M, \mathbb{S}^1_a)$$

and (δ_1, δ_2) is a ring homomorphism.

Differential cohomology in degree 1

Differential cohomology in degree 1

Example

$\pi : P \to M$ be a principal $\mathbb{S}^1_a$-bundle with connection $\theta \in \Omega^1(P, \mathbb{R})$ and curvature $\omega \in \Omega^2(M)$:

$$\pi^* \omega = \mathrm{d}\theta.$$

Differential cohomology in degree 1

Example

$\pi : P \to M$ be a principal $\mathbb{S}^1_a$-bundle with connection $\theta \in \Omega^1(P, \mathbb{R})$ and curvature $\omega \in \Omega^2(M)$:

$$\pi^*\omega = \mathrm{d}\theta.$$

Holonomy of the connection along a loop γ gives an element: $\chi(\gamma) \in \mathbb{S}^1_a$.

Differential cohomology in degree 1

Example

$\pi : P \to M$ be a principal $\mathbb{S}^1_a$-bundle with connection $\theta \in \Omega^1(P, \mathbb{R})$ and curvature $\omega \in \Omega^2(M)$:

$$\pi^*\omega = \mathrm{d}\theta.$$

Holonomy of the connection along a loop γ gives an element: $\chi(\gamma) \in \mathbb{S}^1_a$. Extend χ to any cycle $\gamma + \partial\sigma \in Z_1(M)$ by:

$$\chi(\gamma + \partial\sigma) := \chi(\gamma) + \int_\sigma \omega \pmod{a\mathbb{Z}_a}.$$

Differential cohomology in degree 1

Example

$\pi : P \to M$ be a principal $\mathbb{S}^1_a$-bundle with connection $\theta \in \Omega^1(P, \mathbb{R})$ and curvature $\omega \in \Omega^2(M)$:

$$\pi^*\omega = \mathrm{d}\theta.$$

Holonomy of the connection along a loop γ gives an element: $\chi(\gamma) \in \mathbb{S}^1_a$.
Extend χ to any cycle $\gamma + \partial\sigma \in Z_1(M)$ by:

$$\chi(\gamma + \partial\sigma) := \chi(\gamma) + \int_\sigma \omega \pmod{a\mathbb{Z}_a}.$$

This defines a differential character $\chi \in \hat{H}^1(M, \mathbb{S}^1_a)$ with:

- $\delta_1\chi = \omega \in \Omega^2_{a\mathbb{Z}}(M)$;
- $\delta_2\chi \in H^2(M, a\mathbb{Z})$ the (integral) Chern class of the bundle.

Differential cohomology in degree 1

Example

$\pi : P \to M$ be a principal $\mathbb{S}^1_a$-bundle with connection $\theta \in \Omega^1(P, \mathbb{R})$ and curvature $\omega \in \Omega^2(M)$:

$$\pi^*\omega = \mathrm{d}\theta.$$

Holonomy of the connection along a loop γ gives an element: $\chi(\gamma) \in \mathbb{S}^1_a$.
Extend χ to any cycle $\gamma + \partial\sigma \in Z_1(M)$ by:

$$\chi(\gamma + \partial\sigma) := \chi(\gamma) + \int_\sigma \omega \pmod{a\mathbb{Z}_a}.$$

This defines a differential character $\chi \in \hat{H}^1(M, \mathbb{S}^1_a)$ with:

- $\delta_1\chi = \omega \in \Omega^2_{a\mathbb{Z}}(M)$;
- $\delta_2\chi \in H^2(M, a\mathbb{Z})$ the (integral) Chern class of the bundle.

Note: one can have $\delta_1\chi = \delta_2\chi = 0$ with $\chi \neq 0$ (e.g., if $M = \mathbb{S}^1$).

Differential cohomology in degree 1

Theorem (Cheeger & Simons, 1985)

$$\begin{tikzcd}
\left\{\begin{matrix}\textit{principal } \mathbb{S}^1_a\textit{-bundles} \\ \textit{with connection}\end{matrix}\right\} \arrow[r] \arrow[d] & \hat{H}^1(M, \mathbb{S}^1_a) \\
\left\{\begin{matrix}\textit{isomorphism classes of principal} \\ \mathbb{S}^1_a\textit{-bundles with connection}\end{matrix}\right\} \arrow[ur, dashed, "\simeq"'] &
\end{tikzcd}$$

Differential cohomology in degree 1

Theorem (Cheeger & Simons, 1985)

$$\begin{tikzcd}
\left\{\begin{matrix}\textit{principal } \mathbb{S}^1_a\textit{-bundles} \\ \textit{with connection}\end{matrix}\right\} \arrow[rr] \arrow[d] & & \hat{H}^1(M, \mathbb{S}^1_a) \\
\left\{\begin{matrix}\textit{isomorphism classes of principal} \\ \mathbb{S}^1_a\textit{-bundles with connection}\end{matrix}\right\} \arrow[urr, dashed, "\simeq"'] & &
\end{tikzcd}$$

- Lie groupoid theory leads to a natural section of the horizontal arrow (after a choice of a base point), and hence a simple proof/explanation of the theorem.
- This result generalizes to higher principal bundles and higher degree differential cohomology.

Lie algebroids - the canonical integration

$p : A \to M$ – Lie algebroid with Lie bracket $[\ ,\ ]$ and anchor $\rho : A \to TM$

Lie algebroids - the canonical integration

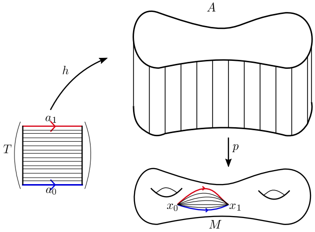
$p : A \to M$ – Lie algebroid with Lie bracket $[\ ,\ ]$ and anchor $\rho : A \to TM$

$$\mathit{\Pi}_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homotopies}} \rightrightarrows M \qquad \begin{cases} A\text{-path: algebroid morphism} \\ \qquad a : TI \to A \\ \\ A\text{-homotopy: algebroid morphism} \\ \qquad h : T(I \times I) \to A \end{cases}$$

Lie algebroids - the canonical integration

$p : A \to M$ – Lie algebroid with Lie bracket $[\ ,\ ]$ and anchor $\rho : A \to TM$

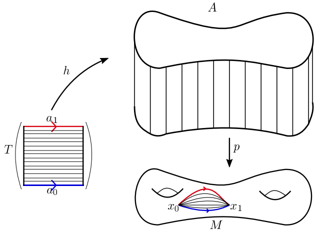
$$\mathit{\Pi}_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homotopies}} \rightrightarrows M \quad \begin{cases} A\text{-path: algebroid morphism} \\ \qquad a : TI \to A \\ \\ A\text{-homotopy: algebroid morphism} \\ \qquad h : T(I \times I) \to A \end{cases}$$



Lie algebroids - the canonical integration

$p : A \to M$ – Lie algebroid with Lie bracket $[\ ,\ ]$ and anchor $\rho : A \to TM$

$$\Pi_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homotopies}} \rightrightarrows M \quad \begin{cases} A\text{-path: algebroid morphism} \\ \qquad a : TI \to A \\ \\ A\text{-homotopy: algebroid morphism} \\ \qquad h : T(I \times I) \to A \end{cases}$$



Topological groupoid with structure maps:

- source: $\mathbf{s}([a]) = p(a(0))$;
- target: $\mathbf{t}([a]) = p(a(1))$;
- product: $[a] \cdot [b] = [a \circ b]$;

Monodromy

For each $x \in M$:

- isotropy Lie algebra: $\mathfrak{g}_x = \ker \rho_x$;
- orbit: $\mathcal{O}_x \subset M$ such that $T_y\mathcal{O} = \mathrm{Im}\,\rho_y$.

and there is a monodromy map:

$$\partial_x : \pi_2(\mathcal{O}_x) \to G(\mathfrak{g}_x)$$

Monodromy

For each $x \in M$:

- isotropy Lie algebra: $\mathfrak{g}_x = \ker \rho_x$;
- orbit: $\mathcal{O}_x \subset M$ such that $T_y\mathcal{O} = \mathrm{Im}\,\rho_y$.

and there is a monodromy map:

$$\partial_x : \pi_2(\mathcal{O}_x) \to G(\mathfrak{g}_x)$$

Theorem (Crainic & RLF, 2003)

The following statements are equivalent:

(i) *A integrates to some Lie groupoid;*

(ii) *$\Pi_1(A)$ is a Lie groupoid;*

(iii) *The monodromy groups $\mathcal{N}_x = \mathrm{Im}\,\partial_x$ are uniformly discrete.*

Prequantization algebroid (Crainic, 2004)

- $\omega \in \Omega^2_{\mathrm{cl}}(M)$ has associated algebroid $A_\omega := TM \oplus \mathbb{R}$:

$$0 \longrightarrow M \times \mathbb{R} \longrightarrow TM \oplus \mathbb{R} \xrightarrow{\rho=\mathrm{pr}} TM \longrightarrow 0$$

with Lie bracket:

$$[(X,f),(Y,g)] := ([X,Y], X(g) - Y(f) + \omega(X,Y)).$$

Prequantization algebroid (Crainic, 2004)

- $\omega \in \Omega^2_{\mathrm{cl}}(M)$ has associated algebroid $A_\omega := TM \oplus \mathbb{R}$:

$$0 \longrightarrow M \times \mathbb{R} \longrightarrow TM \oplus \mathbb{R} \xrightarrow{\rho=\mathrm{pr}} TM \longrightarrow 0$$

with Lie bracket:

$$[(X,f),(Y,g)] := ([X,Y], X(g) - Y(f) + \omega(X,Y)).$$

Monodromy:

$$\partial_x : \pi_2(M,x) \to \mathbb{R}, \quad \sigma \mapsto \int_\sigma \omega$$

$$\mathit{\Pi}_1(A) \text{ is a Lie groupoid} \quad \Longleftrightarrow \quad \mathrm{SPer} \subset \mathbb{R} \text{ is discrete.}$$

Prequantization algebroid (Crainic, 2004)

- $\omega \in \Omega^2_{\mathrm{cl}}(M)$ has associated algebroid $A_\omega := TM \oplus \mathbb{R}$:

$$0 \longrightarrow M \times \mathbb{R} \longrightarrow TM \oplus \mathbb{R} \xrightarrow{\rho=\mathrm{pr}} TM \longrightarrow 0$$

with Lie bracket:

$$[(X,f),(Y,g)] := ([X,Y], X(g) - Y(f) + \omega(X,Y)).$$

Monodromy:

$$\partial_x : \pi_2(M,x) \to \mathbb{R}, \quad \sigma \mapsto \int_\sigma \omega$$

$$\mathit{\Pi}_1(A) \text{ is a Lie groupoid} \quad \Longleftrightarrow \quad \mathrm{SPer} \subset \mathbb{R} \text{ is discrete.}$$

The source fiber $\mathbf{t} : \mathbf{s}^{-1}(x_0) \to M$ is a principal G_{x_0}-bundle, where G_{x_0}:

$$0 \longrightarrow \mathbb{R}/\,\mathrm{SPer}(\omega) \longrightarrow G_{x_0} \longrightarrow \pi_1(M) \longrightarrow 0$$

Prequantization algebroid (continued)

We have the explicit path space description (Crainic, 2004):

$$P = \frac{\{(\gamma, a) : \gamma : I \to M \text{ w/ } \gamma(0) = x_0, a \in \mathbb{R}\}}{\sim} \longrightarrow M, \quad [(\gamma, a)] \mapsto \gamma(1),$$

where $\sim$ is the equivalence relation:

$$(\gamma_1, a_1) \sim (\gamma_2, a_2) \quad \Leftrightarrow \quad \begin{cases} \gamma_2 - \gamma_1 = \partial\sigma, \text{ for } \sigma : \mathbb{D}^2 \to M, \\ a_2 - a_1 = \int_\sigma \omega. \end{cases}$$

Prequantization algebroid (continued)

We have the explicit path space description (Crainic, 2004):

$$P = \frac{\{(\gamma, a) : \gamma : I \to M \text{ w/ } \gamma(0) = x_0, a \in \mathbb{R}\}}{\sim} \longrightarrow M, \quad [(\gamma, a)] \mapsto \gamma(1),$$

where $\sim$ is the equivalence relation:

$$(\gamma_1, a_1) \sim (\gamma_2, a_2) \quad \Leftrightarrow \quad \begin{cases} \gamma_2 - \gamma_1 = \partial\sigma, \text{ for } \sigma : \mathbb{D}^2 \to M, \\ a_2 - a_1 = \int_\sigma \omega. \end{cases}$$

Remarks

- *This bundle has a canonical connection $\theta \in \Omega^1(P)$ induced from the splitting $A_\omega = TM \oplus \mathbb{R}$. It satisfies $\pi^*\omega = \mathrm{d}\theta$.*

Prequantization algebroid (continued)

We have the explicit path space description (Crainic, 2004):

$$P = \frac{\{(\gamma, a) : \gamma : I \to M \text{ w/ } \gamma(0) = x_0, a \in \mathbb{R}\}}{\sim} \longrightarrow M, \quad [(\gamma, a)] \mapsto \gamma(1),$$

where $\sim$ is the equivalence relation:

$$(\gamma_1, a_1) \sim (\gamma_2, a_2) \quad \Leftrightarrow \quad \begin{cases} \gamma_2 - \gamma_1 = \partial\sigma, \text{ for } \sigma : \mathbb{D}^2 \to M, \\ a_2 - a_1 = \int_\sigma \omega. \end{cases}$$

Remarks

- *This bundle has a canonical connection* $\theta \in \Omega^1(P)$ *induced from the splitting* $A_\omega = TM \oplus \mathbb{R}$. *It satisfies* $\pi^*\omega = \mathrm{d}\theta$.
- *If* $\pi_1(M) = \{1\}$ *then* $\mathrm{Per}(\omega) = \mathrm{SPer}(\omega)$ *and* $G_{x_0} = \mathbb{R}/\,\mathrm{Per}(\omega)$. *This gives a principal* $\mathbb{R}/\,\mathrm{Per}(\omega)$*-bundle with connection* θ *satisfying* $\pi^*\omega = \mathrm{d}\theta$. *Note that in this case* $\hat{H}^1(M, \mathbb{S}^1_a) \simeq \Omega^2_{a\mathbb{Z}}(M)$.

Prequantization algebroid (continued)

We have the explicit path space description (Crainic, 2004):

$$P = \frac{\{(\gamma, a) : \gamma : I \to M \text{ w/ } \gamma(0) = x_0, a \in \mathbb{R}\}}{\sim} \longrightarrow M, \quad [(\gamma, a)] \mapsto \gamma(1),$$

where $\sim$ is the equivalence relation:

$$(\gamma_1, a_1) \sim (\gamma_2, a_2) \quad \Leftrightarrow \quad \begin{cases} \gamma_2 - \gamma_1 = \partial\sigma, \text{ for } \sigma : \mathbb{D}^2 \to M, \\ a_2 - a_1 = \int_\sigma \omega. \end{cases}$$

Remarks

- *This bundle has a canonical connection $\theta \in \Omega^1(P)$ induced from the splitting $A_\omega = TM \oplus \mathbb{R}$. It satisfies $\pi^*\omega = \mathrm{d}\theta$.*
- *If $\pi_1(M) = \{1\}$ then $\mathrm{Per}(\omega) = \mathrm{SPer}(\omega)$ and $G_{x_0} = \mathbb{R}/\mathrm{Per}(\omega)$. This gives a principal $\mathbb{R}/\mathrm{Per}(\omega)$-bundle with connection θ satisfying $\pi^*\omega = \mathrm{d}\theta$. Note that in this case $\hat{H}^1(M, \mathbb{S}^1_a) \simeq \Omega^2_{a\mathbb{Z}}(M)$.*
- *If $\pi_1(M) \neq \{1\}$, then the short sequence of G_{x_0} in general will not split, and one cannot find a principal $\mathbb{R}/\mathrm{SPer}(\omega)$-bundle.*

Genus integration

Idea: Replace A-homotopy by A-homology.

Genus integration

Idea: Replace A-homotopy by A-homology.

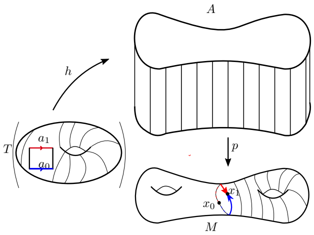
Definition

An A-**homology** between A-paths a_0 and a_1 is an algebroid map

$$h : T\Sigma \to A,$$

with Σ a compact surface with connected boundary $\partial\Sigma$ such that

$$h|_{T(\partial\Sigma)} = a_0 \circ a_1^{-1}.$$



Genus integration

Idea: Replace A-homotopy by A-homology.

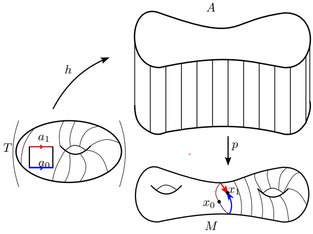
Definition

An A-**homology** between A-paths a_0 and a_1 is an algebroid map

$$h : T\Sigma \to A,$$

with Σ a compact surface with connected boundary $\partial\Sigma$ such that

$$h|_{T(\partial\Sigma)} = a_0 \circ a_1^{-1}.$$



Remarks.

- The genus of Σ is not fixed.

- The A-**homology class** of the A-path a is denoted $[[a]]$

Genus Integration

$$\mathcal{H}_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homologies}} \rightrightarrows M$$

Genus Integration

$$\mathcal{H}_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homologies}} \rightrightarrows M$$

A topological groupoid with structure maps:

- source: $\mathbf{s}([[a]]) = p(a(0))$;
- target: $\mathbf{t}([[a]]) = p(a(1))$;
- product: $[[a]] \cdot [[b]] = [[a \circ b]]$;

Genus Integration

$$\mathcal{H}_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homologies}} \rightrightarrows M$$

A topological groupoid with structure maps:

- source: $\mathbf{s}([[a]]) = p(a(0))$;
- target: $\mathbf{t}([[a]]) = p(a(1))$;
- product: $[[a]] \cdot [[b]] = [[a \circ b]]$;

There is a morphism of topological groupoids:

$$\Pi_1(A) \to \mathcal{H}_1(A), \quad [a] \mapsto [[a]]$$

Genus Integration

$$\mathcal{H}_1(A) = \frac{\{A\text{-paths}\}}{A\text{-homologies}} \rightrightarrows M$$

A topological groupoid with structure maps:

- source: $\mathbf{s}([[a]]) = p(a(0))$;
- target: $\mathbf{t}([[a]]) = p(a(1))$;
- product: $[[a]] \cdot [[b]] = [[a \circ b]]$;

There is a morphism of topological groupoids:

$$\Pi_1(A) \to \mathcal{H}_1(A), \quad [a] \mapsto [[a]]$$

Basic questions:

- What is the meaning of this genus integration?
- When is $\mathcal{H}_1(A)$ smooth?
- If $\mathcal{H}_1(A)$ is smooth, what is its Lie algebroid?

Hurewicz for Lie groupoids

The genus integration $\mathcal{H}_1(A)$ is the set theoretical abelianization of $\Pi_1(A)$

Hurewicz for Lie groupoids

The genus integration $\mathcal{H}_1(A)$ is the set theoretical abelianization of $\Pi_1(A)$

Theorem (Contreras & RLF, 2019)

For any Lie algebroid $A \to M$:

$$\mathcal{H}_1(A) = \frac{\Pi_1(A)}{(\Pi_1(A), \Pi_1(A))},$$

where $(\Pi_1(A), \Pi_1(A)) = \bigcup_{x \in M} (\Pi_1(A)_x, \Pi_1(A)_x)$ is the group bundle formed by the isotropies of $\Pi_1(A)$.

Hurewicz for Lie groupoids

The genus integration $\mathcal{H}_1(A)$ is the set theoretical abelianization of $\Pi_1(A)$

Theorem (Contreras & RLF, 2019)

For any Lie algebroid $A \to M$:

$$\mathcal{H}_1(A) = \frac{\Pi_1(A)}{(\Pi_1(A), \Pi_1(A))},$$

where $(\Pi_1(A), \Pi_1(A)) = \bigcup_{x \in M} (\Pi_1(A)_x, \Pi_1(A)_x)$ is the group bundle formed by the isotropies of $\Pi_1(A)$.

Remarks

- $\mathcal{H}_1(A)$ *need not to be source 1-connected.*
- $\mathcal{H}_1(A)$ *is an example of an abelian groupoid (i.e., isotropy is abelian)*
- *If $\mathcal{H}_1(A)$ is smooth, then its Lie algebroid is abelian, i.e., has abelian isotropy (related to A thorugh abelianization of Lie algebroids)*

Extended Monodromy

Question. When is $\mathcal{H}_1(A)$ smooth?

Simplifying Assumption: A is transitive Lie algebroid.

Extended Monodromy

Question. When is $\mathcal{H}_1(A)$ smooth?

Simplifying Assumption: A is transitive Lie algebroid.

Choose a splitting $\sigma : TM \to A$ of the anchor:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d] & A \arrow[r, "\rho"] \arrow[d, "p"'] & TM \arrow[r] \arrow[l, bend left, "\sigma"] \arrow[dl, dashed, bend left, "\sigma^{\mathrm{ab}}"] & 0 \\
0 \arrow[r] & \mathfrak{g}^{\mathrm{ab}} \arrow[r] & A^{\mathrm{ab}} \arrow[ur] & &
\end{tikzcd}$$

where $\mathfrak{g}_x^{\mathrm{ab}} = \mathfrak{g}_x/[\mathfrak{g}_x, \mathfrak{g}_x]$ and $A^{\mathrm{ab}} = A/[\mathfrak{g}, \mathfrak{g}]$.

Extended Monodromy

Question. When is $\mathcal{H}_1(A)$ smooth?

Simplifying Assumption: A is transitive Lie algebroid.

Choose a splitting $\sigma : TM \to A$ of the anchor:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d] & A \arrow[r, "\rho"] \arrow[d, "p"'] & TM \arrow[r] \arrow[l, bend left, "\sigma"] \arrow[dl, dashed, bend left, "\sigma^{\mathrm{ab}}"] & 0 \\
0 \arrow[r] & \mathfrak{g}^{\mathrm{ab}} \arrow[r] & A^{\mathrm{ab}} \arrow[ur] & &
\end{tikzcd}$$

where $\mathfrak{g}_x^{\mathrm{ab}} = \mathfrak{g}_x / [\mathfrak{g}_x, \mathfrak{g}_x]$ and $A^{\mathrm{ab}} = A/[\mathfrak{g}, \mathfrak{g}]$.

- curvature 2-form $\Omega \in \Omega^2(M, \mathfrak{g}^{\mathrm{ab}})$:

$$\Omega(X, Y) := [\sigma^{\mathrm{ab}}(X), \sigma^{\mathrm{ab}}(Y)] - \sigma^{\mathrm{ab}}([X, Y]).$$

- flat connection ∇ on the bundle $\mathfrak{g}^{\mathrm{ab}} \to M$:

$$\nabla_X \alpha := [\sigma^{\mathrm{ab}}(X), \alpha].$$

Remark. Two different splittings induce the same connection and the same curvature 2-form.

Extended Monodromy

Let $q : \tilde{M}^h \to M$ be the holonomy cover of M relative to ∇, so $q^*\mathfrak{g}^{\mathrm{ab}} \to \tilde{M}$ is trivial with a canonical trivialization.

Extended Monodromy

Let $q : \tilde{M}^h \to M$ be the holonomy cover of M relative to ∇, so $q^*\mathfrak{g}^{\mathrm{ab}} \to \tilde{M}$ is trivial with a canonical trivialization.

Definition

The **extended monodromy homomorphism** at $x \in M$ is the homomorphism of abelian groups:

$$\partial_x^{\mathrm{ext}} : H_2(\tilde{M}^h, \mathbb{Z}) \to G(\mathfrak{g}_x^{\mathrm{ab}}), \quad [\gamma] \mapsto \exp\left(\int_\gamma q^*\Omega\right).$$

$\mathcal{N}_x^{\mathrm{ext}}(A) = \mathrm{Im}\,\partial_x^{\mathrm{ext}}$ is the **extended monodromy group** at x.

Extended Monodromy

Let $q : \tilde{M}^h \to M$ be the holonomy cover of M relative to ∇, so $q^*\mathfrak{g}^{\mathrm{ab}} \to \tilde{M}$ is trivial with a canonical trivialization.

Definition

The **extended monodromy homomorphism** at $x \in M$ is the homomorphism of abelian groups:

$$\partial_x^{\mathrm{ext}} : H_2(\tilde{M}^h, \mathbb{Z}) \to G(\mathfrak{g}_x^{\mathrm{ab}}), \quad [\gamma] \mapsto \exp\left(\int_\gamma q^*\Omega\right).$$

$\mathcal{N}_x^{\mathrm{ext}}(A) = \mathrm{Im}\,\partial_x^{\mathrm{ext}}$ is the **extended monodromy group** at x.

There is a commutative diagram:

$$\begin{tikzcd}
\pi_2(M, x) \arrow[r, "\partial_x"] \arrow[d, "h"'] & G(\mathfrak{g}_x) \arrow[d] \\
H_2(\tilde{M}^h, \mathbb{Z}) \arrow[r, "\partial_x^{\mathrm{ext}}"'] & G(\mathfrak{g}_x^{\mathrm{ab}}) = G(\mathfrak{g}_x)^{\mathrm{ab}}
\end{tikzcd}$$

Extended Monodromy

Theorem (Contreras & RLF, 2019)

Let $A \to M$ be a transitive Lie algebroid with trivial holonomy: $\tilde{M}^h = M$. The following statements are equivalent:

(a) *the genus integration $\mathcal{H}_1(A)$ is smooth;*

(b) *the extended monodromy $\mathcal{N}_x^{\mathrm{ext}}(A)$ groups are discrete;*

(c) *A^{ab} has an abelian integration.*

Extended Monodromy

Theorem (Contreras & RLF, 2019)

Let $A \to M$ be a transitive Lie algebroid with trivial holonomy: $\tilde{M}^h = M$. The following statements are equivalent:

(a) *the genus integration $\mathcal{H}_1(A)$ is smooth;*

(b) *the extended monodromy $\mathcal{N}_x^{\mathrm{ext}}(A)$ groups are discrete;*

(c) *A^{ab} has an abelian integration.*

Remarks

- *An abelian integration of A^{ab} is a Lie groupoid integrating A^{ab} whose isotropy is abelian.*
- *An algebroid with abelian isotropy may not have any abelian integration.*

Prequantization algebroid revisited

The prequantization algebroid $A_\omega := TM \oplus \mathbb{R}$ has trivial holonomy ($\tilde{M}^h = M$) and abelian isotropy ($A^{\mathrm{ab}} = A$):

$$\begin{tikzcd} \pi_2(M, x) \arrow[d, "h"'] \arrow[dr, "\partial_x"] & \\ H_2(M, \mathbb{Z}) \arrow[r, "\partial_x^{\mathrm{ext}}"'] & \mathbb{R} \end{tikzcd} \qquad [\sigma] \longmapsto \int_\sigma \omega$$

Hence:

$$\begin{aligned} \Pi_1(A) \text{ is a Lie groupoid} \quad &\Longleftrightarrow \quad \mathrm{SPer} \subset \mathbb{R} \text{ is discrete} \\ \mathcal{H}_1(A) \text{ is a Lie groupoid} \quad &\Longleftrightarrow \quad \mathrm{Per} \subset \mathbb{R} \text{ is discrete.} \end{aligned}$$

Prequantization algebroid revisited

The prequantization algebroid $A_\omega := TM \oplus \mathbb{R}$ has trivial holonomy ($\tilde{M}^h = M$) and abelian isotropy ($A^{\mathrm{ab}} = A$):

$$\begin{tikzcd} \pi_2(M,x) \arrow[d, "h"'] \arrow[dr, "\partial_x"] & & \\ H_2(M,\mathbb{Z}) \arrow[r, "\partial_x^{\mathrm{ext}}"'] & \mathbb{R} & [\sigma] \longmapsto \int_\sigma \omega \end{tikzcd}$$

Hence:

$$\begin{aligned} \Pi_1(A) \text{ is a Lie groupoid} \quad &\Longleftrightarrow \quad \mathrm{SPer} \subset \mathbb{R} \text{ is discrete} \\ \mathcal{H}_1(A) \text{ is a Lie groupoid} \quad &\Longleftrightarrow \quad \mathrm{Per} \subset \mathbb{R} \text{ is discrete.} \end{aligned}$$

Note: In general, $A \neq A^{\mathrm{ab}}$ and $\tilde{M}^h \neq M$, so the relation between monodromy and extended monodromy is more complicated.

Prequantization algebroid revisited (continued)

The source fiber of $\mathcal{H}_1(A)$ is a principal G_{x_0}-bundle $\mathbf{t} : \mathbf{s}^{-1}(x_0) \to M$ where G_{x_0}:

$$0 \longrightarrow \mathbb{R}/\operatorname{Per}(\omega) \longrightarrow G_{x_0} \longrightarrow H_1(M,\mathbb{Z}) \longrightarrow 0$$

Prequantization algebroid revisited (continued)

The source fiber of $\mathcal{H}_1(A)$ is a principal G_{x_0}-bundle $\mathbf{t} : \mathbf{s}^{-1}(x_0) \to M$ where G_{x_0}:

$$0 \longrightarrow \mathbb{R}/\operatorname{Per}(\omega) \longrightarrow G_{x_0} \longrightarrow H_1(M,\mathbb{Z}) \longrightarrow 0$$

- $G_{x_0} = (\Omega(M, x_0) \times \mathbb{R})/\sim$ where $(\gamma_1, a_1) \sim (\gamma_2, a_2)$ if and only if $\gamma_2 - \gamma_1 = \partial\sigma$ and $a_2 - a_1 = \int_\sigma \omega$, for some $\sigma \in C_2(M)$.

Prequantization algebroid revisited (continued)

The source fiber of $\mathcal{H}_1(A)$ is a principal G_{x_0}-bundle $\mathbf{t} : \mathbf{s}^{-1}(x_0) \to M$ where G_{x_0}:

$$0 \longrightarrow \mathbb{R}/\operatorname{Per}(\omega) \longrightarrow G_{x_0} \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow 0$$

- $G_{x_0} = (\Omega(M, x_0) \times \mathbb{R})/\sim$ where $(\gamma_1, a_1) \sim (\gamma_2, a_2)$ if and only if $\gamma_2 - \gamma_1 = \partial\sigma$ and $a_2 - a_1 = \int_\sigma \omega$, for some $\sigma \in C_2(M)$.
- Since $H_1(M, \mathbb{Z})$ is abelian and $\mathbb{R}/\operatorname{Per}(\omega)$ is a divisible group, this sequence always splits!
- A splitting is the same thing as a choice of differential character

  $$\chi : Z_1(M) \to \mathbb{R}/\operatorname{Per}(\omega) \quad \text{with } \delta_1\chi = \omega.$$

  It realizes $H_1(M, \mathbb{Z})$ as a subgroup of G_{x_0}.

Prequantization algebroid revisited (continued)

After choice of splitting, i.e., of a differential character

$$\chi : Z_1(M) \to \mathbb{R}/\operatorname{Per}(\omega)$$

so that $H_1(M, \mathbb{Z}) \subset G_{x_0}$, we have the quotient groupoid:

$$\mathcal{H}_1(A_\omega)/H_1(M, \mathbb{Z}).$$

Prequantization algebroid revisited (continued)

After choice of splitting, i.e., of a differential character

$$\chi : Z_1(M) \to \mathbb{R}/\operatorname{Per}(\omega)$$

so that $H_1(M, \mathbb{Z}) \subset G_{x_0}$, we have the quotient groupoid:

$$\mathcal{H}_1(A_\omega)/H_1(M, \mathbb{Z}).$$

A source fiber $\mathcal{P}_{\chi, x_0} := \mathbf{s}^{-1}(x_0) \xrightarrow{\mathbf{t}} M$ of this quotient is a principal $\mathbb{R}/\operatorname{Per}(\omega)$-bundle with natural connection θ satisfying $\pi^*\omega = \mathrm{d}\theta$:

Prequantization algebroid revisited (continued)

After choice of splitting, i.e., of a differential character

$$\chi : Z_1(M) \to \mathbb{R}/\operatorname{Per}(\omega)$$

so that $H_1(M,\mathbb{Z}) \subset G_{x_0}$, we have the quotient groupoid:

$$\mathcal{H}_1(A_\omega)/H_1(M,\mathbb{Z}).$$

A source fiber $\mathcal{P}_{\chi,x_0} := \mathbf{s}^{-1}(x_0) \xrightarrow{\mathbf{t}} M$ of this quotient is a principal $\mathbb{R}/\operatorname{Per}(\omega)$-bundle with natural connection θ satisfying $\pi^*\omega = \mathrm{d}\theta$:

$$P_{\chi,x_0} = \frac{\{(\gamma,a) : \gamma : I \to M \text{ w/ } \gamma(0) = x_0, a \in \mathbb{R}/\operatorname{Per}(\omega)\}}{\sim} \longrightarrow M,$$

where $\sim$ is now the equivalence relation:

$$(\gamma_1, a_1) \sim (\gamma_2, a_2) \quad \Leftrightarrow \quad \begin{cases} \gamma_2 - \gamma_1 \in Z_1(M) \\ a_2 - a_1 = \chi(\gamma_2 - \gamma_1) \end{cases}$$

Prequantization algebroid revisited (continued)

After choice of splitting, i.e., of a differential character

$$\chi : Z_1(M) \to \mathbb{R}/\operatorname{Per}(\omega)$$

so that $H_1(M, \mathbb{Z}) \subset G_{x_0}$, we have the quotient groupoid:

$$\mathcal{H}_1(A_\omega)/H_1(M, \mathbb{Z}).$$

A source fiber $\mathcal{P}_{\chi,x_0} := \mathbf{s}^{-1}(x_0) \xrightarrow{\mathbf{t}} M$ of this quotient is a principal $\mathbb{R}/\operatorname{Per}(\omega)$-bundle with natural connection θ satisfying $\pi^*\omega = \mathrm{d}\theta$:

$$P_{\chi,x_0} = \frac{\{(\gamma, a) : \gamma : I \to M \text{ w/ } \gamma(0) = x_0, a \in \mathbb{R}/\operatorname{Per}(\omega)\}}{\sim} \longrightarrow M,$$

where $\sim$ is now the equivalence relation:

$$(\gamma_1, a_1) \sim (\gamma_2, a_2) \quad \Leftrightarrow \quad \begin{cases} \gamma_2 - \gamma_1 \in Z_1(M) \\ a_2 - a_1 = \chi(\gamma_2 - \gamma_1) \end{cases}$$

- This also appears in a recent preprint of Diez, Janssens, Neeb and Vizman, but should be classical...

Conclusion and other on-going exercises

The genus integration produces a natural section

$$\begin{tikzcd} \left\{\begin{array}{c}\text{marked}\\ \text{principal } \mathbb{S}^1_a\text{-bundles}\\ \text{with connection}\end{array}\right\} \arrow[r] & \hat{H}^1(M,\mathbb{S}^1_a) \arrow[l, bend right, dashed] \end{tikzcd}$$

Conclusion and other on-going exercises

The genus integration produces a natural section

$$
\begin{tikzcd}
\left\{ \begin{array}{c} \text{marked} \\ \text{principal } \mathbb{S}^1_a\text{-bundles} \\ \text{with connection} \end{array} \right\} \arrow[r] & \hat{H}^1(M, \mathbb{S}^1_a) \arrow[l, bend right, dashed]
\end{tikzcd}
$$

Remarks

- *Extend this approach to higher degree differential characters in $\hat{H}^k(M, \mathbb{S}^1_a)$ (important, e.g., for ring structure)*

Conclusion and other on-going exercises

The genus integration produces a natural section

$$\begin{tikzcd} \left\{ \begin{array}{c} \text{marked} \\ \text{principal } \mathbb{S}^1_a\text{-bundles} \\ \text{with connection} \end{array} \right\} \arrow[rr] & & \hat{H}^1(M, \mathbb{S}^1_a) \arrow[ll, bend right, dashed] \end{tikzcd}$$

Remarks

- *Extend this approach to higher degree differential characters in $\hat{H}^k(M, \mathbb{S}^1_a)$ (important, e.g., for ring structure)*
- *Construct analogues of genus integration for n-Lie algebroids and L_∞-algebroids*

Conclusion and other on-going exercises

The genus integration produces a natural section

$$\begin{tikzcd} \left\{ \begin{array}{c} \text{marked} \\ \text{principal } \mathbb{S}^1_a\text{-bundles} \\ \text{with connection} \end{array} \right\} \arrow[r] & \hat{H}^1(M, \mathbb{S}^1_a) \arrow[l, bend right, dashed] \end{tikzcd}$$

Remarks

- *Extend this approach to higher degree differential characters in $\hat{H}^k(M, \mathbb{S}^1_a)$ (important, e.g., for ring structure)*
- *Construct analogues of genus integration for n-Lie algebroids and L_∞-algebroids*
- *Extend this approach to torus bundles, symplectic torus bundles, etc.*

Conclusion and other on-going exercises

The genus integration produces a natural section

$$\begin{tikzcd}\left\{\begin{array}{c}\text{marked} \\ \text{principal } \mathbb{S}^1_a\text{-bundles} \\ \text{with connection}\end{array}\right\} \arrow[r] & \hat{H}^1(M, \mathbb{S}^1_a) \arrow[l, dashed, bend right]\end{tikzcd}$$

Remarks

- *Extend this approach to higher degree differential characters in $\hat{H}^k(M, \mathbb{S}^1_a)$ (important, e.g., for ring structure)*
- *Construct analogues of genus integration for n-Lie algebroids and L_∞-algebroids*
- *Extend this approach to torus bundles, symplectic torus bundles, etc.*
- *Extend to the non-abelian case*

Conclusion and other on-going exercises

The genus integration produces a natural section

$$\begin{tikzcd} \left\{ \begin{array}{c} \text{marked} \\ \text{principal } \mathbb{S}^1_a\text{-bundles} \\ \text{with connection} \end{array} \right\} \arrow[r] & \hat{H}^1(M, \mathbb{S}^1_a) \arrow[l, bend right, dashed] \end{tikzcd}$$

Remarks

- *Extend this approach to higher degree differential characters in $\hat{H}^k(M, \mathbb{S}^1_a)$ (important, e.g., for ring structure)*
- *Construct analogues of genus integration for n-Lie algebroids and L_∞-algebroids*
- *Extend this approach to torus bundles, symplectic torus bundles, etc.*
- *Extend to the non-abelian case*

Muito obrigado!