Theoretical Computer Science 947 (2023) 113691

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Improved NP-hardness results for the minimum t-spanner problem on bounded-degree graphs ☆

Renzo Gómez [a,*], Flávio Keidi Miyazawa [a], Yoshiko Wakabayashi [b]

[a] Institute of Computing, University of Campinas, Campinas, Brazil
[b] Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil

A R T I C L E I N F O

Article history:
Received 14 January 2022
Received in revised form 16 June 2022
Accepted 4 January 2023
Available online 9 January 2023
Communicated by D.-Z. Du

Keywords:
Spanner
Sparse spanner
Bounded-degree graph
Planar graph
NP-hardness

A B S T R A C T

For a constant $t \geq 1$, a *t-spanner* of a connected graph G is a spanning subgraph of G in which the distance between any pair of vertices is at most t times its distance in G. This concept, introduced by Peleg and Ullman in 1989, was used in the construction of an optimal synchronizer for the hypercube. We address the problem of finding a t-spanner with minimum number of edges. This problem is called the *minimum t-spanner problem* (MinS$_t$), and is known to be NP-hard for every $t \geq 2$ even on bounded-degree graphs. Our main contribution is to improve the previous results, by showing that MinS$_t$ is NP-hard even on planar graphs with maximum degree at most 4 (resp. 5) when $t \geq 4$ (resp. $t = 3$). We also show that with a slight modification of a result presented by Kobayashi (2018), MinS$_2$ remains NP-hard on planar graphs with maximum degree 7.

1. Introduction

Throughout this text, we consider that the input graph is always connected and may have multiple edges linking the same pair of vertices, but has no loops (even if this is not stated explicitly). The *distance* between two vertices u and v in a graph G is the minimum length of a path between u and v in G, and it is denoted by $d_G(u,v)$. Let $t \geq 1$ be a rational constant. A *t-spanner* of a graph G is a spanning subgraph H of G in which the distance between any pair of vertices is at most t times its distance in G. That is, $d_H(u,v) \leq t \cdot d_G(u,v)$, for all $u, v \in V$.

In 1985, Awerbuch [2] designed an efficient simulation technique, called a synchronizer, for distributed algorithms in asynchronous networks. Peleg & Ullman [15], in 1989, introduced the concept of spanners and studied its relation with the synchronizer of a given network. They constructed a 3-spanner of a hypercube which implies an optimal synchronizer for this kind of network. After that, spanners have appeared in multiple practical applications, such as distributed systems and communication networks (synchronization, building succint and efficient routing tables [16], distance oracles [17,3], roadmap planning [20]), computational geometry, robotics, etc. We refer the interested reader to Ahmed et al. [1] for an extensive literature review on graph spanners.

We address the Minimum t-spanner problem (MinS$_t$), also known as the *sparsest t-spanner problem*. Given a graph G, this problem seeks a t-spanner of G with minimum number of edges. As noted by Cai & Corneil [8], it suffices to study MinS$_t$ only when t is an integer number (since from a result for this case we may derive results for the case t is a rational

☆ This article belongs to Section A: Algorithms, automata, complexity and games, Edited by Paul Spirakis.

* Corresponding author.
E-mail address: rgomez@ic.unicamp.br (R. Gómez).

https://doi.org/10.1016/j.tcs.2023.113691

Table 1
Computational complexity of $\textsc{MinS}_t$ on subclasses of perfect graphs.

Graph class	$t=2$	$t=3$	$t=4$	$t\geq 5$
chordal	NP-hard [19]	NP-hard [19]	NP-hard [19]	NP-hard [19]
strongly chordal	open	open	P [5]	P [5]
interval	open	P [13]	P [13]	P [13]
split	NP-hard [19]	P [19]	P [19]	P [19]
permutation	open	P [13]	P [13]	P [13]
bipartite	P	NP-hard [7]	NP-hard [7]	NP-hard [7]
chordal bipartite	P	open	open	NP-hard [6]
ATE-free	P	P [6]	P [6]	P [6]
convex bipartite	P	P [19]	P [19]	P [19]

Table 2
Computational complexity of $\textsc{MinS}_t$ on graphs with maximum degree at most k. All hardness results remain valid on planar graphs. The shaded cells indicate the results we obtained (some combined with previously known ones).

max degree $\leq k$	$t=2$	$t=3$	$t=4$	$t\geq 5$
$k=3$	P [9]	P [11]	open	open
$k=4$	P [9]	open	NP-hard	NP-hard
$k=5$	open	NP-hard	NP-hard	NP-hard
$k=6$	open	NP-hard [12]	NP-hard [12]	NP-hard
$k=7$	NP-hard	NP-hard [12]	NP-hard [12]	NP-hard
$k=8$	NP-hard [12]	NP-hard [12]	NP-hard [12]	NP-hard
$k\geq 9$	NP-hard [9]	NP-hard [9]	NP-hard [9]	NP-hard [9]

number). The first computational complexity result on $\textsc{MinS}_t$ was established in 1989 by Peleg & Schäffer [14], who showed that $\textsc{MinS}_2$ is NP-hard. In 1994, Cai [7] showed that $\textsc{MinS}_t$ is NP-hard for $t\geq 2$, extending the previous result. Since then, the complexity of $\textsc{MinS}_t$ has been investigated in subclasses of perfect graphs. We summarize in Table 1 the main results in this line of research. Regarding this table, we make the following observations. First, a t-spanner of a bipartite graph, for t even, must be a $(t-1)$-spanner of the graph. Second, if a graph admits a t-spanner that is a tree, this is a minimum t-spanner for the graph. Deciding whether a graph admits such tree is known as the Tree t-spanner problem ($\textsc{TreeS}_t$) in the literature. Thus, given a class of graphs, we can derive further implications for $\textsc{MinS}_t$ by showing either that (a) $\textsc{TreeS}_t$ is NP-hard; or that (b) every graph (in the class) admits a tree t-spanner, and that we can find such tree in polynomial time. These observations imply some of the results that are shown in Table 1.

We focus on the computational complexity of $\textsc{MinS}_t$ on bounded-degree graphs. We denote by $\Delta(G)$ the maximum degree of a graph G (and write simply Δ when referring to an arbitrary graph). A first result on this class was obtained by Cai & Keil [9] in 1994. They showed that $\textsc{MinS}_2$ can be solved in polynomial time if $\Delta\leq 4$. Moreover, they showed that $\textsc{MinS}_t$ is NP-hard when $t\geq 2$ and $\Delta\leq 9$. Recently, Kobayashi [12] improved this result showing that, even on planar graphs, $\textsc{MinS}_t$ is NP-hard when $t=2$ and $\Delta\leq 8$; and also when $3\leq t\leq 4$ and $\Delta\leq 6$. Inspired by the results obtained by Kobayashi, we were able to improve these results showing that $\textsc{MinS}_t$ on planar graphs is NP-hard when $t=2$ and $\Delta\leq 7$; when $t=3$ and $\Delta\leq 5$; and also when $t\geq 4$ and $\Delta\leq 4$. In Table 2, we summarize the main results known in the literature regarding the computational complexity status of $\textsc{MinS}_t$ on bounded-degree graphs.

In what follows, we describe the organization of this text. In Section 2, we present the main concepts and results that will be used here. In particular, we define the *dominating set with degree-k-constraint problem* (DSC_k), which plays an important role in the NP-hardness proofs of $\textsc{MinS}_t$ on planar graphs. In Section 3, we show that DSC_5 is NP-hard on plane graphs with face-degree at most five. This result implies that $\textsc{MinS}_3$ is NP-hard on planar graphs with maximum degree at most five. We also observe that, by slightly modifying the reduction given by Kobayashi, we can improve his result for $\textsc{MinS}_2$. In Section 4, we improve the previously known result for $t\geq 4$, showing that $\textsc{MinS}_t$ remains hard even if $\Delta\leq 4$. Finally, in Section 5, we mention some concluding remarks and directions for future work.

2. Preliminaries

Let $G=(V,E)$ be a graph. The *length* of a path or cycle in G is its number of edges. Let $S\subseteq V$. We denote by $E(S)$ the set of edges in G with both ends in S. We say that S is a *vertex cover* of G if every edge in G has at least one of its ends in S, that is, $E(V\setminus S)=\emptyset$. We say that S is a *dominating set* of G if, for every vertex $v\in V\setminus S$, there exists a vertex in S that is adjacent to v.

In the *minimum vertex cover problem* (VCover), the aim is to find a vertex cover of minimum cardinality. In the *minimum dominating set* (DomS) problem, one seeks a dominating set of minimum cardinality. In this text, we are interested in the following generalization of DomS, called *minimum dominating set with degree-k-constraint problem* (DSC_k), in which one seeks a dominating set of minimum cardinality that contains every vertex of degree at least k in the graph. For simplicity, we use the short notation dsc_k to refer to a set that is a feasible solution to DSC_k.

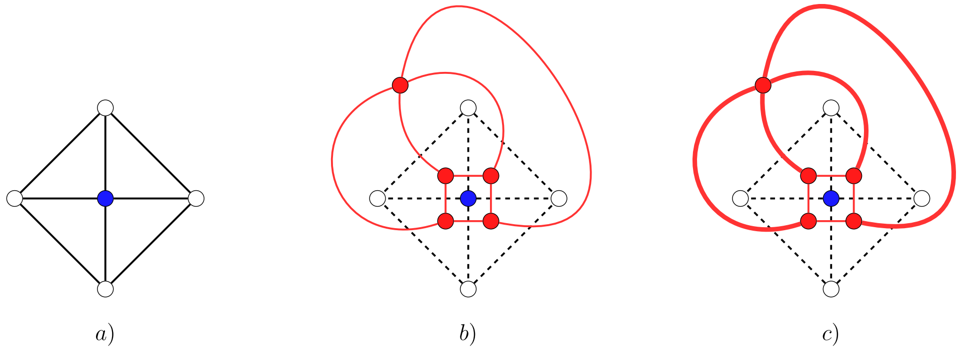

Fig. 1. *a*) a nearly 4-edge-connected plane graph G; *b*) the dual graph G^* (in red); and *c*) a minimum 2-spanner of G^* (the subgraph induced by the thick red edges). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

It is known that DomS is NP-hard on planar subcubic graphs [10]. We note that DomS on subcubic planar graphs can be reduced to DSC$_k$, for $k \geq 5$, as follows. Let G be a planar graph such that $\Delta(G) \leq 3$, and let $K_{1,k}$ be the star on $k+1$ vertices. Now, let H be the graph obtained from $G \cup K_{1,k}$ by linking a vertex in G to a vertex of degree one in $K_{1,k}$. Observe that any dsc$_k$ of H contains the center of $K_{1,k}$. Then, a minimum dsc$_k$ of H, when restricted to G, is a minimum dominating set of G. Therefore, DSC$_k$ is NP-hard on planar graphs, for $k \geq 5$.

Let $G = (V, E)$ be a connected graph, and let $X \subseteq V$. We denote by $\delta_G(X)$ the set of edges in E with one end in X and the other in $V \setminus X$. Moreover, $\delta(G)$ denotes the minimum degree of G. Let $k \geq 2$ be a positive integer. We say that G is *nearly k-edge-connected* if it satisfies the following two conditions:

(i) $\delta(G) \geq k-1$, and
(ii) for every $X \subseteq V$, if $|\delta_G(X)| \leq k-1$, then $|X| = 1$ or $|V \setminus X| = 1$.

Kobayashi [12] showed the following important result that relates DSC$_{t+2}$ on a nearly $(t+2)$-edge-connected planar graph G to MinS$_t$ on its dual graph G^*.

Theorem 1 *(Kobayashi, 2018). Let $G = (V, E)$ be a nearly $(t+2)$-edge-connected plane graph, and let G^* be its dual. Then,* DSC$_{t+2}$ *on G can be reduced in polynomial time to* MinS$_t$ *on G^*. Furthermore, if D^* (resp. S^*) is a minimum dsc$_{t+2}$ (resp. minimum t-spanner) of G (resp. G^*), we have that*

$$|E(S^*)| = |D^*| - |V| + |E|.$$

We show, in Fig. 1, an example of a nearly 4-edge-connected plane graph G, its dual graph G^* (in red) and a minimum 2-spanner of G^* (the subgraph induced by the thick red edges). The vertex in blue is a minimum dsc$_4$ of G.

We recall that the dual graph G^* of a plane graph G is also planar. Furthermore, the degree of a vertex in G^* is the number of edges on the boundary of the face f that corresponds to it in G. We call this number the *face-degree* of f. Kobayashi [12] showed that, for $k \in \{5, 6\}$, DSC$_k$ is NP-hard on nearly k-edge-connected plane graphs whose face-degree is at most six. By Theorem 1 and the previous observation, we have the following.

Theorem 2 *(Kobayashi, 2018). For $t \in \{3, 4\}$,* MinS$_t$ *is* NP*-hard on planar graphs G with $\Delta(G) \leq 6$.*

To derive hardness results for MinS$_t$ on graphs with smaller maximum degree, we follow the same approach. For that, we show NP-hardness results for DSC$_k$ on plane graphs with smaller face-degree.

3. Improved hardness results for MinS$_3$ and MinS$_2$

We show first that DSC$_5$ is NP-hard on nearly 5-edge-connected plane graphs with face-degree at most five. To this end, we show a reduction from VCover to DSC$_5$.

Uehara [18] showed that VCover is NP-hard on planar 3-connected cubic simple graphs. As noted by Kobayashi [12], we can extend this result to k-regular planar multigraphs, for every $k \geq 3$. To see this, consider a planar 3-connected cubic graph G. By Petersen's Theorem [4], G has a perfect matching, say M. Let G' be the graph obtained from G by adding $k-3$ parallel edges, for each edge in M. Note that G' is planar, 3-connected and k-regular. Moreover, every vertex cover of G is a vertex cover of G', and vice versa. Thus, the following holds.

Corollary 1 *(Kobayashi, 2018).* VCover *is* NP*-hard on planar* 3*-connected k-regular multigraphs, for every $k \geq 3$.*

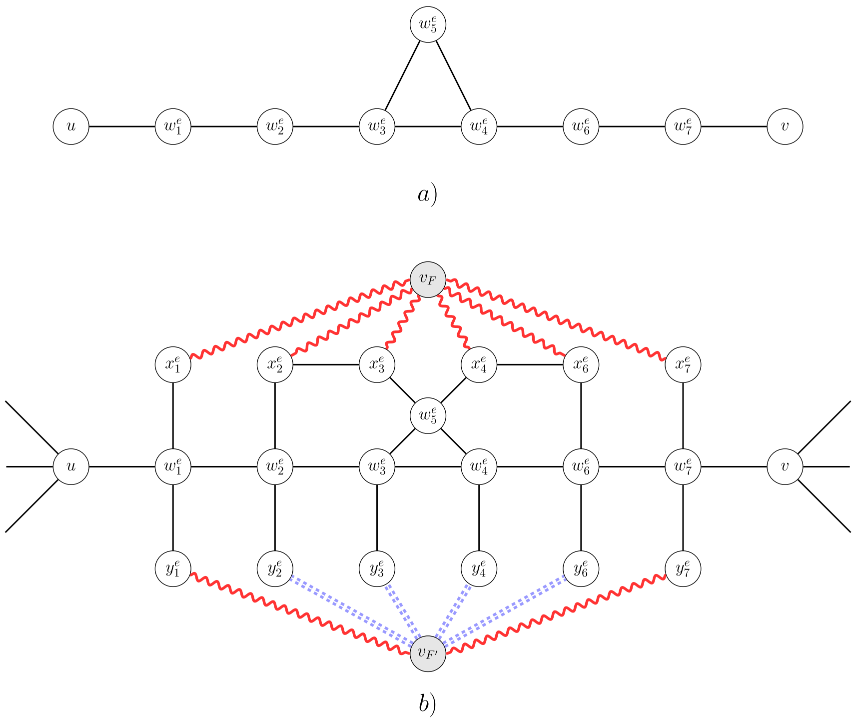

Fig. 2. *a*) the graph H_e; and *b*) the graph obtained from H_e after adding vertices and edges.

We note that the above result refers to VCover on k-regular multigraphs, $k \geq 3$, and the reduction from VCover to DSC_5 — to be shown in what follows — will lead to multigraphs of bounded face-degree. As we mentioned, the latter is related to MinS_t (cf. Theorem 1), and this way we will be able to show hardness of MinS_t on bounded-degree graphs. It is immediate that MinS_t on a multigraph can be solved on its underlying simple graph. Thus when reducing to MinS_t, it does not matter whether we obtain a multigraph.

Definition of $\mathcal{D}_5$-reduction of G, where G is a 3-connected 4-regular graph

Let $G = (V, E)$ be a plane 3-connected 4-regular graph, and let $\mathcal{F}$ be the set of faces of G. The $\mathcal{D}_5$-*reduction* of G is the planar graph $G' = (V', E')$ obtained from G as follows. For each face $F \in \mathcal{F}$, we create a vertex v_F. Let $V_{\mathcal{F}} = \{v_F : F \in \mathcal{F}\}$ be the set of these vertices. For each edge $e = uv \in E$, let H_e be the graph with vertex set $V(H_e) = \{u, w_1^e, w_2^e, w_3^e, w_4^e, w_5^e, w_6^e, w_7^e, v\}$ and edge set as shown in Fig. 2 *a*). To obtain G', first we replace each edge e, in G, with H_e. After that,

(i) for $i \in \{1, 2, 3, 4, 6, 7\}$, we add vertices x_i^e and y_i^e;
(ii) we link w_i^e to x_i^e and y_i^e, for $i \in \{1, 2, 6, 7\}$; and
(iii) add the edge set $\{w_5^e x_3^e,\ w_5^e x_4^e,\ w_3^e y_3^e,\ w_4^e y_4^e,\ x_2^e x_3^e,\ x_4^e x_6^e\}$.

Since G is 3-connected, every edge $e \in E$ belongs to the boundary of exactly two faces, say F and F', of G. We continue the construction of G' by linking the vertices x_i^e to v_F by two parallel edges. Analogously, we link y_1^e and y_7^e to $v_{F'}$ by two parallel edges. Next, we link y_i^e to $v_{F'}$ by three parallel edges, for $i \in \{2, 3, 4, 6\}$. We show an example of this construction in Fig. 2 *b*). The wavy edges and dotted edges represent two and three parallel edges, respectively.

Before we conclude the construction, we recall that our objective is to restrict the face-degree of G'. As one can see in Fig. 2 *b*), this is valid locally for every gadget that replaces an edge $e \in E$. We observe that this is not true when we consider two adjacent gadgets that contain the same vertex $v_F \in V_{\mathcal{F}}$. To forbid this, let us first introduce the following definition. Let e and f be distinct edges in E. We say that x_i^e and x_j^f are *close* if

1. x_i^e and x_j^f are both adjacent to a vertex $v_F \in V_{\mathcal{F}}$, and
2. w_i^e and w_j^f are both adjacent to a vertex $u \in V$.

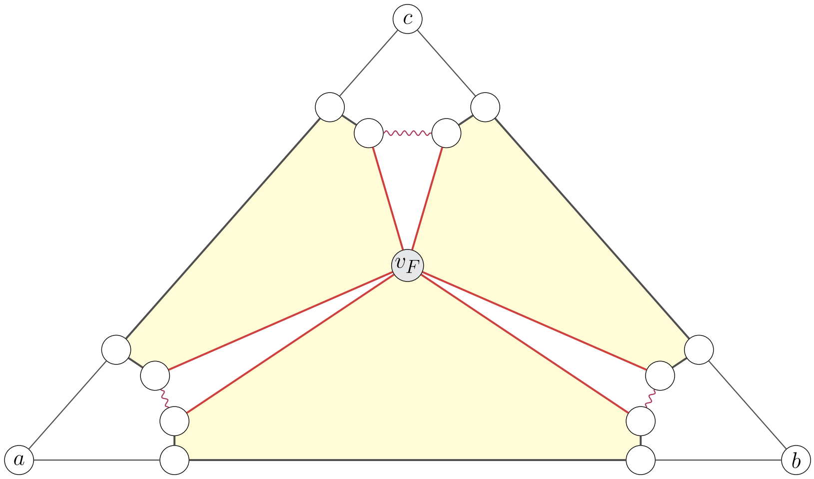

Fig. 3. Configuration obtained from face F after adding edges between close vertices.

Analogously, we extend this definition to pairs of vertices x_i^e, y_j^f and y_i^e, y_j^f. To finish the construction of G', we link every pair of close vertices by a single edge. In Fig. 3, we show the configuration obtained from face F after adding the edges between close vertices (depicted by the wavy edges). In this figure, we omit the vertices and edges inside the cycle (depicted by the shaded pentagon) defined by the vertices w_1^e, x_1^e (resp. y_1^e), v_F, x_7^e (resp. y_7^e), and w_7^e. We observe that G' satisfies the following properties:

a) every vertex in $V' \setminus V_{\mathcal{F}}$ has degree 4, and
b) every face has face-degree at most 5.

In what follows, we show that G' is nearly 5-edge-connected.

Lemma 1. *Let G be a 3-connected 4-regular plane graph, and let G' be the $\mathcal{D}_5$-reduction of G. Then, G' is nearly 5-edge-connected.*

Proof. Let $G = (V, E)$ and $G' = (V', E')$. We have already argued that $\delta(G') \geq 4$. Thus, we only need to show that if

$$\delta_{G'}(X) \leq 4, \text{ then } |X| = 1 \text{ or } |V' \setminus X| = 1. \tag{1}$$

Let $X \subseteq V'$ be one such set. First, we show that either $V_{\mathcal{F}} \subseteq X$ or $V_{\mathcal{F}} \subseteq V' \setminus X$. Let F_1 and F_2 be two faces in $\mathcal{F}$ that share an edge $e \in E$. As shown in Fig. 2 *b*) (considering $F_1 = F$ and $F_2 = F'$), there are six edge-disjoint paths between v_{F_1} and v_{F_2} (each one going through a vertex x_i^e). Thus, either $\{v_{F_1}, v_{F_2}\} \subseteq X$ or $\{v_{F_1}, v_{F_2}\} \subseteq V' \setminus X$. Applying the previous argument on each pair of adjacent faces, we have that $V_{\mathcal{F}} \subseteq X$ or $V_{\mathcal{F}} \subseteq V' \setminus X$. By symmetry, we may suppose that $V_{\mathcal{F}} \subseteq V' \setminus X$. Thus, we only need to show that $|X| = 1$. Observe that, if we show that X is an independent set, then we may conclude that $|X| = 1$, since $\delta(G') \geq 4$. Thus, it suffices to show the following:

$$\text{for every edge } ab \in E', \text{ we have } |\{a, b\} \cap X| \leq 1. \tag{2}$$

Let ab be an edge of E'. Since $V_{\mathcal{F}} \subseteq V' \setminus X$, we may suppose that a and b belong to $V' \setminus V_{\mathcal{F}}$. We will show that there exist at least five edge-disjoint paths between $\{a, b\}$ and $V_{\mathcal{F}}$ which implies (2). To shorten notation, we write *edps* to refer to *edge-disjoint paths*. We distinguish the following cases.

Case 1: $a = x_i^e$ and $b = w_i^e$, for $i \notin \{3, 4\}$.

Note that every vertex x_i^e has three edps linking it to some vertex $v_F \in V_{\mathcal{F}}$. Two of them are parallel edges, and the other one contains a neighbor x_j^e (or a close neighbor x_j^f). Similarly, the vertex w_i^e has three additional edps linking it to $v_{F'}$, where F' is the face that shares the edge e with the face F.

We observe that analogous arguments as above show that there exists six edps between $V_{\mathcal{F}}$ and each of the following sets: $\{w_5^e, x_4^e\}$, $\{w_5^e, x_3^e\}$, and $\{w_i^e, y_i^e\}$, for $i \in \{1, \ldots, 7\}$.

Case 2: $a = w_i^e$ and $b = w_{i+1}^e$, for $i \notin \{4, 5\}$.

If $i < 3$ or $i > 5$, the vertex w_i^e has three edps: the first contains the edge $w_i^e x_i^e$, the second contains the edge $w_i^e y_i^e$, and the third contains the vertex w_{i-1}^e (or a vertex in $V(G)$). A similar argument shows that w_{i+1}^e has three additional edps. We show an example for the case $i = 2$ in Fig. 4 *a*). Observe that, in the case $i = 3$, w_3^e and w_4^e are not adjacent to a vertex

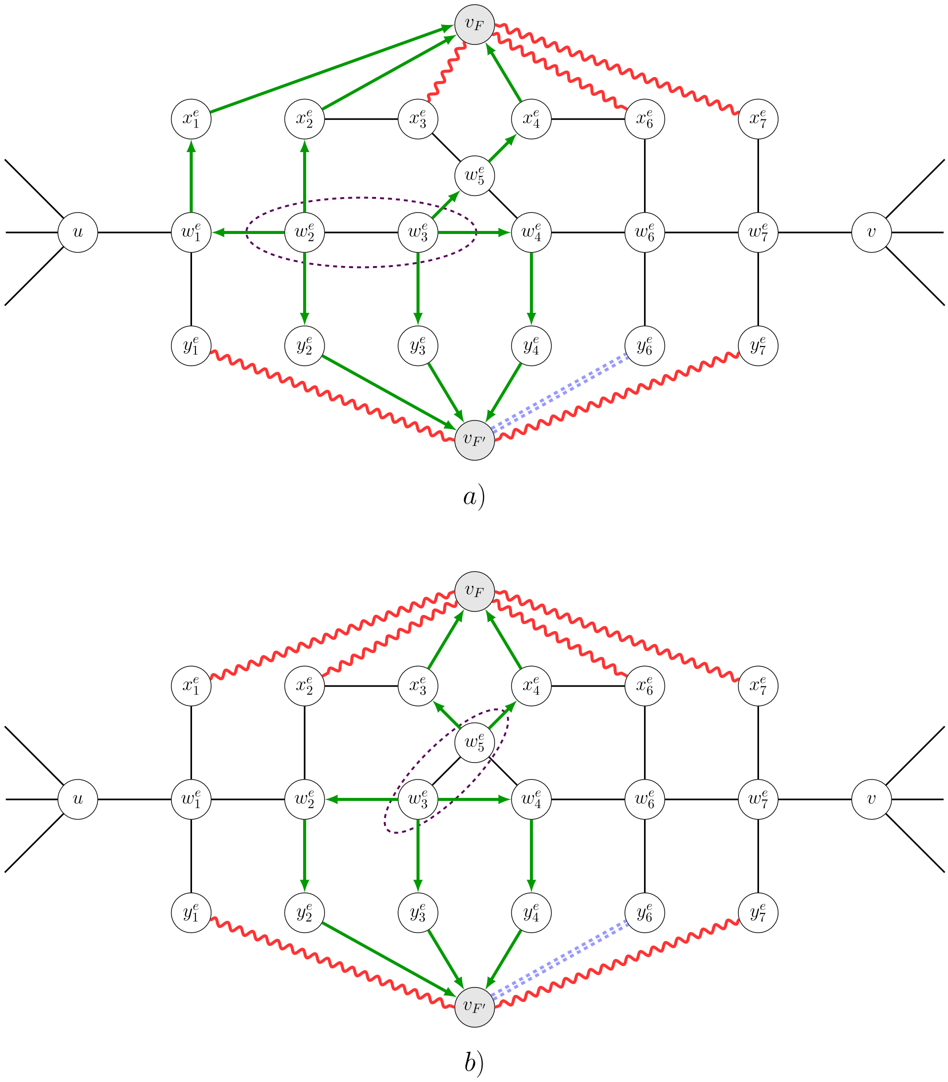

Fig. 4. a) Six edge-disjoint paths between $\{w^e_2, w^e_3\}$ and $V_{\mathcal{F}}$; and b) five edge-disjoint paths between $\{w^e_3, w^e_5\}$ and $V_{\mathcal{F}}$. The green arcs indicate the edges that belong to those paths.

x^e_i. In this case, we can use the paths $\langle w^e_3, w^e_5, x^e_3, v_F\rangle$ and $\langle w^e_4, w^e_5, x^e_4, v_F\rangle$ to obtain six edps. We observe that a similar argument applies to the case $a = w^e_4$ and $b = w^e_6$.

Case 3: $a = w^e_i$ and $b = u$, for $u \in V$.

In this case u has three edps that do not use the edge uw^e_i. Each one contains one of the edges incident to u (different from $w^e_i u$). Moreover, w^e_i has three additional edps: the first one contains x^e_i, the second one contains y^e_i, and the last one contains either w^e_{i-1} or w^e_{i+1} (depending on which vertex exists).

Case 4: $a = x^e_i$ and $b = x^e_{i+1}$ (or a and b are close neighbors).

The vertex x^e_i has three edps: two parallel edges linking it to a vertex $v_F \in V_{\mathcal{F}}$, and another path that contains the vertex w^e_i and links x^e_i to $v_{F'}$, where F' is the face that shares edge e with F. An analogous argument shows that x^e_{i+1} has three additional edps.

Case 5: $ab = w^e_3 w^e_5$ or $ab = w^e_4 w^e_5$.

Without loss of generality, suppose that $a = w^e_3$ and $b = w^e_5$. Then, we have two edps linking w^e_5 to a vertex $v_F \in V_{\mathcal{F}}$. One contains the vertex x^e_3 and the other contains the vertex x^e_4. Additionally, we have three edps linking w^e_3 to $v_{F'}$, where F' is the face that shares the edge e with the face F. Thus, in total we have five edps. We depict this case in Fig. 4 b). □

Next, we show the relation between a vertex cover of G and a dsc$_5$ in G'. Since every vertex in $V_{\mathcal{F}}$ has degree at least five, this set of vertices must be included in every dsc$_5$ of G'. A key observation to show the correctness of our reduction is the following. There exists a minimum dsc_5 of G' that, when restricted to H_e, induces a minimum dominating set of it that contains one of the ends of e, for every $e \in E$. To show that, the following result is important.

Lemma 2. *Let $e \in E$ and $D \subseteq V(H_e)$ be a set that dominates $\{w_i^e : 1 \le i \le 7\}$. Then, $|D| \ge 3$ and $|D \cap \{w_i^e : 1 \le i \le 7\}| \ge 2$.*

Proof. First, since D dominates w_1^e, w_5^e and w_7^e, it contains at least one vertex in each of the following sets: $\{u, w_1^e, w_2^e\}$, $\{w_3^e, w_4^e, w_5^e\}$ and $\{w_6^e, w_7^e, v\}$. As the previous sets are disjoint, we have that $|D| \ge 3$. As D dominates w_2^e and w_6^e, then D contains at least one vertex in the following sets: $\{w_1^e, w_2^e, w_3^e\}$ and $\{w_4^e, w_6^e, w_7^e\}$. As these sets are disjoint and do not contain u or v, we have that $|D \cap \{w_i^e : 1 \le i \le 7\}| \ge 2$. □

The following result shows that there is a minimum dsc_5 of G' that induces a minimum vertex cover of G.

Proposition 1. *Let $G = (V, E)$ be a 3-connected 4-regular plane graph, and let G' be the $\mathcal{D}_5$-reduction of G. Moreover, let D^* be a minimum dsc_5 of G', and let C^* be a minimum vertex cover of G. Then,*

$$|D^*| = |C^*| + |\mathcal{F}| + 2|E|,$$

where $\mathcal{F}$ is the set of faces of G.

Proof. First, since every vertex v_F has degree greater than four, we have that $V_{\mathcal{F}} \subseteq D^*$. Furthermore, if x_i^e (resp. y_i^e) belongs to D^*, we can exchange x_i^e (resp. y_i^e) with w_i^e to obtain another minimum dsc_5 of G'. This follows from the fact that $V_{\mathcal{F}}$ dominates every vertex x_i^e and y_i^e, and the vertex w_i^e is the unique neighbor of x_i^e (or y_i^e) that is not already dominated by $V_{\mathcal{F}}$. Thus, we may suppose that

$$D^* \setminus V_{\mathcal{F}} \subseteq V \cup \{w_i^e : 1 \le i \le 7,\ e \in E\}. \tag{3}$$

In what follows, we show that there exists an optimal solution D^* that induces a vertex cover of G, such that

$$D^* \cap \{u, v\} \ne \emptyset, \text{ for each } uv \in E. \tag{4}$$

Let $e = uv \in E$, and consider H_e. Let $D_e^* = D^* \cap V(H_e)$. Since the neighborhood of each w_i^e is contained in $V(H_e)$, (3) implies that D_e^* is a dominating set of $\{w_i^e : 1 \le i \le 7\}$. Thus, by Lemma 2, we have that $|D_e^*| \ge 3$. In the case that $D_e^* \cap \{u, v\} = \emptyset$, we can exchange D_e^* with $\{u, w_3^e, w_7^e\}$ since this set also dominates H_e. Therefore, we may suppose that (4) holds.

Let $C = D^* \cap V$. Finally, by Lemma 2, $D_e^* = D^* \cap V(H_e)$ contains at least two vertices in $\{w_i^e : 1 \le i \le 7\}$, for each edge $e \in E$. That is, $|D^* \cap \{w_i^e : 1 \le i \le 7\}| \ge 2$, for each $e \in E$. The previous arguments imply that

$$\begin{aligned} |D^*| &\ge |C| + |\mathcal{F}| + 2|E| \\ &\ge |C^*| + |\mathcal{F}| + 2|E|, \end{aligned}$$

where C^* is a minimum vertex cover of G. To show the reverse inequality, we construct a set D that is a dsc_5 of G', as follows.

a) first, let $D = V_{\mathcal{F}} \cup C^*$,
b) then, for each edge $e = uv \in E$, add $\{w_3^e, w_7^e\}$ to D if $u \in C^*$, otherwise add $\{w_4^e, w_1^e\}$.

Since $|D^*| \le |D| = |C^*| + |\mathcal{F}| + 2|E|$, the claim follows. □

Since G' is nearly 5-edge-connected with face-degree at most 5, by Proposition 1 and Corollary 1, we have the following.

Proposition 2. DSC_5 *is* NP-*hard on nearly* 5-*edge-connected plane graphs whose face-degree is at most* 5.

Combining Theorem 1 and Proposition 2, we obtain the following result.

Theorem 3. $\textsc{MinS}_3$ *is* NP-*hard on planar graphs G with $\Delta(G) \le 5$.*

We conclude this section with the following remark. Our idea of linking close neighbors is used to control the face-degree of G'. To see this, just consider removing a wavy edge in Fig. 3. We observe that this idea can be used in the construction given by Kobayashi [12] to improve his result for DSC_4. In fact, in his construction, the boundary of every face inside the gadget (that replaces each edge of G) contains at most six edges. A boundary with eight edges appears when we consider a face that contains v_F and two close neighbors. Since in the original construction the vertices x_i^e are linked to v_F by at least two parallel edges, we can exchange one of these edges to link a pair of close neighbors. This situation is depicted in Fig. 5. We leave to the reader to check that, after this modification, the reduction of Kobayashi [12] remains valid, and conclude that the following holds.

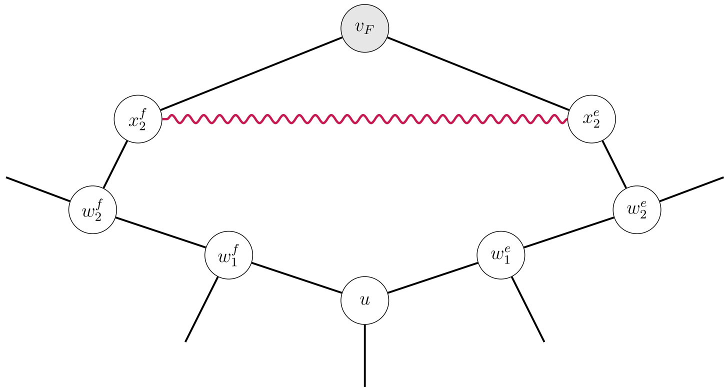

Fig. 5. Linking close neighbors.

Theorem 4. MinS_2 *is* NP-*hard on planar graphs* G *with* $\Delta(G) \leq 7$.

4. Improved hardness result for MinS_t, $t \geq 4$

As in the previous section, we first improve the hardness results for DSC_k, $k \geq 6$, showing that it remains NP-hard on plane graphs whose face-degree is at most 4. Similarly, we reduce VCover on planar graphs to DSC_k, $k \geq 6$, on planar nearly k-edge-connected graphs.

Regarding the gadget of the previous section, the main changes in this case are the following. We create copies of x_i^e and y_i^e in order to satisfy the degree lower bound for each vertex w_i^e. Moreover, we contract every pair of close vertices and add edges to forbid the appearance of faces with face-degree greater than 5, in particular, when we consider gadgets that correspond to adjacent edges in G. To achieve this, it is crucial that $\delta(G') \geq k-1 \geq 5$. In what follows, we formalize our construction.

As in the previous section, $G = (V, E)$ denotes a 3-connected $(k-1)$-regular plane graph, and $\mathcal{F}$ denotes the set of faces of G. For $k \geq 6$, the $\mathcal{D}_k$*-reduction* of G is the planar graph $G' = (V', E')$ obtained as follows. First, we create a vertex v_F for each face $F \in \mathcal{F}$. Then, we replace each edge $e = uv \in E$ by H_e. Let $\ell := \lfloor (k-3)/2 \rfloor$ and $c := \lceil (k-3)/2 \rceil$. For each $e \in E$, we add the following set of vertices:

$$
\begin{aligned}
X_A &= \left\{x_{i,j}^e : i \in \{1,7\},\ j \in \{1,\dots,c-1\}\right\},\\
X_B &= \left\{x_{i,j}^e : i \in \{2,6\},\ j \in \{1,2,\dots,c\}\right\},\\
X_C &= \left\{x_{5,j}^e : j \in \{1,2,\dots,k-3\}\right\},\\
Y_A &= \left\{y_{i,j}^e : i \in \{1,7\},\ j \in \{1,\dots,\ell-1\}\right\},\\
Y_B &= \left\{y_{i,j}^e : i \in \{2,6\},\ j \in \{1,\dots,\ell\}\right\},\\
Y_C &= \left\{y_{i,j}^e : i \in \{3,4\},\ j \in \{1,\dots,k-5\}\right\}.
\end{aligned}
$$

We observe that when $k = 6$, we have $Y_A = \emptyset$. In what follows, we denote by C_F the cycle that is the boundary of the face $F \in \mathcal{F}$ in G. For each $F \in \mathcal{F}$ and each $u \in V(C_F)$, we add a vertex z_F^u to G'. Let $Z = \{z_F^u : F \in \mathcal{F},\ u \in V(C_F)\}$. Now, we add edges in the following way:

(i) we link w_i^e to the vertices $x_{i,j}^e$ and $y_{i,j}^e$, for $i \in \{1,7\}$ and $j \in \{1,\dots,c-1\}$;
(ii) we link w_i^e to the vertices $x_{i,j}^e$ and $y_{i,j}^e$, for $i \in \{2,6\}$ and $j \in \{1,\dots,c\}$;
(iii) we link w_5^e and each vertex in X_C;
(iv) we link w_i^e and $y_{i,j}^e$, for $i \in \{3,4\}$ and $j \in \{1,\dots,k-5\}$;
(v) we link w_3^e (resp. w_4^e) and $x_{5,1}^e$ (resp. $x_{5,k-3}^e$);
(vi) we link $x_{1,c-1}^e$ and $x_{2,1}^e$; $x_{2,c}^e$ and $x_{5,1}^e$; $x_{5,k-3}^e$ and $x_{6,1}^e$; $x_{6,c}^e$ and $x_{7,1}^e$;
(vii) we link $y_{1,\ell-1}^e$ and $y_{2,1}^e$; $y_{2,\ell}^e$ and $y_{3,1}^e$; $y_{3,k-5}^e$ and $y_{4,1}^e$; $y_{4,k-5}^e$ and $y_{6,1}^e$; $y_{6,\ell}^e$ and $y_{7,1}^e$.

Next, we describe how we link the vertices of Z in G'. Let F and F' be the faces whose boundaries contain $e = uv$. We link both z_F^u and z_F^v to v_F by $k-3$ edges. On the other hand, if $k = 6$, we link both $z_{F'}^u$ and $z_{F'}^v$ to $v_{F'}$ by $k-4$ edges, otherwise we link them by $k-3$ edges to $v_{F'}$. Without loss of generality, suppose that w_1^e (resp. w_7^e) is adjacent to u (resp. v) in H_e. In case $k = 6$, we link $z_{F'}^u$ (resp. $z_{F'}^v$) to $y_{2,1}^e$ (resp. $y_{6,1}^e$). Finally, we link the vertices z_F^u and $z_{F'}^u$ (resp. z_F^v and $z_{F'}^v$) to w_1^e (resp. w_7^e).

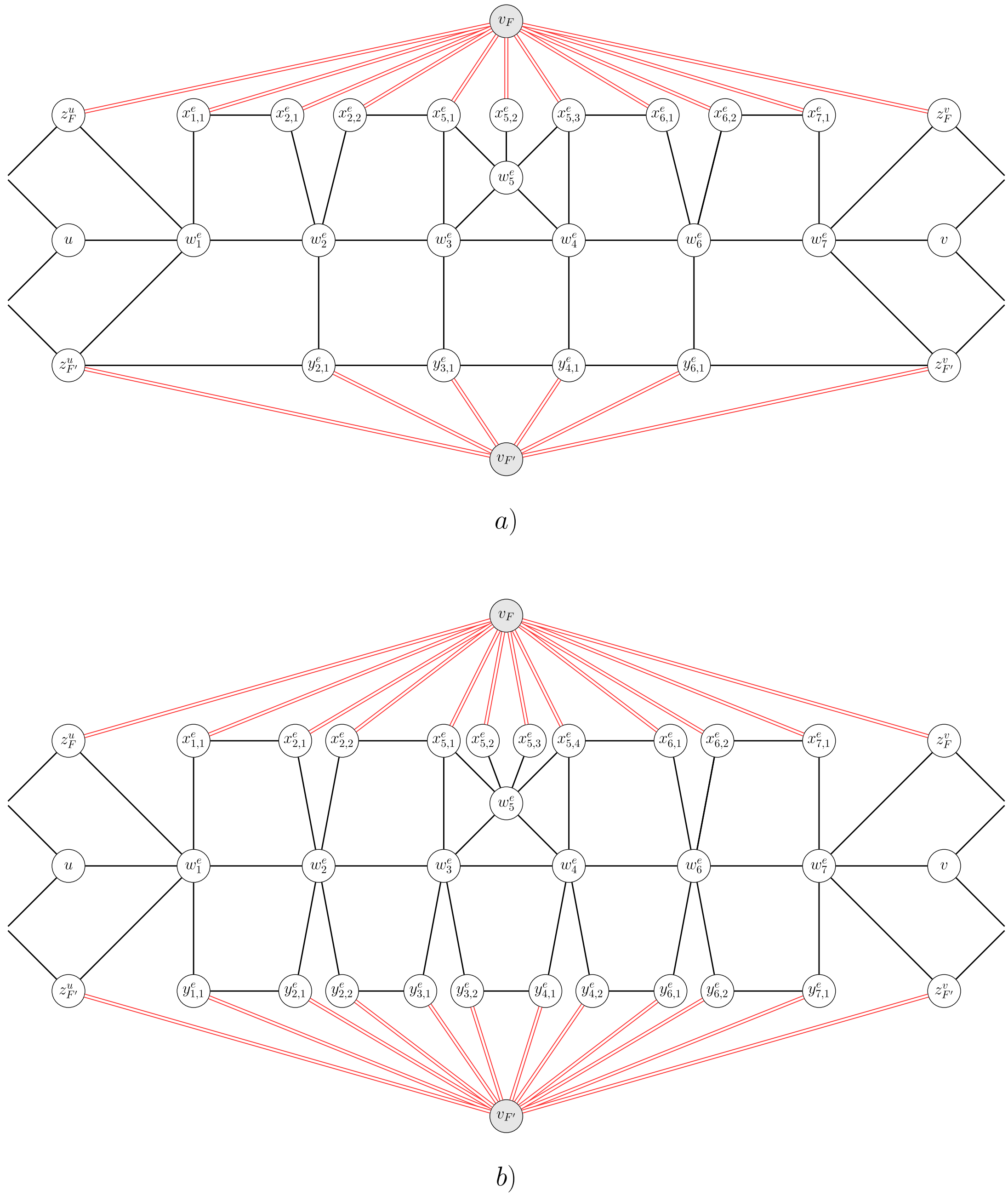

Fig. 6. Gadgets used to obtain the $\mathcal{D}_k$-reduction G' for $a)$ $k=6$; and $b)$ $k=7$.

To conclude the construction of G', we link every $x^e_{i,j}$ (resp. $y^e_{i,j}$) to v_F (resp. $v_{F'}$) by (multiple) edges such that $x^e_{i,j}$ (resp. $y^e_{i,j}$) achieves degree $k-1$ in G'. We show an example for $k=6$ and $k=7$ in Fig. 6. We now show that G' is nearly k-edge connected.

Lemma 3. *Let G be a 3-connected $(k-1)$-regular plane graph, $k \geq 6$, and let G' be the $\mathcal{D}_k$-reduction of G. Then, G' is nearly k-edge-connected.*

Proof. Let $G=(V,E)$ and $G'=(V',E')$. By the way G' is constructed, we have that $\delta(G') \geq k-1$. Let us show that, if $\delta_{G'}(X) \leq k-1$, then $|X|=1$ or $|V' \setminus X|=1$. Let $X \subseteq V'$ be one such set. First, we show that either $V_{\mathcal{F}} \subseteq X$ or $V_{\mathcal{F}} \subseteq V' \setminus X$. We recall that, we write *edp* (*edps*) to refer to edge-disjoint path(s). Let F_1 and F_2 be two faces in $\mathcal{F}$ that share an edge

$e = uv \in E$. Without loss of generality suppose that the vertices $x^e_{i,j}$ (resp. $y^e_{i,j}$) are adjacent to v_{F_1} (resp. v_{F_2}). We now prove that there are at least k edps between v_{F_1} and v_{F_2} in G'. By construction, we have

i) $\ell - 1$ edps, each of which contains $x^e_{i,j}$, w_i and $y^e_{i,j}$, for $i \in \{1, 7\}$;
ii) ℓ edps that contain $x^e_{i,j}$, w_i and $y^e_{i,j}$, for $i \in \{2, 6\}$;
iii) one edp that contains $z^u_{F_1}$ and $z^u_{F_2}$ (resp. $z^v_{F_1}$ and $z^v_{F_2}$);
iv) the edps $\langle v_{F_1}, x^e_{5,1}, w^e_3, y^e_{3,1}, v_{F_2}\rangle$ and $\langle v_{F_1}, x^e_{5,k-3}, w^e_4, y^e_{4,1}, v_{F_2}\rangle$.

Hence, there are at least

$$\begin{aligned} 2(\ell-1)+2\ell+2+2 &= 2\lfloor (k-5)/2\rfloor + 2\lfloor (k-3)/2\rfloor + 4 \\ &\geq (k-6)+(k-4)+4 \\ &= 2k-6 \end{aligned}$$

edps between F_1 and F_2 in G'. Since $k \geq 6$, we have that $2k - 6 \geq k$. Thus, v_{F_1} and v_{F_2} belong either to X or to $V' \setminus X$. Applying the previous argument on each pair of adjacent faces, we have that $V_{\mathcal{F}} \subseteq X$ or $V_{\mathcal{F}} \subseteq V' \setminus X$.

Without loss of generality, we may suppose that $V_{\mathcal{F}} \subseteq V' \setminus X$. Thus, we only need to show that $|X| = 1$. By the same arguments as those given in the proof of Lemma 1, it suffices to show that, for every edge $ab \in E'$, $a, b \in V' \setminus V_{\mathcal{F}}$, there exist at least k edps between $\{a, b\}$ and $V_{\mathcal{F}}$. We distinguish the following cases.

Case 1: $a = w^e_i$ and $b = w^e_{i+1}$, for $i \notin \{3, 4\}$.

First, observe that each w^e_i is adjacent to $k-3$ vertices that are different from its neighbors in H_e. We can obtain $k-3$ edps that contain each one of these vertices. Moreover, we can obtain $k-3$ additional edps from w^e_{i+1} by similar arguments. Thus, we have at least $2k-6$ edps between $\{w^e_i, w^e_{i+1}\}$ and $V_{\mathcal{F}}$. As $2k-6 \geq k$, for $k \geq 6$, the claim follows. We observe that the same argument applies to the edge $w^e_4 w^e_6$.

Case 2: $a = w^e_i$ and $b = x^e_{i,j}$.

First, note that there are at least $k-3$ edps between $x^e_{i,j}$ and $V_{\mathcal{F}}$. These edps are edges linking $x^e_{i,j}$ to $V_{\mathcal{F}}$ or a path (of length two) that contains a neighbor $x^e_{i',j'}$. Moreover, every vertex w^e_i is adjacent to at least three vertices different from $x^e_{i,j}$. Thus, we can obtain at least k edps. By analogous arguments, we can show that there exist k edps between $V_{\mathcal{F}}$ and the set $\{w^e_i, y^e_{i,j}\}$. We show an example of this case for $k = 6$ in Fig. 7 *a*). In this figure, we depict six edps between $\{w^e_6, x^e_{6,1}\}$ and $V_{\mathcal{F}}$.

Case 3: $a = x^e_{i,j}$ and $b = x^e_{i',j'}$ or $a = z^u_F$ and $b = y^e_{2,1}, y^e_{6,1}$.

In this case, both a and b have at least $k-4$ edges linking them to a vertex in $V_{\mathcal{F}}$. Moreover, we have two additional edps, one that contains w^e_i and other that contains $w^e_{i'}$. Thus, we have $2k-6 \geq k$ edps. Analogous arguments show that there exist k edps between $V_{\mathcal{F}}$ and two adjacent vertices $y^e_{i,j}$ and $y^e_{i',j'}$.

Case 4: $a = w^e_i$ and $b = z^u_F$.

First, the vertex z^u_F has $k-3$ edges linking it to $v_F \in V_{\mathcal{F}}$. Moreover, we can obtain three additional edps starting from w^e_i, and containing either $x^e_{i,1}$, w^e_{i+1} or w^e_{i-1} (depending on which one exists), or $z^u_{F'}$, where F' is the face that shares the edge e with the face F.

Case 5: $a = u \in V$ and $b = w^e_i$.

In this case, we can obtain $k-2$ edps starting from u, each containing a neighbor of u different from w^e_i. Moreover, we can obtain at least two additional edps that start from w^e_i and contain a neighbor of w^e_i different from u.

Case 6: $a = w^e_3$, w^e_4 and $b = w^e_5$.

First, let F and F' be the faces that share the edge e in G. Moreover, suppose that the vertices $x^e_{i,j}$ (resp. $y^e_{i,j}$) are adjacent to v_F (resp. $v_{F'}$) in G'. Then, we have $k-3$ edps of the form $\langle w^e_5, x^e_{5,j}, v_F\rangle$, for $j \in \{1, \ldots, k-3\}$. Without loss of generality, suppose that $a = w^e_3$. Then, we have the following three additional edps: $\langle w^e_3, w^e_2, y^e_{2,1}, v_{F'}\rangle$, $\langle w^e_3, w^e_4, y^e_{4,1}, v_{F'}\rangle$ and $\langle w^e_3, y^e_{3,1}, v_{F'}\rangle$.

Case 7: $a = w^e_3$ and $b = w^e_4$.

Let F and F' be the faces that share the edge e in G. Without loss of generality, suppose that the vertices $x^e_{i,j}$ (resp. $y^e_{i,j}$) are adjacent to v_F (resp. $v_{F'}$) in G'. First, we show that there exist $k-3$ edps starting in w^e_3. We have $k-5$ edps of the form $\langle w^e_3, y^e_{3,j}, v_{F'}\rangle$, for $j \in \{1, \ldots, k-5\}$. The additional two edps are $\langle w^e_3, x^e_{5,1}, v_F\rangle$ and $\langle w^e_3, w^e_2, y^e_{2,1}, v_{F'}\rangle$. By symmetric arguments, we have $k-3$ additional edps starting in w^e_4. Thus, we have $2k-6 \geq k$ edps. We show an example of this case for $k = 6$ in Fig. 7 *b*). $\square$

To conclude our reduction, we show that there is an optimal dsc_k of G' that induces a minimum vertex cover of G.

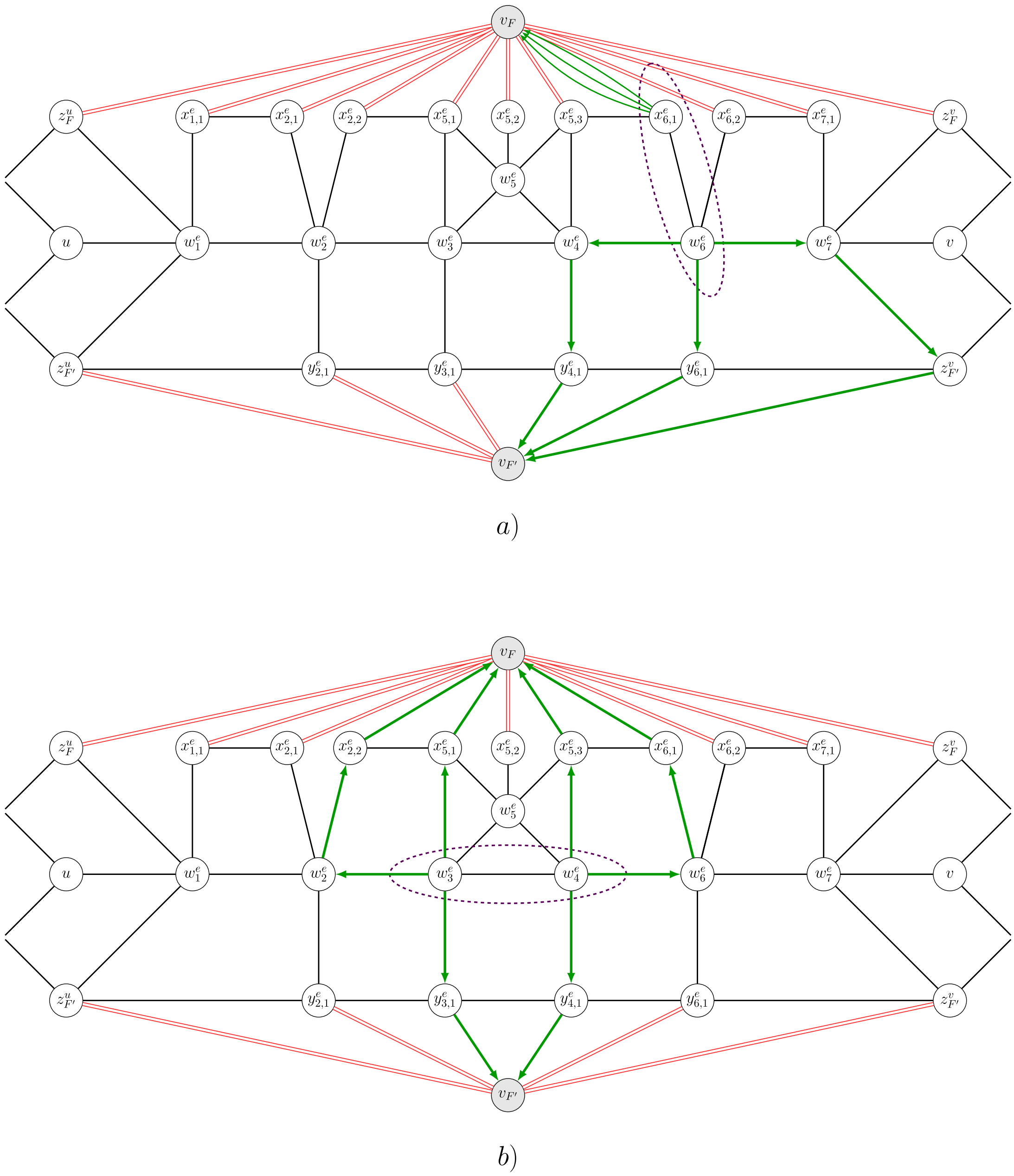

Fig. 7. *a*) Six edge-disjoint paths between $\{w_6^e, x_{6,1}^e\}$ and $V_{\mathcal{F}}$; and *b*) six edge-disjoint paths between $\{w_4^e, w_6^e\}$ and $V_{\mathcal{F}}$. The green arcs indicate the edges that belong to those paths.

Proposition 3. *Let $G = (V, E)$ be a 3-connected $(k-1)$-regular plane graph, and let G' be the $\mathcal{D}_k$-reduction of G. Moreover, let D^* be a minimum dsc$_k$ of G', and let C^* be a minimum vertex cover of G. Then,*

$$|D^*| = |C^*| + |\mathcal{F}| + 2|E|,$$

where $\mathcal{F}$ is the set of faces in G.

Proof. First, we show that every vertex $v_F \in V_{\mathcal{F}}$ has degree at least k. Note that, the cycle C_F has length at least two. Moreover, we have $k-3$ vertices in X_C and $2k-10$ vertices in Y_C, and at least two vertices in Z adjacent to v_F. Then, the degree of v_F is at least

$$2 \cdot \min\{k-3, 2k-10\} + 2 \geq k, \text{ for } k \geq 6.$$

Thus, $V_{\mathcal{F}} \subseteq D^*$. In what follows, we show that

$$D^* \setminus V_{\mathcal{F}} \subseteq V \cup \{w_i^e : 1 \le i \le 7,\ e \in E\}. \tag{5}$$

Observe that, if $y_{i,j}^e$ belongs to D^*, we can exchange this vertex with w_i^e. This follows from the fact that $V_{\mathcal{F}}$ already dominates every $y_{i,j}^e$, and w_i^e is the unique vertex in the closed neighborhood of $y_{i,j}^e$ that is not dominated by $V_{\mathcal{F}}$. The same argument applies for the vertices $x_{i,j}^e$, for $i \in \{1,2,6,7\}$, and $x_{5,j}^e$, for $1 < j < k-3$. Also observe that, if $x_{5,1}^e \in D^*$ (resp. $x_{5,k-3}^e \in D^*$), we can exchange this vertex with w_3^e (resp. w_4^e). This holds because w_3^e (resp. w_4^e) also dominates w_5^e (and itself), which are the unique vertices in the closed neighborhood of $x_{5,1}^e$ (resp. $x_{5,k-3}^e$) that are not dominated by $V_{\mathcal{F}}$. By an analogous argument, if a vertex $z_F^u \in D^*$, we can exchange this vertex with $u \in V$. Thus, we may assume that (5) holds. Finally, using the same arguments as those mentioned in the proof of Proposition 1, we can conclude the proof. □

Since G' is nearly k-edge-connected with face-degree at most 4, by Proposition 3 and Corollary 1, we have the following.

Proposition 4. *For $k \ge 6$, DSC$_k$ is NP-hard on nearly k-edge-connected plane graphs whose face-degree is at most 4.*

The previous proposition combined with Theorem 1 implies the following result.

Theorem 5. *For $t \ge 4$, MinS$_t$ is NP-hard on planar graphs G with $\Delta(G) \le 4$.*

5. Concluding remarks

The computational complexity results for MinS$_t$ on graphs of bounded degree were improved by the insightful proofs shown by Kobayashi in 2018. The further improvements we have shown in this paper (indicated by the shaded cells in Table 2) were inspired by the results obtained by Kobayashi. It remains now a few open questions on bounded-degree graphs, as one can see in this table.

We observe that the polynomial-time algorithms that exist for MinS$_t$ on bounded-degree graphs rely on the decomposition of the input graph into a family of basic subgraphs. In these cases, a minimum t-spanner of a basic subgraph H has size either $|V(H)|-1$ or $|V(H)|$. Thus, the size of a minimum t-spanner is bounded by the size of a spanning tree of H plus a constant. For MinS$_3$, we have carried out some computational experiments that suggest that this is not true for graphs with bounded-degree 4. Therefore, we conjecture that MinS$_3$ is NP-hard on graphs with maximum degree at most 4. We consider this a challenging and interesting question to answer.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We thank the referees for the careful reading and valuable comments and suggestions. This research has been partially supported by FAPESP - São Paulo Research Foundation (Proc. 2015/11937-9). R. Gómez is supported by FAPESP (Proc. 2019/14471-1); F. K. Miyazawa is supported by FAPESP (Proc. 2016/01860-1) and CNPq (Proc. 314366/2018-0); Y. Wakabayashi is supported by CNPq (Proc. 311892/2021-3 and 423833/2018-9).

References

[1] R. Ahmed, G. Bodwin, F.D. Sahneh, K. Hamm, M.J.L. Jebelli, S. Kobourov, R. Spence, Graph spanners: a tutorial review, Comput. Sci. Rev. 37 (2020) 100253.
[2] B. Awerbuch, Complexity of network synchronization, J. ACM 32 (4) (1985) 804–823.
[3] S. Baswana, S. Sen, Approximate distance oracles for unweighted graphs in expected $O(n^2)$ time, ACM Trans. Algorithms 2 (4) (2006) 557–577.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008.
[5] A. Brandstädt, V. Chepoi, F. Dragan, Distance approximating trees for chordal and dually chordal graphs, J. Algorithms 30 (1) (1999) 166–184.
[6] A. Brandstädt, F. Dragan, H. Le, V. Le, R. Uehara, Tree spanners for bipartite graphs and probe interval graphs, Algorithmica 47 (1) (2007) 27–51.
[7] L. Cai, NP-completeness of minimum spanner problems, Discrete Appl. Math. 48 (2) (1994) 187–194.
[8] L. Cai, D. Corneil, Tree spanners, SIAM J. Discrete Math. 8 (3) (1995) 359–387.
[9] L. Cai, M. Keil, Spanners in graphs of bounded degree, Networks 24 (4) (1994) 233–249.
[10] M. Garey, D. Johnson, Computers and intractability, in: A Guide to the Theory of NP-Completeness, in: A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif, 1979.
[11] R. Gómez, F.K. Miyazawa, Y. Wakabayashi, Minimum t-spanners on subcubic graphs, in: Petra Mutzel, Md. Saidur Rahman, Slamin (Eds.), WALCOM: Algorithms and Computation, in: Lecture Notes in Computer Science, vol. 13174, Springer, 2022, pp. 365–380.
[12] Y. Kobayashi, NP-hardness and fixed-parameter tractability of the minimum spanner problem, Theor. Comput. Sci. 746 (2018) 88–97.

[13] M. Madanlal, G. Venkatesan, C. Pandu Rangan, Tree 3-spanners on interval, permutation and regular bipartite graphs, Inf. Process. Lett. 59 (2) (1996) 97–102.
[14] D. Peleg, A. Schäffer, Graph spanners, J. Graph Theory 13 (1) (1989) 99–116.
[15] D. Peleg, J. Ullman, An optimal synchronizer for the hypercube, SIAM J. Comput. 18 (4) (1989) 740–747.
[16] D. Peleg, E. Upfal, A trade-off between space and efficiency for routing tables, J. ACM 36 (3) (1989) 510–530.
[17] M. Thorup, U. Zwick, Approximate distance oracles, J. ACM 52 (1) (2005) 1–24.
[18] R. Uehara, NP-complete problems on a 3-connected cubic planar graph and their applications, Technical Report TWCU-M-0004, Tokyo Woman's Christian University, October 1996.
[19] G. Venkatesan, U. Rotics, M. Madanlal, J. Makowsky, C. Pandu Rangan, Restrictions of minimum spanner problems, Inf. Comput. 136 (2) (1997) 143–164.
[20] W. Wang, D. Balkcom, A. Chakrabarti, A fast online spanner for roadmap construction, Int. J. Robot. Res. 34 (11) (2015) 1418–1432.