# A note on a min-max conjecture of Woodall

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**Abstract**

In 1978 Woodall [6], conjectured the following: in a planar digraph the size of a shortest cycle is equal to the maximum cardinality of a collection of disjoint tranversals of cycles. We prove that this conjecture is true when the digraph is series-parallel. In fact, we prove a stronger weighted version that gives the latter result as a corollary.

**Keywords.** Series-parallel digraph, girth, dicut, dijoin, min-max theorem.

## 1 Introduction

Let $D = (V, A)$ be a digraph. A *cycle* in $D$ is a closed (directed) walk in which the origin and all internal vertices are different. A subset $B$ of arcs is a *dicut* of $D$ if there exists a subset $X$ of $V$ such that $B$ consists exactly of those arcs of $D$ leaving $X$ and $D$ has no arc entering $X$. A *transversal of cycles* is a subset of arcs which intersects every cycle in $D$. For simplicity, here *transversal* means *transversal of cycles*. A *dijoin* in $D$ is a subset of arcs which intersects every dicut. Note that for plane digraphs there is a correspondence between dicuts (dijoins) and cycles (transversals) in the corresponding dual digraphs.

The study of dijoins and dicuts has always been fruitful for the development of graph theory and combinatorial optimization. Perhaps the most famous result relating these concepts is the celebrated min-max theorem proved by Lucchesi and Younger [3, 2]: *in any digraph, the cardinality of a maximum collection of disjoint dicuts is equal to the size of a minimum dijoin.*

We are concerned with the related problem that arises in this context: the one obtained by interchanging the roles of dijoins and dicuts. Clearly, the cardinality of a maximum collection of disjoint dijoins is at most the size of a minimum dicut. In

1978 Woodall ([6], pp. 629) conjectured that these extremal quantities are always equal for any digraph.

**Conjecture 1.1 (Woodall, 1978).** *In any digraph the size of a minimum dicut is equal to the cardinality of a maximum collection of disjoint dijoins.*

This conjecture is still unsettled. The best known result is due to Feofiloff & Younger [1], and Schrijver [5]. They proved independently that this conjecture holds for source-sink connected digraphs (these are the digraphs for which there exists a directed path from each source to each sink).

If we restrict ourselves to planar digraphs, then by duality we have the following equivalent conjecture.

**Conjecture 1.2 (Woodall, 1978).** *In a planar digraph the size of a shortest cycle is equal to the cardinality of a maximum collection of disjoint transversals.*

It is known that the min-max relation stated in the above conjecture does not hold for nonplanar digraphs. We learned from D.H. Younger a counterexample (a tournament with 15 vertices) given by C. Thomassen.

We show that this conjecture is true for *series-parallel* digraphs. We say that a digraph is series-parallel if its underlying graph does not contain a subdivision of $K_4$. It is easy to see that a series-parallel digraph is planar and that its dual digraph is also series-parallel. Thus, the above conjectures are equivalent for series-parallel digraphs and it suffices to prove Conjecture 1.2. In fact, we prove a stronger result (a more general weighted version), defined as follows.

Let $(D, l)$ denote a pair consisting of a digraph $D = (V, A)$ and a nonnegative-integer-valued function $l$ defined on the arcs of $D$. We denote by $\mathrm{girth}(D, l)$ the length of a shortest cycle (with respect to the length function $l$) in $(D, l)$, and by $\mathrm{transv}(D, l)$ the cardinality of a maximum collection of transversals such that each arc $a$ is contained in at most $l(a)$ members of this collection (such a collection is called *$l$-disjoint*).

Clearly, $\mathrm{transv}(D, l) \leq \mathrm{girth}(D, l)$. We prove that if $D$ is series-parallel then $\mathrm{girth}(D, l) = \mathrm{transv}(D, l)$. We note that this min-max equality does not hold for planar digraphs (see [4]). As a last remark we also mention that the weighted version of Conjecture 1.1 —in the same sense as above— is valid for source-sink connected digraphs (this was proved by Feofiloff & Younger [1], and Schrijver [5]).

To simplify the notation, if $(x, y)$ is an arc, we denote it by $xy$.

## 2 The main result

**Theorem 2.1.** *Let $D$ be a series-parallel digraph and $l$ a nonnegative-integer-valued length function defined on the arcs of $D$. Suppose $D$ has at least one cycle. Then $girth(D, l) = transv(D, l)$.*

*Proof.* Clearly $\text{transv}(D,l) \le \text{girth}(D,l)$. Thus we need only prove the reverse inequality. And in fact, it suffices to prove it only for strongly connected digraphs without cut vertices. To see that, suppose that $D$ has $p \ge 2$ strong components (maximal strongly connected subgraphs), say $D_1, D_2, \ldots, D_p$. Clearly, each cycle of $D$ lies in one of the components $D_i$. Suppose that $\text{girth}(D,l) = k$. Thus, if $\text{transv}(D_i,l_i) \ge \text{girth}(D_i,l_i)$ for $i = 1,\ldots,p$, where $l_i$ is the restriction of $l$ to $D_i$, then we can take in each $D_i$ a collection of $k$ $l$-disjoint transversals, and combine these collections properly to obtain a collection of $k$ $l$-disjoint transversals of $D$. Now we proceed to show that $\text{girth}(D,l) \le \text{transv}(D,l)$. This will be done by induction on the number of arcs of $D$.

Let $D = (V,A)$. In what follows we shall always reduce the pair $(D,l)$ to a pair $(D',l')$ such that $\text{girth}(D',l') = \text{girth}(D,l)$.

If $D$ has only two vertices the result is immediate. So suppose that $D$ has at least three vertices and $\text{girth}(D,l) > 0$, since otherwise the result is immediate.

If $a = xy$ and $b = yz$ are two arcs in series (that is, $y$ has degree 2, and $x \ne z$), then define a new digraph $D' := D - y + f$, where $f := xz$ is a new arc. Let $l'$ be a length function defined on the arcs of $D'$ as follows.

$$l'(e) := \begin{cases} l(a) + l(b), & \text{if } e = f; \\ l(e), & \text{otherwise.} \end{cases}$$

Note that $\text{girth}(D',l') = \text{girth}(D,l)$. Moreover, $\text{transv}(D',l') \le \text{transv}(D,l)$. In fact, given an $l'$-disjoint collection $F'_1,\ldots,F'_p$ of transversals of $D'$ we can obtain from this an $l$-disjoint collection $F_1,\ldots,F_p$ of transversals of $D$ as follows. For each $F'_i$ that contains $f$ we replace $f$ by $a = xy$ or $b = yz$ in such a way that $a$ does not appear in more than $l(a)$ sets and $b$ does not appear in more than $l(b)$ sets. It is easy to see that each member of the resulting collection is a transversal of $D$ and that this collection is $l$-disjoint. By the induction hypothesis, $\text{girth}(D',l') = \text{transv}(D',l')$. Thus,

$$\text{girth}(D,l) = \text{girth}(D',l') = \text{transv}(D',l') \le \text{transv}(D,l),$$

and the result follows.

If $a = xy$ and $b = xy$ are two parallel arcs with $l(a) \le l(b)$, then take $D' := D - b$ with length function $l'$ on its arcs defined as $l'(e) := l(e)$ for $e \in A - \{b\}$. Note that $\text{girth}(D',l') = \text{girth}(D,l)$. Furthermore, $\text{transv}(D',l') \le \text{transv}(D,l)$. In fact, given an $l'$-disjoint collection $F'_1,\ldots,F'_p$ of transversals of $D'$ we can obtain from this an $l$-disjoint collection $F_1,\ldots,F_p$ of transversals of $D$ by taking $F_i := F'_i \cup \{b\}$, if $F'_i$ contains $a$; and $F_i := F'_i$, otherwise. Using the induction hypothesis and the facts previously mentioned, we have

$$\text{girth}(D,l) = \text{girth}(D',l') = \text{transv}(D',l') \le \text{transv}(D,l),$$

and the result follows.

So suppose that $D$ contains neither arcs in series nor parallel arcs, and let $k := \text{girth}(D,l)$.

To ease our presentation we shall assume that $(D,l)$ is *critical*, that is, every arc $a$ with positive length is contained in some shortest cycle. If this is not the case for some arc $a = xy$ then we can decrease $l(a)$ conveniently (define $l''(a) = \max\{0, k-\alpha\}$ where $\alpha$ is the length of a shortest path which goes from $y$ to $x$), and repeat this operation for every arc until we obtain a critical digraph $(D,l'')$ for some $l'' \leq l$. Thus, it suffices to show that there is an $l''$-disjoint collection of $k$ transversals of $D$, as this collection will also be $l$-disjoint.

So, we assume that $(D,l)$ is critical. In particular, $l(a) \leq k$ for every arc $a$ of $D$.

It is well known that every 2-connected series-parallel graph has a vertex with exactly two neighbours. Let $y$ be a vertex in $D$ with exactly two neighbours, say $x$ and $z$. Because $D$ is strongly connected with no series or parallel arcs, we have two possibilities.

**Case 1.** $xy, yz, zy \in A$ and $yx \notin A$. (The case $yx, yz, zy \in A$ and $xy \notin A$ can be treated similarly.)

Consider the digraph $D' := D - zy$ and set $l'(e) := l(e)$ for $e \in A - \{zy\}$. Since $D$ is critical, it follows that $\text{girth}(D,l) = \text{girth}(D',l') = l(yz) + l(zy) = k$, where the last equality follows from the fact that $\{yz, zy\}$ is the only cycle in $D$ containing the arc $zy$.

By the induction hypothesis, there exists an $l'$-disjoint collection $\mathcal{F} = \{F_1', \ldots, F_k'\}$ of $k$ transversals of $D'$. We can assume that $yz$ is contained in exactly $l(yz)$ transversals in $\mathcal{F}$. From $\mathcal{F}$ we can obtain an $l$-disjoint collection $F_1, \ldots, F_k$ of transversals of $D$ as follows. We take

$$F_i := \begin{cases} {F_i}', & \text{if } {F_i}' \text{ contains } yz; \\ {F_i}' \cup \{zy\}, & \text{if } {F_i}' \text{ does not contain } yz. \end{cases}$$

Thus, $\text{transv}(D,l) \geq k = \text{girth}(D,l)$, and the result follows.

**Case 2.** $xy, yx, yz, zy \in A$.

Let $C_1 = \{xy, yx\}$ and $C_2 = \{zy, yz\}$. Note that any cycle other than $C_1$ that contains the arc $xy$ must also contain the arc $yz$. Similarly, any cycle other than $C_2$ that contains the arc $yz$ must also contain the arc $xy$. Similar arguments apply to the arcs $yx$ and $zy$.

Consider the digraph $D' := D - y + \{xz, zx\}$, and set $l'(e) := l(e)$ for $e \in A - \{xy, yx, yz, zy\}$, $l'(xz) := l(xy) + l(yz)$ and $l'(zx) := l(zy) + l(yx)$.

It is immediate that $\text{girth}(D,l) \leq \text{girth}(D',l')$. If $\text{girth}(D,l) < \text{girth}(D',l')$, then every arc distinct from $xy, yx, yz, zy$ has zero length (recall that $D$ is critical). Moreover, this also implies that $C_1, C_2$ are two shortest cycles and every cycle in $D$ contains one of the arcs $xy, yx, yz, zy$ (otherwise $\text{girth}(D,l) = 0$ and the result is obvious). It can be easily shown in this case that $\text{girth}(D,l) = \text{transv}(D,l)$.

Suppose now that $\text{girth}(D,l) = \text{girth}(D',l')$. By the induction hypothesis, there exists an $l'$-disjoint collection $F_1', \ldots, F_k'$ of transversals of $D'$.

Note that for each $i$, $F''_i := F'_i - \{xz, zx\}$ is almost a transversal of $D$: it may only not cover cycles that go through some of the arcs $xy, yx, yz, zy$. Thus, $F''_i \cup \{xy, zy\}$ as well as $F''_i \cup \{yx, yz\}$ are transversals of $D$. Furthermore, if for instance $zx \notin F'_i$ then $F''_i \cup \{xy, yz\}$ is also a transversal of $D$. Now we show how to obtain an $l$-disjoint collection $F_1, \ldots, F_k$ of transversals from $F'_1, \ldots, F'_k$ using that type of extension. By symmetry, we only have to analyse two cases.

**Case 2a.** $l(xy) \geq l(zy)$ and $l(yz) \leq l(yx)$.

Let $t := l(zy)$. For $i = 1, \ldots, t$ define $F_i := (F'_i - \{xz, zx\}) \cup \{xy, zy\}$. For $i = t+1, \ldots, k$ define $F_i := (F'_i - \{xz, zx\}) \cup \{yx, yz\}$. Clearly, $F_1, \ldots, F_k$ form an $l$-disjoint collection since $l(zy) + l(yz) \geq k$, $l(yz) \leq l(yx)$ and $l(zy) \leq l(xy)$.

**Case 2b.** $l(xy) \geq l(zy)$ and $l(yz) \geq l(yx)$.

Suppose that, after relabeling the sets $F'_i$, the arc $zx \in F'_i$ for $i = 1, \ldots, q$ and $zx \notin F'_i$ for $i = q+1, \ldots, k$. Note that $q \leq l'(zx) = l(zy) + l(yx)$. Furthermore, we have $l(zy) + l(yx) \leq k$, since otherwise either $xy$ or $yz$ is not contained in a shortest cycle, contradicting the fact that $(D, l)$ is critical.

Without loss of generality, suppose that $q = l(zy) + l(yx)$. This can be done since $q \leq k$ and hence, we can add the arc $zx$ to some tranversals $F'_i$ to obtain the desired equality. Let $t := l(zy)$. For $i = 1, \ldots, t$ define $F_i := (F'_i - \{xz, zx\}) \cup \{zy, xy\}$; and for $i = t+1, \ldots, q$ define $F_i := (F'_i - \{xz, zx\}) \cup \{yx, yz\}$. Clearly, $F_1, \ldots, F_q$ is an $l$-disjoint collection of transversals of $D$.

Now we want to define the $k - q$ remaining transversals. Note that we cannot use the arcs $yx, zy$ any more, if we want to obtain an $l$-disjoint collection. On the other hand, as $zx$ does not belong to any $F'_i$ for $q + 1 \leq i \leq k$, we can define $F_i := (F'_i - \{xz\}) \cup \{xy, yz\}$ for $i = q + 1, \ldots, k$. As we have noted above, each $F_i$ is a transversal of $D$. To see that this collection is $l$-disjoint we need only check that $l(xy) - l(zy) \geq k - q$ and $l(yz) - l(yx) \geq k - q$. In fact, as $k$ is the girth of $(D, l)$, we have $k \leq l(xy) + l(yx)$, that is, $l(xy) \geq k - l(yx)$. Subtracting $l(zy)$ from both sides of this last inequality, and using the fact that $q = l(yx) + l(zy)$, we have

$$l(xy) - l(zy) \geq k - l(yx) - l(zy) = k - q.$$

Analogously, we have $l(yz) - l(yx) \geq k - q$.

In both cases we conclude that $\operatorname{transv}(D, l) \geq k = \operatorname{girth}(D, l)$. This completes the proof. ■

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