Randomly Eulerian Graphs

Consider the graph in Fig. 7.46, which is an Euler graph.

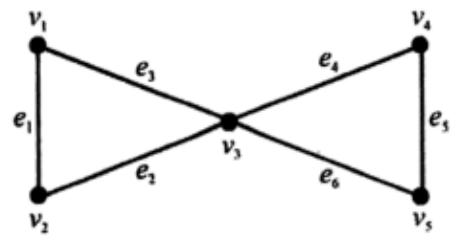

Fig: 7.46

Suppose that we start from vertex v_1 and trace the path v_1, e_1, v_2, e_2, v_3. Now at v_3 we have three choices of going to v_1, v_4 or v_5. If we take the first choice, we will only trace the circuit $v_1, e_1, v_2, e_2, v_3, e_3, v_1$, which is not an Euler line. Thus starting from the vertex v_1, we cannot cover the entire Euler line simply by moving along any edge that has not already been traversed.

We say a graph G is *randomly Eulerion from a vertex* v if whenever we start from vertex v and traverse along the edges of G in an arbitrary way never using any edge twice, we eventually obtain an Euler line.

For example, the graph in Fig. 7.46 is randomly Eulerian from vertex v_3 but not from any other vertex. The graph in Fig 7.47 (a) is not randomly Eulerian from any vertex although it is Euler graph. The graph in Fig 7.47(b) is randomly Eulerian from all its vertices.

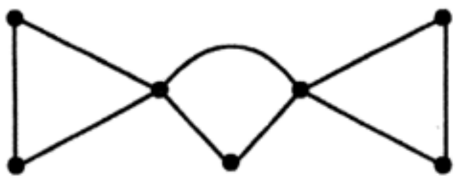

Fig. 7.43(a) Euler graph which is not randomly Eulerian from any vertex

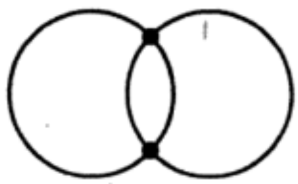

Fig. 7.43(b) Randomly Eulerian from all vertices

Fig: 7.47

The following theorem gives a characterization of a graph which is randomly Eulerian from a vertex v.

Theorem 4: An Euler graph G is randomly Eulerian from a vertex v if and only if every circuit in G contains v.

Proof: We first suppose that G is randomly Eulerian from a vertex v. If possible, we assume that there is a circuit C in G which does not contain v. Consider the subgraph $H = G - C$. Every vertex in H (not necessarily connected) is of even degree. Consider the component of H which contains v. This component is an Euler graph and it contains all the edges incident on v. Thus in this component, there exists an Euler line T starting and ending at vertex v. This Euler line contains all the edges incident on v. Therefore, it cannot be extended to include the edges of C. This contradicts the fact that G is randomly Eulerian from v.

Conversely, suppose vertex v in an Euler graph G is present in every circuit of G. If possible, suppose that G is not randomly Eulerian from v. Then there exists a closed walk T, starting and ending at v, containing all the edges incident on v, but not containing all the edges of G. Moreover, since G is connected, there exists a vertex u different from v such that it is the end vertex of an edge not in T.

Let H be the subgraph of G obtained by removing edges of T from G. Then v is an isolated vertex in H. In H, every vertex is of even degree. So the component of H which contains u is an Euler graph. By corollary of Theorem 2, there is a circuit containing u. This circuit obviously does not contain vertex v. This contradicts the fact that v is in every circuit of G. Thus G must be randomly Eulerian from v. This completes the proof.

We next give a method for constructing an Euler line in a given Euler graph. This method is known as **Fleury's Algorithm**. We require an additional definition before stating the algorithm. An edge in a connected graph G is said to be a *bridge* if the removal of e from G leaves the graph G disconnected. For example, the edge joining the vertices v_4 and v_5 in the graph of Fig. 7.48 is a bridge.

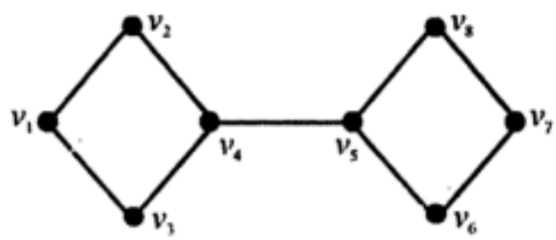

Fig: 7.48