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Three Short Proofs in Graph Theory

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The aim of this note is to give short proofs of three well-known theorems of graph theory.

BROOKS' THEOREM. *If G is a graph with maximum degree n ($n \geqslant 3$) and G contains no complete ($n + 1$)-graph, then G is n-colorable.*

Proof[1]. Suppose G contains two points a, b at distance 2 such that $G - a - b$ is connected (this is satisfied, for example, if G is 3-connected). Let v be a point adjacent to a and b.

As $G - a - b$ is connected, we can arrange its points in a sequence $x_1 = v, x_2, \ldots, x_{m-2}$ such that each point x_i, $i \geqslant 2$ is adjacent to an earlier point; in fact, if $x_1, \ldots, x_i$ have already been chosen let x_{i+1} be any point not yet listed and adjacent to one of them.

We define an n-coloring of G as follows. Let a, b get color 1 (this is legitimate, since they are nonadjacent). We successively color x_{m-2}, $x_{m-3}, \ldots, x_2$ with one of the colors $1, \ldots, n$. This is always possible since each has fewer than n neighbours previously colored. Although this may not be true for $x_1 = v$, it has two neighbors, a and b, of the same color and so we can find a color for v different from the colors of its neighbors.

What's left is to find appropriate points a, b for nontrivial cases. As noted, this is trivially possible if G is 3-connected (since it cannot be a complete graph). One way to finish is to say that 2-separable graphs can easily be broken into smaller pieces whose n-colorings can be put together. Another possibility is this. We may assume G is 2-connected. Let x be a point which is not adjacent to all the other points but has degree at least 3 (we may assume such a point exists). If $G - x$ is still 2-connected, let $a = x$ and let b be any point at distance 2 from x. If $G - x$ is separable, consider 2 endblocks B_1, B_2 (an endblock B_i is a 2-connected component containing a point z_i such that for any other 2-connected component

[1] A related idea was used by J. Ponstein, A new proof of Brooks' chromatic number theorem for graphs, *J. Combinatorial Theory* 7 (1969), 255–257.

B' either B_i and B' are disjoint or z_i is their only common vertex). Since G is 2-connected, there are $a \in B_1 - z_1$, $b \in B_2 - z_2$ adjacent to x. Now a, b satisfy the requirements.

The other two proofs are based on the idea that if a theorem indicates the structure of certain "extremal" graphs then a proof of the theorem may sometimes be obtained by verifying this structure directly.

KÖNIG'S THEOREM. *The maximum number* $\nu(G)$ *of independent edges of a bipartite graph equals to the minimum number* $\tau(G)$ *of points covering all edges.*

TUTTE'S THEOREM. *A graph G has a 1-factor if and only if* $G - X$ *has at most* $|X|$ *odd components for all* $X \leqslant V(G)$.

Both theorems have a trivial half, which hardly have different proofs: $\nu(G) \leqslant \tau(G)$ for any bipartite graph (in fact, for any graph) and the condition given in Tutte's theorem is necessary. We only give the non-trivial parts in detail.

Proof of König's Theorem ($\nu(G) \geqslant \tau(G)$). Let G' be a minimal subgraph of G with the property $\tau(G') = \tau(G)$. We claim G' consists of independent edges. This will finish the proof as the number of these edges is, obviously, at least $\tau(G') = \tau(G)$.

Suppose, to the contrary, that G' has a vertex x adjacent to y_1 and y_2. By the minimality of G', $\tau(G' - (x, y_i)) < \tau(G)$ and so, there is a set $S_i \subseteq V(G)$, $|S_i| = \tau(G) - 1$ which covers all edges of $G' - (x, y_i)$. Since S_i cannot cover (x, y_i), we have $x, y_i \notin S_i$.

Set $S = S_1 \cap S_2$, $|S| = t$, $R = (S_1 - S) \cup (S_2 - S) \cup \{x\}$. Then $|R| = 2(\tau(G) - 1 - t) + 1 = 2(\tau(G) - t) - 1$. R induces a bipartite subgraph G'' of G' (since any subgraph of G is bipartite). Let T be the smaller of the two color classes of G''. Then $|T| \leqslant [\frac{1}{2}|R|] = \tau(G) - t - 1$. Observe that $T \cup S$ covers all edges of G': if an edge is induced by R, T covers it and if it is not, it can meet both S_1 and S_2 only if it has an endpoint in $S_1 \cap S_2 = S$.

Now

$$|T \cup S| = \tau(G) - t - 1 + t = \tau(G) - 1 < \tau(G'),$$

a contradiction.

Proof of Tutte's Theorem (Sufficiency). Assume G is a graph which satisfies the condition that the number of odd components of $G - X$ is at most $|X|$, but has no 1-factor. The condition with $X = \phi$ yields that $|V(G)|$ is even. Let G' be a maximal graph on $V(G)$ containing all edges of G and having no 1-factor.

Let V_1 be the set of those points of G' which are connected to every other point and let $V_2 = V(G) - V_1$. Let G'' be the subgraph of G' induced by V_2.

We claim G'' consists of disjoint complete graphs, i.e., adjacency is an equivalence relation on V_2. Suppose, to the contrary that there are a, b, $c \in V_2$ with (a, b), $(b, c) \in V(G')$ and $(a, c) \notin V(G')$. As $b \in V_2$, we find a point d such that $(b, d) \notin V(G)$.

By the maximality of G', $G' + (a, c)$ has a 1-factor F_1 and $G' + (b, d)$ has a 1-factor F_2. Obviously, $(a, c) \in F_1$, $(b, d) \in F_2$ but $(a, c) \notin F_2$ and $(b, d) \notin F_1$.

$F_1 \cup F_2$ decomposes into disjoint cycles and edges (these are the edges of $F_1 \cap F_2$).

Let C be the cycle of $F_1 \cup F_2$ containing (a, c). If $(b, d) \notin C$, exchange the edges of $F_1 \cap C$ for the edges of $F_2 \cap C$ in F_1. The new 1-factor does not contain (a, c) or (b, d) and is therefore a 1-factor of G', a contradiction. So $(b, d) \in C$.

Removing (b, d) and (a, c) from C we get two paths, one of them having d as an end point. Let P be this path. We may assume without loss of generality that the other endpoint of P is a. Then $C' = P + (b, d) + (a, b)$ is a cycle which alternates with respect to F_2. So removing the edges of $F_2 \cap C'$ from F_2 but adding the other edges of C', we again get a 1-factor of G', a contradicition.

So we have shown that G'' consists of disjoint complete subgraphs; there must be more than $| V_1 |$ odd ones among these, otherwise G' obviously has a 1-factor. Thus $G' - V_1$ has $> | V_1 |$ odd components and therefore, so does $G - V_1$. Thus, the condition is not satisfied.