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One of the core topics in graph theory deals with matchings in graphs, or, more generally, degree-constrained sub-graphs of a graph. In this paper, the first in a series of three papers on this topic, the second of which has appeared [J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 125–130], the author describes a good algorithm for finding a maximum cardinality matching in an arbitrary graph. (A matching in a graph is a set of edges no two of which meet the same vertex.) On the face of it, finding a maximum matching is a simple task. Pure mathematicians show little interest in algorithms for such obviously finite problems. Practical people show little awareness of what is to be desired for such problems. The author articulates the theoretically concrete and practical goal of "cracking exponentiality", in adopting the term "good" for algorithms in which the amount of work increases at worst algebraically, rather than exponentially, relative to some reasonable measure of problem size. His algorithm for maximum matching is good in this sense. It is good in other ways as well. For example, it yields an interesting generalization to arbitrary graphs of the König theorem concerning maximum matchings in bipartite graphs; suitably extended, it also yields a characterization, in terms of half-spaces, of the convex hull of all matchings in a graph, where a matching is viewed as a $(0,1)$-vector having one component for each edge of the graph, the 1's picking out edges of the matching.

The author's generalization of the König theorem runs as follows. A set consisting of one vertex covers an edge if the edge meets the vertex; a set consisting of $2k+1$ vertices $(k=1,2,\cdots)$ covers an edge if both of its end vertices are in the set. The capacity of a 1-element cover is 1; the capacity of a $(2k+1)$-element cover is k. An odd-set cover of a graph G is a family of odd sets of vertices such that each edge of G is covered by a member of the family. Theorem: The maximum cardinality of a matching in G equals the minimum capacity sum of an odd-set cover in G. While this theorem can be shown to be equivalent to a previously known result about maximum matching [C. Berge, *Théorie des graphes et ses applications*, Chapitre 18, Théorème 5, Dunod, Paris, 1958; MR0102822 (21 #1608)], it is now in a much more attractive form, one that strongly suggests, via the duality theorem of linear programming, the following characterization of matchings in terms of linear inequalities. With each edge e of a graph G, associate a nonnegative variable x_e, and impose constraints of the following two types (corresponding to the two types of odd-set covers): (1) the sum of the x's corresponding to edges which meet a vertex v is at most 1, for each v; (2) for any set R of $2k+1$ vertices $(k=1,2,\cdots)$ in G, the sum of the x's corresponding to edges with both ends in R is at most k. It is obvious that a $(0,1)$-vector corresponding to a matching satisfies these inequalities. The author asserts in the present paper (the proof being deferred to the second paper of the series) that every extreme point of the polyhedron thus defined corresponds to a matching.

Reviewed by *D. R. Fulkerson*