Brooks' Theorem and Beyond

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Abstract: We collect some of our favorite proofs of Brooks' Theorem, highlighting advantages and extensions of each. The proofs illustrate some of the major techniques in graph coloring, such as greedy coloring, Kempe chains, hitting sets, and the Kernel Lemma. We also discuss standard strengthenings of vertex coloring, such as list coloring, online list coloring, and Alon–Tarsi orientations, since analogs of Brooks' Theorem hold in each context. We conclude with two conjectures along the lines of Brooks' Theorem that are much stronger, the Borodin–Kostochka Conjecture and Reed's Conjecture. © 2015 Wiley Periodicals, Inc. J. Graph Theory 80: 199–225, 2015

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1. INTRODUCTION

Brooks' Theorem is among the most fundamental results in graph coloring. In short, it characterizes the (very few) connected graphs for which an obvious upper bound on the chromatic number holds with equality. It has been proved and reproved using a wide range of techniques, and the different proofs generalize and extend in many directions. In this article we share some of our favorite proofs. In addition to surveying Brooks' Theorem, we aim to illustrate many of the standard techniques in vertex coloring[1]; furthermore, we prove versions of Brooks' Theorem for standard strengthenings of vertex coloring, including list coloring, online list coloring, and Alon–Tarsi orientations. We present the proofs roughly in order of increasing complexity, but each section is self-contained and the proofs can be read in any order. Before we state the theorem, we need a little background.

A *proper coloring* assigns colors, denoted by positive integers, to the vertices of a graph so that endpoints of each edge get different colors. A graph G is k-colorable if it has a proper coloring with at most k colors, and its *chromatic number* $\chi(G)$ is the minimum value k such that G is k-colorable. If a graph G has maximum degree Δ, then $\chi(G) \leq \Delta + 1$, since we can repeatedly color an uncolored vertex with the smallest color not already used on its neighbors. Since the proof of this upper bound is so easy, it is natural to ask whether we can strengthen it. The answer is yes, nearly always.

A *clique* is a subset of vertices that are pairwise adjacent. If G contains a clique K_k on k vertices, then $\chi(G) \geq k$, since all clique vertices need distinct colors. Similarly, if G contains an odd length cycle, then $\chi(G) \geq 3$, even when $\Delta = 2$. In 1941, Brooks [10] proved that these are the only two cases in which we *cannot* strengthen our trivial upper bound on $\chi(G)$.

Brooks' Theorem. *Every graph G with maximum degree Δ has a Δ-coloring unless either (i) G contains $K_{\Delta+1}$ or (ii) $\Delta = 2$ and G contains an odd cycle.*

We typically emphasize the case $\Delta \geq 3$, and often write the following: "Every graph G satisfies $\chi \leq \max\{3, \omega, \Delta\}$," where ω is the size of the largest clique in G. This formulation follows Brooks' original article, and it has the benefit of saving us from considering odd cycles in every proof. Before the proofs, we introduce our notation, which is fairly standard. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v. The *degree* $d(v)$ of a vertex v is $|N(v)|$. For a graph G with vertex set $V(G)$, we often write $|G|$ to denote $|V(G)|$. The *clique number* $\omega(G)$ of G is the size of its largest clique. We denote the maximum and minimum degrees of G by $\Delta(G)$ and $\delta(G)$. We write simply ω, Δ, δ, or χ when referring to the original graph G, rather than to any subgraph. The subgraph of G induced by vertex set V_1 is the graph with vertex set V_1 and with every edge of G that has both endpoints in V_1; it is denoted $G[V_1]$. The subgraph induced by a clique is *complete*.

To *color greedily* is to consider the vertices in some order, and color each vertex v with the smallest color not yet used on any of its neighbors. For every graph, there exists a vertex order under which greedy coloring is optimal (given an optimal coloring, consider the vertices in order of increasing color). However, considering each of the $n!$ vertex orderings for an n-vertex graph is typically impractical.

[1] One common coloring approach notably absent from this article is the probabilistic method. Although the so-called naive coloring procedure has been remarkably effective in proving numerous coloring conjectures asymptotically, we are not aware of any probabilistic proof of Brooks' Theorem. For the reader interested in this technique, we highly recommend the monograph of Molloy and Reed "Graph Colouring and the Probabilistic Method" [48].

In our proofs, we often assume that a counterexample exists, and this assumption leads us to a contradiction. A *minimum counterexample* G to Brooks' Theorem is one minimizing $|G|$. To avoid repetition later, we note the following here, which is helpful in multiple proofs. We will use it later without further comment. Any minimum counterexample G must satisfy $\chi > \max\{\omega, \Delta\}$, and thus $\chi = \Delta + 1$. If H is a proper induced subgraph of G, then minimality of G gives $\chi(H) \le \max\{3, \omega(H), \Delta(H)\} \le \Delta$. So for every vertex v, $G - v$ is Δ-colorable; since G is not Δ-colorable, v must have a neighbor in all Δ color classes in any Δ-coloring of $G - v$. Since this is true for every vertex v, it follows that G must be Δ-regular.

2. GREEDY COLORING

To motivate our first two proofs, we return to the observation that every graph satisfies $\chi \le \Delta + 1$. The idea is to color the vertices greedily in an arbitrary order. To prove $\chi \le \Delta$, it suffices to order the vertices so that at the point when each vertex gets colored, it still has an uncolored neighbor. Of course, this is impossible, since in any order the final vertex will have no uncolored neighbors. Nonetheless, we *can* find an order such that every vertex except the last still has an uncolored neighbor when it gets colored. Such an order yields a Δ-coloring of $G - v$ for each choice of a final vertex v. Our first two proofs show two different ways to ensure that we can extend this Δ-coloring to G.

In 1975, Lovász published a three-page article [43] entitled "Three Short Proofs in Graph Theory." It included the following proof of Brooks' Theorem by coloring greedily in a good order. The proof needs a few notions of connectedness. A *cutset* is a subset $V_1 \subset V$ such that $G \setminus V_1$ is disconnected. If a single vertex is a cutset, then it is a *cutvertex*. A graph is k-connected if every cutset has size at least k. A *block* is a maximal 2-connected subgraph. The *block graph* of a graph G has the blocks of G as its vertices and has two blocks adjacent if they intersect. It is easy to see that every block graph is a forest; each leaf of a block graph corresponds to an *endblock* in the original graph.

Lemma 2.1. *Let G be a 2-connected graph with $\delta(G) \ge 3$. If G is not complete, then G contains an induced path on three vertices, say uvw, such that $G \setminus \{u, w\}$ is connected.*

Proof. Since G is connected and not complete, it contains an induced path on three vertices. If G is 3-connected, any such path will do. Otherwise, let $\{v, x\} \subset V(G)$ be a cutset. Since $G - v$ is not 2-connected, it has at least two endblocks B_1, B_2. Since G is 2-connected, each endblock of $G - v$ has a noncutvertex adjacent to v (see Figure 1). Let $u \in B_1$ and $w \in B_2$ be such vertices. Now $G \setminus \{u, w\}$ is connected since $d(v) \ge 3$. So uvw is our desired induced path. ■

Proof 2 of Brooks' Theorem. Let G be a connected graph. First, suppose that G has a vertex v with $d(v) < \Delta$. We color greedily in order of decreasing distance to v (breaking ties arbitrarily). For each vertex u other than v, when u gets colored, some neighbor w on a shortest path in G from u to v is uncolored, so we use at most Δ colors. Since $d(v) < \Delta$, we can color v last. Similarly, suppose G has a cutvertex v. Now for each component H of $G - v$, we can Δ-color $H + v$, since v has fewer than Δ neighbors in H. By permuting the color classes to agree on v, we get a Δ-coloring of G.

Now assume that G is Δ-regular, 2-connected, and not complete. Let uvw be the induced path guaranteed by Lemma 2.1. Color u and w with color 1; now as before,

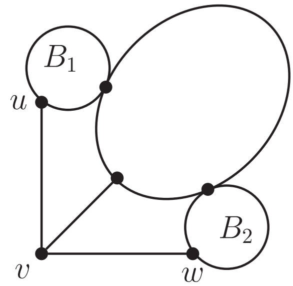

FIGURE 1. G contains an induced path uvw such that $G \setminus \{u, w\}$ is connected.

color the remaining vertices greedily in order of decreasing distance in $G \setminus \{u, w\}$ from v. Again we can color v last, this time because it has two neighbors with the same color. ∎

Lovász's proof has many variations. Bondy [5] used a depth-first search tree to construct the path in Lemma 2.1, and Bryant [11] used yet another method. Schrijver's proof [60] skips Lemma 2.1 by using greedy coloring only for 3-connected graphs and handling two-vertex cutsets by patching together colorings of the components.

3. KEMPE CHAINS

The most famous theorem in graph theory is the 4 Color Theorem: Every planar graph is 4-colorable. In 1852, Guthrie asked whether this was true. Two years later the problem appeared in *Athenæum* [28, 44], a London literary journal, where it attracted the attention of mathematicians. In 1879, Kempe published a proof. Not until 1890 did Heawood highlight a flaw in Kempe's purported solution.[2] Fortunately, Heawood largely salvaged Kempe's ideas, and proved the 5 Color Theorem. The key tool in this work is now called a Kempe chain. This technique is among the most well-known in graph coloring. It yields a short proof that every bipartite graph has a proper Δ-edge-coloring, and similar ideas show that *every* graph has a proper $(\Delta + 1)$-edge-coloring (see Section 5.3 of Diestel [18]).

In 1969 Mel'nikov and Vizing [45] used Kempe chains to give the following elegant proof of Brooks' Theorem. We phrase this proof in terms of a minimal counterexample G, and for an arbitrary vertex v, we color $G - v$ by minimality. To turn the proof into an algorithm, we can simply color greedily toward v, as in our first proof; thus, the "hard part" is once again showing how to color this final vertex v.

For a proper coloring of a graph G, an (i, j)-*Kempe chain* is a component of the subgraph of G induced by the vertices of colors i and j. A *swap* in an (i, j)-Kempe chain H swaps the colors on H; each vertex in H colored i is recolored j and vice versa. Such a swap yields another proper coloring.

Proof 3 of Brooks' Theorem. Suppose the theorem is false and let G be a minimum counterexample. Choose an arbitrary vertex v. By minimality, G is Δ-regular. Further,

[2] In 1880, Tait published a second proof. But, alas, it, too, was founding wanting.

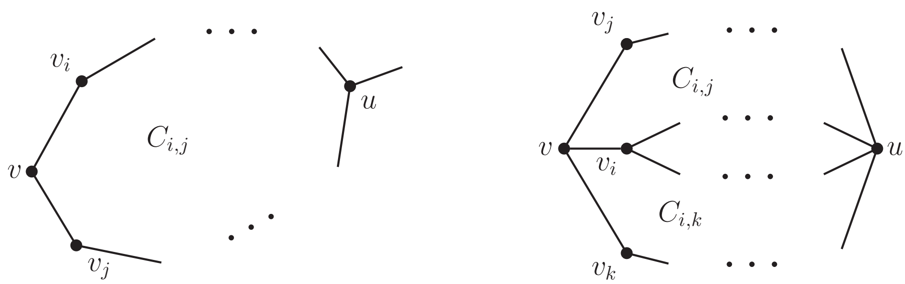

FIGURE 2. The left figure shows Claim 2 and the right figure shows Claim 3 in Proof 3 of Brooks' Theorem.

in each Δ-coloring of $G - v$, each color appears on some neighbor of v. Fix an arbitary Δ-coloring C of $G - v$. For each $i \in \{1, \dots, \Delta\}$, let v_i be the neighbor of v using color i. By a similar argument, for each v_i, each color other than i appears on a neighbor of v_i; for otherwise we could recolor v_i and color v with i. For each pair of colors i and j, let $C_{i,j}$ denote the (i, j)-Kempe chain containing v_i.

Claim 1. *For all i and j*, $C_{i,j} = C_{j,i}$. If not, then after a swap in $C_{i,j}$, two neighbors of v use j and none use i, so we can color v with i.

Claim 2. *For all i and j*, the (i, j)-*Kempe chain* $C_{i,j}$ *containing* v_i and v_j *is a path.* If not, then $C_{i,j}$ has a vertex of degree at least 3 (since v_i and v_j have degree 1); let u be the unique such vertex in $C_{i,j}$ that is closest to v_i. At most $\Delta - 2$ colors appear on neighbors of u, so we can recolor u. Now i and j violate Claim 1.

Claim 3. *For all i, j, and k, we have* $C_{i,j} \cap C_{i,k} = v_i$. Suppose not and choose $u \in C_{i,j} \cap C_{i,k}$, with $u \neq v_i$. Since u has color i, colors j and k each appear on two neighbors of u; so we can recolor u. Now i and j (and also i and k) violate Claim 1.

Claim 4. *Brooks' Theorem is true.* If the neighbors of v form a clique, then $G = K_{\Delta+1}$ and there is nothing to prove. So instead there exist some nonadjacent v_i and v_j, say v_1 and v_2 by symmetry. Let u be the neighbor of v_1 in $C_{1,2}$. Now perform a swap in $C_{1,3}$. Call this new coloring C', and define v'_i and $C'_{i,j}$ analogously to v_i and $C_{i,j}$ for C. Since u still uses color 2, clearly $u \in C'_{2,3}$. By Claim 3, the swap on $C_{1,3}$ did not disrupt $C_{1,2}$ except at v_1; so $u \in C'_{1,2}$. Now $u \in C'_{1,2} \cap C'_{2,3}$, which violates Claim 3, and gives the desired contradiction. ■

Kostochka and Nakprasit [40] took this Kempe chain proof further by showing that when extending the coloring to v, we can ensure that only one color class changes size.[3] Catlin [13] proved that the Δ-coloring given by Brooks' Theorem can be chosen so that one of the color classes is a maximum independent set, and this result is a quick corollary of the theorem of Kostochka and Nakprasit. Much earlier, Mitchem [46] gave a short

[3]They proved the following. For $k \geq 3$, let G be a K_{k+1}-free graph with $\Delta(G) \leq k$. Suppose that $G - v$ has a k-coloring with color classes $M_1, \dots, M_k$. Then G has a k-coloring with color classes $M'_1, \dots, M'_k$ such that $|M_i| \neq |M'_i|$ for exactly one i.

proof of Catlin's result by modifying an existing Δ-coloring via Kempe chains. Recently, Sivaraman [61] gave a short inductive proof of Catlin's Theorem.

4. REDUCING TO THE CUBIC CASE

A natural idea for proving Brooks' Theorem is induction on the maximum degree. For a graph G with maximum degree Δ and clique number at most Δ, suppose we have some independent set I such that $G - I$ has maximum degree and clique number each at most $\Delta - 1$. If we can color $G - I$ with $\Delta - 1$ colors, then we can extend the coloring to G with one extra color. This approach forms the basis for our proofs in the next two sections. Of course we must provide a base case for the induction, and we must also show how to find this very useful set I.

Before proving Brooks' Theorem again, we give an easy lemma, which covers the base case in our inductive proof. We often prove coloring results by repeatedly extending a partial coloring. During this process, the lists of valid colors for two uncolored vertices may differ, depending on the colors already used on their neighbors. This motivates the notion of *list coloring*. Later we develop this idea further, but for now we need only the following lemma.

Lemma 4.1. *If each vertex of a cycle C has a list of two colors, then C has a proper coloring from its lists unless C has odd length and all lists are identical.*

Proof. Denote the vertices by $v_1, \ldots, v_n$ and let the lists be as specified. If all lists are identical and n is even, then we alternate colors on C. So suppose that two lists differ. This implies that the lists differ on some pair of adjacent vertices, say on v_1 and v_n. Color v_1 with a color not in the list for v_n. Now color the vertices greedily in order of increasing index. ■

The following proof is due to the second author [50]. In some ways it is simpler than the first two proofs, since this one needs neither connectivity concepts nor recoloring arguments.

Proof 4 of Brooks' Theorem. Suppose the theorem is false and let G be a minimum counterexample. Recall that G must be Δ-regular.

First, suppose G is 3-regular. A *diamond* is K_4 minus an edge. If G contains an induced diamond D, then by minimality we 3-color $G - D$. The two nonadjacent vertices in D each still have two colors available, so we color them with a common color, and then finish the coloring. So G cannot contain diamonds. Since $\delta(G) \geq 2$, G contains an induced cycle C. Each vertex of C has one neighbor outside of C. Since $\Delta = 3$ and G does not contain K_4, two vertices of C have distinct neighbors outside of C; call the neighbors x and y (see Figure 3). When x and y are adjacent, let $H = G - C$; otherwise, let $H = (G - C) + xy$. Since G does not contain diamonds, H does not contain K_4. Since G is minimum, H is 3-colorable. That is, $G - C$ has a 3-coloring where x and y get different colors. Each vertex of C loses one color to its neighbor outside of C, and so still has two colors available. Since x and y use different colors, by Lemma 4.1 we can extend the coloring to $V(C)$, and hence to all of G.

Now instead suppose $\Delta \geq 4$. Since G is not complete, it has an induced 3-vertex path, uvw. By minimality, $G - v$ has a Δ-coloring; choose a color class I of this Δ-coloring with $u, w \notin I$. Now $\omega(G - I) \leq \Delta - 1$; this is because any K_Δ in $G - I$ would have to

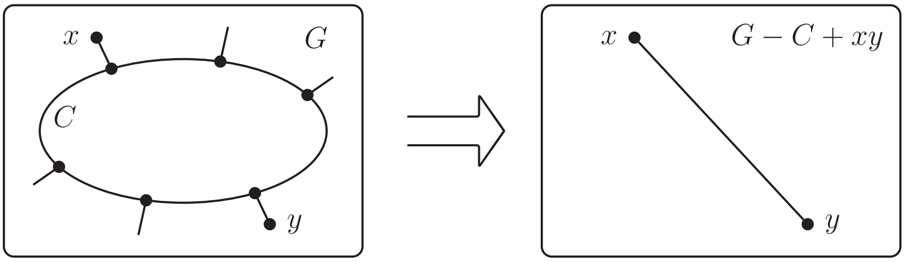

FIGURE 3. How to Δ-color a graph with no $K_{\Delta+1}$ when $\Delta = 3$.

contain v and all of its neighbors in $G - I$, but its neighbors u and w are nonadjacent. Form I' by expanding I to a maximal independent set, and let $H = G - I'$. Since I' is maximal, each vertex in H has a neighbor in I', so $\Delta(H) \leq \Delta - 1$. If $\Delta(H) = \Delta - 1$, then $\omega(H) \leq \Delta(H)$, so Brooks' Theorem holds for H, and $\chi(H) \leq \Delta - 1$. Otherwise, $\Delta(H) \leq \Delta - 2$. Now a greedy coloring gives $\chi(H) \leq \Delta(H) + 1 \leq \Delta - 1$. In each case $\chi(H) \leq \Delta - 1$; now we use one more color on I' to get $\chi(G) \leq \chi(H) + 1 \leq \Delta$. ■

In one variation of Proof 4 we do not reduce to the cubic case [53]. Instead, we note that, similarly to Lemma 4.1, a K_k has a proper coloring from lists of size $k - 1$ unless all the lists are identical. Hence if G contains a K_Δ (or an odd cycle when $\Delta = 3$), then we get a Δ-coloring of G by minimality in the same way as in the 3-regular case of Proof 4. Otherwise, removing any maximal independent set yields a smaller counterexample.

A *hitting set* is an independent set that intersects every maximum clique. The reduction to the cubic case in the previous proof is an immediate consequence of more general lemmas on the existence of hitting sets [38, 51, 33, 64]. Schmerl [59] extended Brooks' Theorem to all locally finite graphs, by constructing a recursive hitting set.[4] Tverberg also modified his own earlier proof [64] to give a shorter constructive proof [65] of Brooks' Theorem for locally finite graphs. We will see this earlier proof in Section 5.

5. K-TREES

Tverberg used k-trees [64] to give a short proof of Brooks' Theorem. Similar to the proof in Section 4, this one inductively colors $G - I$ by minimality, where I is a hitting set. The differences in the two proofs lie in how we find I and how we handle the base case.

We define *k-trees*[5] as follows. For $k = 3$, an odd cycle is a k-tree and for $k \geq 4$, a K_k is a k-tree. Additionally, any graph formed by adding an edge between vertices of degree $k - 1$ in disjoint k-trees is again a k-tree. For convenience, we write T_k to mean K_k when $k \geq 4$ and to mean an odd cycle when $k = 3$.

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[4]By *recursive hitting set*, we mean an infinite set S of vertices, which is both a hitting set and a *recursive* set; more formally, S is recursive if for any vertex v, we can decide whether v is in S in finite time.

[5]Often the term *k-tree* is used for a different (and inequivalent) notion. In presenting Tverberg's proof, we keep with his use of this term, even though it is admittedly nonstandard.

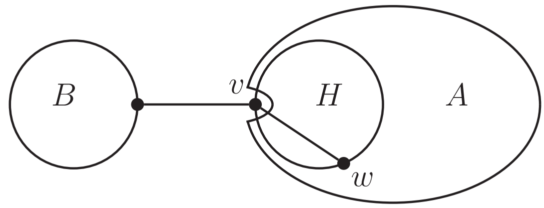

FIGURE 4. Components A and B of $G - v$ in the final case of Lemma 5.1.

The name k-tree comes from the fact that contracting each T_k to a single vertex yields a tree T. A *leaf* in a k-tree is a T_k corresponding to a leaf in T. Each leaf in a k-tree has only one vertex of degree k in G, so it is easy to show by induction that each k-tree other than T_k has at least $k + 1$ vertices of degree $k - 1$. Now we can state Tverberg's lemma.

Lemma 5.1. *Let G be a connected graph and let $k = \Delta(G)$. If $k \geq 3$ and G is neither a k-tree nor K_{k+1}, then G has a vertex v of degree k such that no component of $G - v$ is a k-tree.*

Proof. Suppose the lemma is false and let G be a counterexample. Now G contains a T_k, for otherwise any vertex v of degree k suffices. Let H be a copy of T_k with the minimum number of vertices of degree k in G. First, suppose H has only one vertex v of degree k, and note that v is a cutvertex of G. Since G is not a k-tree and H is a k-tree, $G - H$ is not a k-tree. Since neither $G - H$ nor $H - v$ is a k-tree, v is the desired vertex, which is a contradiction.

So instead H has at least two vertices of degree k. Pick neighbors $v, w \in V(H)$, where v has degree k and w has degree as small as possible. When $G - v$ is connected, let $A = G - v$; otherwise $G - v$ has two components, A and B, where $H - v \subseteq A$. Since G is a counterexample, either A or B is a k-tree.

Suppose B is a k-tree. Now we find a copy of T_k with at most one vertex of degree k, which contradicts the minimality of H. If $B = T_k$, then B will do. Otherwise, we choose a leaf of B with no vertex adjacent to v in G. Thus B is not a k-tree; so instead A is a k-tree.

Since A is a k-tree, w has degree at least $k - 1$ in A and hence at least k in G. By our choice of w, every vertex in H has degree k in G. Hence, by the minimality of H, every vertex in a T_k in G has degree k. But every vertex of A is in a T_k and hence has degree k in A unless it is a neighbor of v. So, A is a k-tree with at most k vertices of degree $k - 1$. The only such k-tree is T_k, so $A = T_k$ and $G = K_{k+1}$, a contradiction. ■

Proof 5 of Brooks' Theorem. Suppose the theorem is false and choose a minimum counterexample G. If G is a Δ-tree, then let v be a cutvertex. For each component H of $G - v$, we can Δ-color $H + v$, and then permute the colors so the colorings agree on v. So G is not a Δ-tree. Now apply Lemma 5.1 with $k = \Delta$ recursively on components until all components have maximum degree less than k; let $v_1, \ldots, v_r$ be the vertices used by the lemma and let $I = \{v_1, \ldots, v_r\}$. Note that I is independent. If $k = 3$, then $G - I$ consists of even cycles and paths, so we can 2-color it. If $k \geq 4$, then by minimality of $|G|$, we can $(k - 1)$-color $G - I$. Now we finish by coloring I with a new color, a contradiction. ■

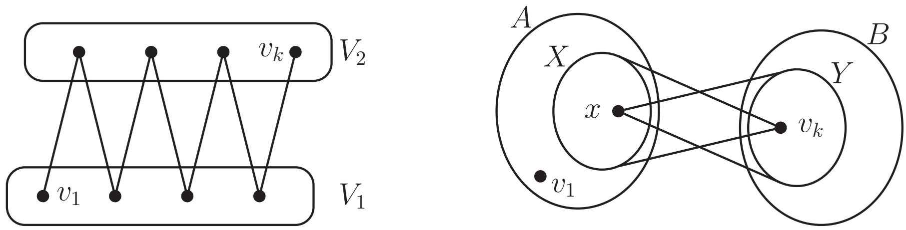

FIGURE 5. The left figure shows an example of a *P*-acceptable path. The right figure shows sets *X* and *Y* constructed from components *A* and *B*.

6. PARTITIONED COLORING

The *coloring number* $col(G)$ is defined by $col(G) = 1 + \max_{H \subseteq G} \delta(H)$, where H ranges over all subgraphs of G. Suppose we color a graph G as follows: delete a vertex v of minimum degree, recursively color $G - v$, and greedily color v. This method shows that $\chi(G) \leq col(G)$. In 1976, Borodin [6] proved the following generalization of Brooks' Theorem.

Theorem 6.1. *Let G be a graph not containing a $K_{\Delta+1}$. If $s \geq 2$ and $r_1, \dots, r_s$ are positive integers with $r_1 + \dots + r_s \geq \Delta \geq 3$, then $V(G)$ can be partitioned into sets $V_1, \dots, V_s$ such that $\Delta(G[V_i]) \leq r_i$ and $col(G[V_i]) \leq r_i$ for all $i \in \{1, \dots, s\}$.*

Brooks' Theorem follows from Theorem 6.1 by taking $s = \Delta$ and $r_1 = \dots = r_s = 1$.[6] One way to prove Theorem 6.1 is to extend the idea of the fundamental result of Lovász [42] along the lines of Catlin [14] and Bollobás and Manvel [4], where a partition is repeatedly modified by moving vertices from one part to another. In his dissertation [52] the second author gave the following proof of Brooks' Theorem, where this dynamic process is specialized and made static. (The heart of this proof is Lemma 6.2, which is similar to the special case of Theorem 6.1 when $s = 2$, $r_1 = 1$, and $r_2 = \Delta - 1$.)

Let G be a graph. A partition $P = (V_1, V_2)$ of $V(G)$ is *normal* if it minimizes the value of $(\Delta - 1)\|V_1\| + \|V_2\|$, where $\|V_i\|$ denotes $|E(G[V_i])|$. Note that if P is a normal partition, then $\Delta(G[V_1]) \leq 1$ and $\Delta(G[V_2]) \leq \Delta - 1$, since if some vertex v has degree that is too high in its part, then we can move it to the other part. The *P-components* of G are the components of $G[V_1]$ and $G[V_2]$. A P-component is an *obstruction* if it is a K_2 in $G[V_1]$ or a K_Δ in $G[V_2]$ or an odd cycle in $G[V_2]$ when $\Delta = 3$. (Note that if a P-component contains an obstruction, then the obstruction is the whole P-component.)

A path $v_1 \dots v_k$ is *P-acceptable* (see Figure 5) if v_1 is in an obstruction and for all $i, j \in \{1, \dots, k\}$, v_i and v_j are in different P-components. A P-acceptable path is *maximal* if it is not contained in a larger P-acceptable path. This means that a P-acceptable path $v_1 \dots v_k$ is maximal if and only if every neighbor of v_k is in the same P-component as some vertex in the path. Given a partition P, to *move* a vertex u is to move it to the other part of P. Note that if P is normal and u is in an obstruction, then the partition formed

[6]Section 9 gives another extension of Brooks' Theorem (also due to Borodin [7]) that classifies the graphs that are colorable when each vertex v is allowed a list of colors $L(v)$ and $|L(v)| = d(v)$. Borodin, Kostochka, and Toft [9] proved an intruiging common generalization of these two, seemingly unrelated, results.

by moving u is again normal since u had maximum degree in its original part. For a subgraph H of G and vertex $u \in V(G)$, let $N_H(u) = N(u) \cap V(H)$.

Lemma 6.2. *Let G be a graph with $\Delta \geq 3$. If G does not contain $K_{\Delta+1}$, then $V(G)$ has an obstruction-free normal partition.*

Proof. Suppose the lemma is false and let G be a counterexample. Among the normal partitions of G with the minimum number of obstructions, choose $P = (V_1, V_2)$ and a maximal P-acceptable path $v_1 \ldots v_k$ so as to minimize k. Throughout the proof, we often move some vertex u in an obstruction A. Since this destroys A, the minimality of P implies that the move creates a new obstruction, which must contain u. So if u has a neighbor w, initially in the other part, then w is in this new obstruction. Finally, the new partition has the minimum number of obstructions.

Let A and B be the P-components containing v_1 and v_k respectively. Let $X = N_A(v_k)$. If $|X| = 0$, then moving v_1 creates a new normal partition P'. Since v_1 is adjacent to v_2, the new obstruction contains v_2. So $v_2v_3 \ldots v_k$ is a maximal P'-acceptable path, violating the minimality of k. Hence $|X| \geq 1$.

Pick any vertex $x \in X$, and form P' by moving x. The new obstruction contains v_k and hence all of B, since each obstruction is a whole P'-component. So $\{x\} \cup V(B)$ induces an obstruction. Let $Y = N_B(x)$; see Figure 5. Since $x \in X$ was arbitrary, the argument works for all $z \in X$. Since obstructions are regular, $N_B(z) = Y$ for all $z \in X$, which implies that X is joined to Y in G. Also since obstructions are regular, $|Y| = \delta(B) + 1$.

First, suppose $|X| \geq 2$. Similar to above, form P' from P by moving x and v_k. Now $\{v_k\} \cup V(A - x)$ induces an obstruction (v_k is in a P'-component with v_1, since $|X - x| \geq 1$). Because obstructions are regular, $|N_{A-x}(v_k)| = \Delta(A)$ and hence $|X| \geq \Delta(A) + 1$. Since X and Y are disjoint, $|X \cup Y| \geq (\Delta(A) + 1) + (\delta(B) + 1) = \Delta(G) + 1$; the equality holds because A and $B + x$ are obstructions. (If $\Delta > 3$, then $\Delta(A) + 1 = |A|$ and $\delta(B) + 1 = |B|$ and the sizes of obstructions in distinct parts sums to $\Delta + 2$; the case $\Delta = 3$ is a little different, since now one obstruction is an odd cycle.)

Suppose X is not a clique and pick nonadjacent $x_1, x_2 \in X$. It is easy to check that moving x_1, v_k, x_2, violates the choice of P. Hence X is a clique. Similarly, suppose Y is not a clique and pick nonadjacent $y_1, y_2 \in Y$ and any $x' \in X - \{x\}$. Now moving x, y_1, x', y_2 again violates the choice of P. Hence Y is a clique. But X is joined to Y, so $X \cup Y$ induces $K_{\Delta+1}$ in G, a contradiction.

So instead $|X| = 1$. Suppose $X \neq \{v_1\}$, and first suppose A is K_2. Now moving x creates another normal partition P' with the minimum number of obstructions. In P', the path $v_kv_{k-1} \ldots v_1$ is a maximal P'-acceptable path, since the P'-components containing v_2 and v_k contain all neighbors of v_1 in that part. Repeating the above argument using P' in place of P gets us to the same point with A not K_2. Hence we may assume A is not K_2.

Move each of $v_1, \ldots, v_k$ in turn. The obstruction A is destroyed by moving v_1, and for $1 \leq i < k$, the obstruction created by moving v_i is destroyed by moving v_{i+1}. So after the moves, v_k is in an obstruction. The minimality of k implies that $\{v_k\} \cup V(A - v_1)$ induces an obstruction and hence $|X| \geq 2$, since A is not K_2. This contradicts $|X| = 1$.

Therefore $X = \{v_1\}$. Now moving v_1 creates an obstruction containing both v_2 and v_k, so $k = 2$. Since v_1v_2 is maximal, v_2 has no neighbor in the other part besides v_1. But now moving v_1 and v_2 creates a partition violating the choice of P. ■

Proof 6 of Brooks' Theorem. Suppose Brooks' Theorem is false and choose a counterexample G minimizing Δ. Clearly $\Delta \geq 3$. By Lemma 6.2, $V(G)$ has an obstruction-free normal partition (V_1, V_2). Note that V_1 is an independent set, since $\Delta(G[V_1]) \leq 1$ and $G[V_1]$ contains no K_2. Since $G[V_2]$ is obstruction-free, the minimality of Δ gives $\chi(G[V_2]) \leq \Delta(G[V_2]) \leq \Delta - 1$. Using one more color on V_1 gives $\chi(G) \leq 1 + \chi(G[V_2]) \leq \Delta$, a contradiction. ■

7. SPANNING TREES WITH INDEPENDENT LEAF SETS

In this section, we take a scenic route. We combine two lemmas of independent interest, to give an unexpected proof of Brooks' Theorem. The union of two forests F_1 and F_2 (on the same vertex set) is 4-colorable, since we can color each vertex v with a pair (a_1, a_2), where a_i is the color of v in a proper 2-coloring of F_i. A *star forest* is a disjoint union of stars. Sauer conjectured that the union of a forest and a star forest is always 3-colorable, and Stiebitz [62] verified this conjecture. The main component of his proof is a lemma that allows for extending a k-coloring of an induced subgraph to the whole graph when it has a spanning forest with certain properties.

Böhme et al. [3] classified the graphs with a spanning tree whose leaves form an independent set; by combining this result with Stiebitz's coloring lemma, they gave an alternative proof of Brooks' Theorem. In this section, we prove both the lemma of Stiebitz and that of Böhme et al., as well as show how they easily yield Brooks' Theorem.

We need two definitions. An *independency tree* is a spanning tree in which the leaves form an independent set. Let $v_1, \ldots, v_n$ be an order of the vertices of a graph G; call it σ. We define a *depth-first-search tree*, or *DFS tree*, with respect to σ as follows. We iteratively grow a tree. At each step i, we have a tree T_i and an *active vertex* x_i. Let $T_1 = v_1$ and $x_1 = v_1$. We grow the tree as follows. If x_i has a neighbor not in T_i, then choose the first such neighbor w, with respect to σ. Now let $T_{i+1} = T_i + x_i w$ and $x_{i+1} = w$. If x_i has no neighbor outside the tree, then let w be the neighbor of x_i on a path in T_i to v_1. Now let $T_{i+1} = T_i$ and $x_{i+1} = w$. This algorithm terminates only when $x_k = v_1$ and all neighbors of v_1 are in the tree. It is easy to check that when this happens T_k is a spanning tree. A *DFS independency tree* is both a DFS tree and an independency tree. We begin with the following elegant lemma, from Böhme et al. [3]; about 25 years earlier, Dirac and Thomassen [19] proved a variation containing (1) and (4), as well as other equivalent conditions (but not (2) or (3)).

Lemma 7.1. *For a connected graph G, the following four conditions are equivalent.*

(1) *G is C_n, K_n, or $K_{n/2,n/2}$ for n even.*
(2) *G has no independency tree.*
(3) *G has no DFS independency tree.*
(4) *G has a Hamiltonian path, and every Hamiltonian path is contained in a Hamiltonian cycle.*

Proof.

(1) $\Rightarrow$ (2): If G is C_n or K_n, then any spanning tree has all leaves pairwise adjacent, so G has no independency tree. So Let G be $K_{n/2,n/2}$. If all leaves are in the same part, then each vertex in the other part has degree at least 2. This requires at least n edges, which is too many for an n-vertex tree; so we get a contradiction.
(2) $\Rightarrow$ (3): This implication is immediate from the definitions.

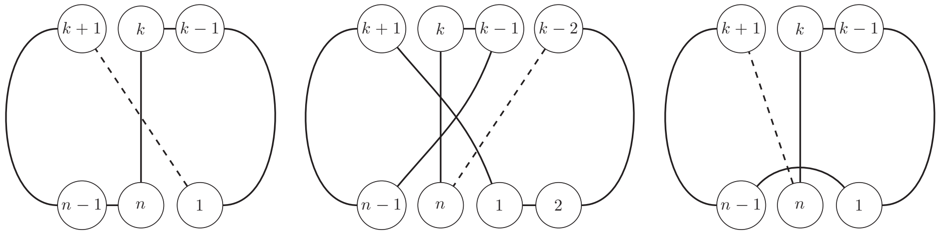

FIGURE 6. Key steps in the proof that (4) ⇒ (1): On the left, G must contain $v_1 v_{k+1}$. In the center, G must contain $v_n v_{k-2}$. A symmetric argument shows that G must contain $v_n v_{k+2}$. On the right, G must contain $v_n v_{k+1}$.

(3) ⇒ (4): We prove the contrapositive. If G has a Hamiltonian path P that is not contained in a Hamiltonian cycle, then P is an independency tree. Considering the vertices in the order in which they appear in P shows that P is a DFS independency tree. So suppose instead that G has no Hamiltonian path. Let $P = v_1 \dots v_k$ be a maximum path in G. The maximality of P implies that v_1 and v_k are nonadjacent to each $u \in V \setminus V(P)$. Similarly v_1 and v_k are nonadjacent (if not, let u be a neighbor of some v_i not on P; now the path $v_{i+1} \dots v_k v_1 \dots v_i u$ is longer than P). Now consider a DFS tree T that begins with $v_1, \dots, v_k$. Since it has an independent leaf set, T is a DFS independency tree.

(4) ⇒ (1): Suppose that G has a Hamiltonian path P, and P is contained in a Hamiltonian cycle C, where $C = v_1 \dots v_n$. If C has no chords, then $G = C_n$, and (1) holds. So assume C has a chord. The *length* of a chord $v_i v_j$ is $\min(|i-j|, n-|i-j|)$.

The left part of Figure 6 shows that if G contains a chord of C of a given length, then G contains all chords of C of that length; the central part shows that if G contains a chord of C of odd (even) length, then it contains all chords of C of odd (even) length.

Suppose n is odd. If G has a chord, then it has either $v_n v_{(n+3)/2}$ or $v_n v_{(n-1)/2}$ (since one chord has odd length and the other even length). By the middle of Figure 6, the presence of one of these chords implies the presence of the other. So G has all chords and $G = K_n$. Suppose instead that n is even. If G has an even chord, then G has a chord of length 2. For any chord $v_n v_k$ of length at least 3, the right part of Figure 6 shows that G also contains $v_n v_{k+1}$; hence G contains chords of both parities, so $G = K_n$. In the final case, with n even and no even chord, all odd chords exist, so $G = K_{n/2,n/2}$. ■

Next we state Stiebitz's lemma for extending a k-coloring of a subgraph to the whole graph.

Lemma 7.2. *Let H be an induced subgraph of a graph G with $\chi(H) \le k$ for some $k \ge 3$. Then $\chi(G) \le k$ if G has a spanning forest F where*

(1) *for each component C of H, $F[V(C)]$ is a tree; and*
(2) *$d_G(v) \le d_F(v) + k - 2$ for every $v \in V(G-H)$.*

Before proving Lemma 7.2, we use Lemmas 7.1 and 7.2 to give a short proof of Brooks' Theorem.

Proof 7 of Brooks' Theorem. It suffices to consider connected graphs. If a graph G is C_n, K_n, or $K_{n/2,n/2}$ for n even, then $\chi(G)$ satisfies the desired bound. So suppose G is none of these graphs. By Lemma 7.1, G has an independency tree T. Let I denote the

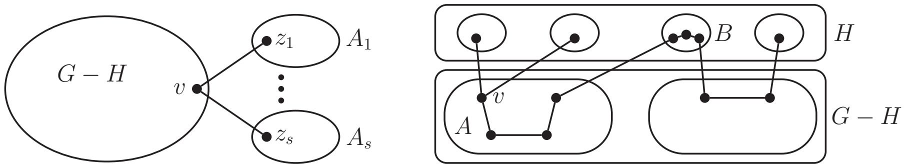

FIGURE 7. The left figure shows Claim 1. The right figure shows Claim 3.

independent leaf set of T. To apply Lemma 7.2, let $H = G[I]$, let $F = T$, and let $k = \Delta$. Clearly $\chi(H) = 1 \leq k$. Each component C of H is an isolated vertex, which is a tree, so (1) holds. Finally, each vertex $v \in V(G - H)$ is a nonleaf in F, so $d_F(v) \geq 2$. Thus $d_G(v) \leq k \leq d_F(v) + (k - 2)$. So Lemma 7.2 implies that $\chi(G) \leq k = \Delta$. ■

The proof of Lemma 7.2 yields an algorithm which extends the coloring of H to one of G, by adding vertices to H one at a time (this is a rough approximation; we give more precise details below). The proof is by contradiction and it relies on fives claims; any vertex v violating a claim allows us to make progress in extending the coloring. One obvious route is to color v immediately and add it to H. When this is not possible, we form a smaller graph G' from $G - v$ by identifying some vertices, and we color G' recursively; afterwards, we color v greedily. Clearly G' (and F' and H') must satisfy the hypotheses of the lemma. We also must ensure that the resulting coloring of $G - v$ uses at most $k - 1$ colors on neighbors of v, so that we can extend the coloring to v.

Proof of Lemma 7.2. For any graphs U and W, we write $U - W$ for the subgraph of U induced by $V(U) \setminus V(W)$. If $uv \in E(F)$, then u is an F-neighbor of v, and u and v are F-adjacent. Suppose the lemma is false and choose a counterexample pair G, H minimizing $|G - H|$. Note that each vertex v in $G - H$ must have a neighbor in H, since otherwise we can add v to H. Thus $|H| \geq 1$.

Claim 1. *If there exists $v \in V(G - H)$ adjacent to components $A_1, \ldots, A_s$ of H with $d_G(v) \leq s + k - 2$, then there exist i and j, with $i \neq j$, and a path in $F - v$ from A_i to A_j.*

Suppose not and choose such a $v \in V(G - H)$. We will find a k-coloring of G. For each $i \in \{1, \ldots, s\}$, let z_i be a neighbor of v in A_i. Form G', F', H' from G, F, H (repectively) by deleting v and identifying all z_i as a single new vertex z. Now $\chi(H') \leq k$, since by permuting colors in each component we can get a k-coloring of H where all the z_i use the same color. Also, F' is a spanning forest in G' since we are assuming there is no path in $F - v$ from A_i to A_j whenever $i \neq j$. It is easy to check that Conditions [cond1](1) and [cond2](2) hold for G', F', H'. Now $|G' - H'| < |G - H|$, so by minimality of $|G - H|$, we have a k-coloring of G'. This gives a k-coloring of $G - v$ where $z_1, \ldots, z_s$ all get the same color. So v has at most $d_G(v) - (s - 1) \leq k - 1$ colors used on its neighborhood, leaving a color free to finish the k-coloring on G, a contradiction.

Claim 2. *Every leaf of F is in H and every vertex not in H has an F-neighbor not in H.* We can rewrite this formally: $d_F(v) \geq 2$ and $d_{F-H}(v) \geq 1$ for all $v \in V(G - H)$. Applying Claim 1 with $s = 1$ implies $d_G(v) \geq k$. Now [cond2]Condition (2) gives $d_F(v) \geq d_G(v) + 2 - k \geq 2$. Suppose $d_{F-H}(v) = 0$ for some $v \in V(G - H)$. Since F is a forest, [cond1]Condition (1) implies that all F-neighbors of v must be in different components of H. Moreover there can be no path between two of these components in $F - v$. [cond2]Condition (2) gives $d_G(v) \leq d_F(v) + k - 2$, so applying Claim 1 with $s = d_F(v)$ gives a contradiction. Thus $d_{F-H}(v) \geq 1$ for all $v \in V(G - H)$.

Claim 3. *There exists v in $G - H$ with $d_{F-H}(v) = 1$ such that every component of H that is F-adjacent to v is not F-adjacent to any other vertex in $G - H$.*

Form a bipartite graph F' from F by contracting each component of H and each component of $F - H$ to a single vertex. Since F is a forest, [cond1]Condition (1) implies that F' is also a forest. So some vertex contracted from a component A of $F - H$ has at most one neighbor of degree at least 2; say this neighbor is contracted from B, where $B \subseteq (F \cap H)$. (If not, then we can walk between components of H and $F - H$ until we get a cycle in F.) Let v be a leaf of A that is not F-adjacent to B; this gives $d_{F-H}(v) = d_A(v) \leq 1$. Claim 2 gives $d_{F-H}(v) \geq 1$, so in fact $d_{F-H}(v) = 1$ as desired.

Claim 4. *If the v in Claim 3 is adjacent to a component of H, then it is F-adjacent to that component.*

Let $A_1, \ldots, A_r$ be the components of H that are F-adjacent to v, where $r = d_F(v) - 1$. Suppose there is another component A_{r+1} of H that is adjacent to v. Since no vertex of $G - H$ besides v is F-adjacent to any of $A_1, \ldots, A_r$, there can be no F-path in $F - v$ between any pair among $A_1, \ldots, A_r, A_{r+1}$. Now the contrapositive of Claim 1 implies that $d_G(v) > (r + 1) + k - 2 = d_F(v) + k - 2$; this inequality contradicts [cond2]Condition (2).

Claim 5. *The lemma holds.*

Let $H' = G[V(H) \cup \{v\}]$, with v as in Claims 3 and 4. By Claim 4, [cond1]Condition (1) of the hypotheses holds for H'. [cond2]Condition (2) clearly holds and F is still a forest. Also, by permuting colors in the components we can get a k-coloring of H where all F-neighbors of v get the same color. Hence v has at most $d_H(v) - (d_F(v) - 2) \leq d_G(v) - 1 - (d_F(v) - 2) = d_G(v) - d_F(v) + 1 \leq k - 1$ colors on its neighborhood. Hence H' is k-colorable. But then, by minimality of $|G - H|$, G is k-colorable, a contradiction. ■

8. KERNEL PERFECTION

In the late 1970s, Vizing [66] and, independently, Erdős, Rubin, and Taylor [26] introduced the notion of list coloring, which is the subject of Sections 8 and 9. An f-list assignment gives to each vertex v a list $L(v)$ of $f(v)$ allowable colors. A proper L-coloring is a proper coloring where each vertex gets a color from its list. A graph G is f-choosable (or f-list colorable) if it has a proper L-coloring for each f-list assignment L. We are particularly interested in two cases of f. If G is f-choosable and f is constant, say $f(v) = k$ for all v, then G is k-choosable. The minimum k such that G is k-choosable is its *choice number* χ_ℓ.[7] If $f(v) = d(v)$ for all v and G is f-choosable, then G is *degree-choosable*. All our remaining proofs show that Brooks' Theorem is true even for list coloring,[8] which is a stronger result.

In 2009, Schauz [58] and Zhu [67] introduced *online list coloring*. In this variation, list sizes are fixed (each vertex v gets $f(v)$ colors), but the lists themselves are provided

[7] Erdős, Rubin, and Taylor noted that bipartite graphs can have arbitrarily large choice number. Let $m = \binom{2k-1}{k}$ and let $G = K_{m,m}$. If we assign to the vertices of each part the distinct k-subsets of $\{1, \ldots, 2k - 1\}$, then G has no good coloring. Thus $\chi_\ell(G) > k$.

[8] Formally: If G is a connected graph other than an odd cycle or a clique, then $\chi_\ell(G) \leq \Delta$.

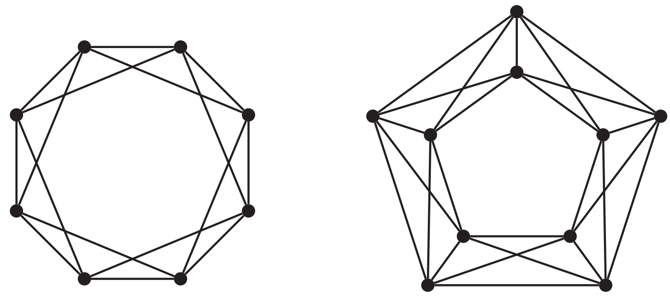

FIGURE 8. The only two connected K_Δ-free graphs where $\alpha = \frac{|G|}{\Delta}$.

online by an adversary. In round 1, the adversary reveals the set of vertices whose lists contain color 1. The algorithm then uses color 1 on some independent subset of these vertices (and cannot change this set later). In each subsequent round k, the adversary reveals the subset of uncolored vertices with lists containing k. Again the algorithm chooses an independent subset of these vertices to use color k. The algorithm wins if it succeeds in coloring all vertices. And the adversary wins if it reveals a vertex v on each of $f(v)$ rounds, but the algorithm never colors it. A graph is online k-list colorable (or k-paintable) if some algorithm can win whenever $f(v) = k$ for all v. The minimum k such that a graph G is online k-list colorable is its *online choice number*, denoted χ_{OL}, (or *paint number*).[9]

Any proof of the following lemma yields a proof of Brooks' Theorem for coloring, list coloring, and even online list coloring using the kernel ideas below (we write $\alpha(G)$ to denote the maximum size of an independent set, or stable set, in G). Note that this lemma follows immediately from Brooks' Theorem for coloring and also from Brooks' Theorem for fractional coloring (defined below). Surprisingly, all known short proofs of Lemma 8.1 rely on some version of Brooks' Theorem.

Lemma 8.1. *If G is a graph with $\Delta > 2$ and G does not contain $K_{\Delta+1}$, then $\alpha(G) \geq \frac{|G|}{\Delta}$.*

It is natural to ask when we can improve the bound in Lemma 8.1. When G can be partitioned into disjoint copies of K_Δ, clearly we cannot. But Albertson, Bollobas, and Tucker [1] did improve the bound when G is connected and K_Δ-free, except when G is one of the two exceptional graphs shown in Figure 8. Along these lines, Fajtlowicz [27] proved that every graph G satisfies $\alpha(G) \geq \frac{2|G|}{\omega+\Delta+1}$. In general, this bound is weaker than Lemma 8.1, but when $\omega \leq \Delta - 2$ it is stronger.

A closely related problem is *fractional coloring*, where independent sets are assigned nonnegative weights so that for each vertex v the sum of the weights on the sets containing v is 1. In a fractional k-coloring, the sum of all weights on the independent sets is k; the *fractional chromatic number* is the minimum value k allowing a fractional k-coloring. (A standard vertex coloring is the special case when the weight on each set is 0 or 1.)

[9] At first glance, an adversary assigning lists to vertices seems to have much more power in the context of online list coloring than in list coloring. However, in practice the choice number and online choice number are often equal. In fact, it is unknown [12] whether there exists a graph with $\chi_{OL} > \chi_\ell + 1$.

King, Lu, and Peng [34] strengthened the result of Albertson et al. [1] by showing that every connected K_Δ-free graph with $\Delta \geq 4$ (except for the two graphs in Figure 8) has fractional chromatic number at most $\Delta - \frac{2}{67}$; this result was further strengthened by Edwards and King [23, 22]. Recently Dvořák, Sereni, and Volec [21] proved fascinating results on fractional coloring of triangle-free cubic graphs. Specifically, they proved that all such graphs have fractional chromatic number at most $\frac{14}{5}$, which is best possible. (For numerous earlier results on this problem, see the references in [21].)

Kostochka and Yancey [37] gave a simple, yet powerful, application of the Kernel Lemma to a list coloring problem. A *kernel* in a digraph D is an independent set $I \subseteq V(D)$ such that each vertex in $V(D) - I$ has an edge into I. A digraph in which every induced subdigraph has a kernel is *kernel-perfect*, and kernel-perfect orientations can be very useful for list coloring; Alon and Tarsi [2], Remark 2.4] attribute this result to Bondy, Boppana, and Siegel (here $d^+(v)$ denotes the out-degree of v in D).

Kernel Lemma. *Let G be a graph and $f : V(G) \to \mathbb{N}$. If G has a kernel-perfect orientation such that $f(v) \geq d^+(v) + 1$ for each $v \in V(G)$, then G is f-choosable.*

The proof of the Kernel Lemma is by induction on the total number of colors in the union of all the lists. For an arbitrary color c, let H be the subdigraph induced by the vertices with c in their lists. We use c on the vertices of some kernel of H, and invoke the induction hypothesis to color the remaining uncolored subgraph. The same proof works for online list coloring, since we simply choose c to be color 1.

All bipartite digraphs are kernel-perfect, and the next lemma [37] generalizes this fact.

Lemma 8.2. *Let A be an independent set in a graph G and let $B = V(G) - A$. Any digraph D created from G by replacing each edge in $G[B]$ by a pair of opposite arcs and orienting the edges between A and B arbitrarily is kernel-perfect.*

Proof. Let G be a minimum counterexample, and let D be a digraph created from G that is not kernel-perfect. To get a contradiction it suffices to construct a kernel in D, since each subdigraph has a kernel by minimality. Either A is a kernel or there is some $v \in B$ which has no outneighbors in A. In the latter case, each neighbor of v in G has an inedge to v, so a kernel in $D - v - N(v)$ together with v is a kernel in D. ■

Now we can show that Lemma 8.1 implies Brooks' Theorem. For simplicity we only prove this for list coloring, but a minor modification of the proof works for online list coloring. This proof is a special case of a result of the second author and Kierstead [32].

Theorem 8.3. *Every graph satisfies $\chi_l \leq \max\{3, \omega, \Delta\}$.*

Proof 8 of Brooks' Theorem. Suppose the theorem is false and let G be a minimum counterexample. The minimality of G implies $\chi_l(G - v) \leq \Delta$ for all $v \in V(G)$. So G is Δ-regular. Lemma 8.1 implies $\alpha(G) \geq \frac{|G|}{\Delta}$. Let A be a maximum independent set in G and let $B = V(G) - A$. For each subgraph H, let $A_H = A \cap V(H)$ and $B_H = B \cap V(H)$. The number of edges between A and B is $\alpha(G)\Delta$, which is at least $|G|$. So there exists a nonempty induced subgraph H of G with at least $|H|$ edges between A_H and B_H, since $H = G$ is one example. Pick such an H minimizing $|H|$. See Figure 9.

For all $v \in V(H)$, let d_v be the number of edges incident to v between A_H and B_H. We show that $d_v = 2$ for all v. If $d_v \leq 1$ for some v, then $H - v$ violates the minimality of H. The same is true if $d_v < d_w$ for some v and w, as we now show. Let k be the minimim degree in H, and choose v and w with $d_v = k$ and $d_w > d_v$. Let $\|H\|$

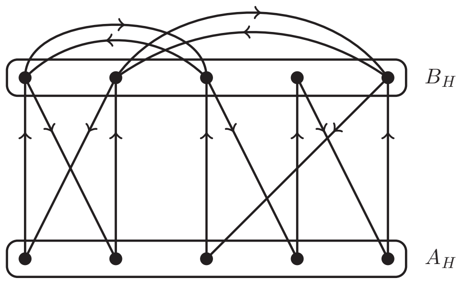

FIGURE 9. The bipartite graph H with parts A_H and B_H.

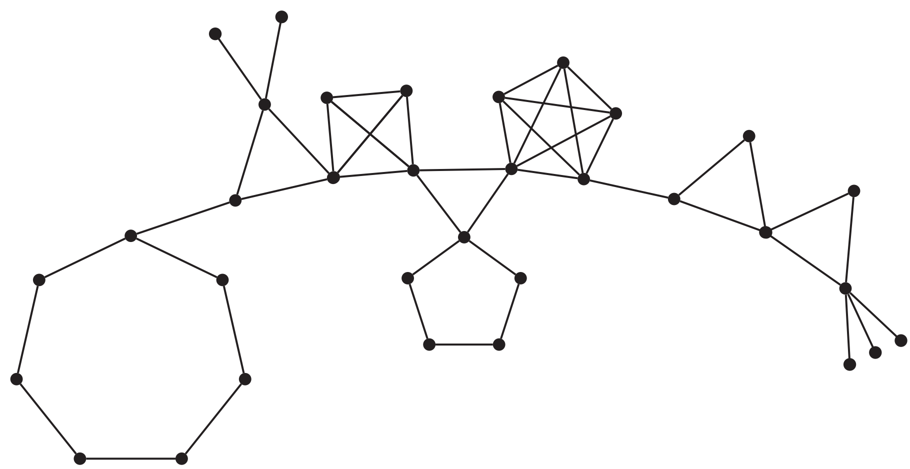

FIGURE 10. A Gallai tree with 15 blocks.

denote the number of edges in H. Since $d_w > k$, we have $2\,\|H\| \geq k|H| + 1$. Deleting v gives $2\,\|H - v\| \geq k\,|H| + 1 - 2k = k(|H| - 1) + (1 - k) = 2(|H| - 1) + (1 - k) + (k - 2)(|H| - 1)$. When $k \geq 3$ the final term is at least k, since $|H| \geq k + 1$; so $\|H - v\| \geq |H - v|$, a contradiction. If instead $k = 2$, then $2\,\|H - v\| \geq 2(|H| - 1) - 1$. Since the left side is even, we conclude $2\,\|H - v\| \geq 2(|H| - 1)$, again reaching a contradiction.

So $d_v = d_w$ for all v and w; we call this common value d, and note that $d \geq 2$. Now there are $(d/2)|H|$ edges between A_H and B_H. It is easy to check that $(d/2)|H| - d \geq |H| - 1$ for $d > 2$, so the minimality of $|H|$ shows that $d = 2$. Thus the edges between A_H and B_H induce a disjoint union of cycles (in fact, just a single cycle, by the minimality of H).

Create a digraph D from H by replacing each edge in $H[B_H]$ by a pair of opposite arcs and orienting the edges between A_H and B_H consistently along the cycles. By Lemma 8.2, D is kernel-perfect. Since $d^+(v) \leq d(v) - 1$ for each $v \in V(H)$, the Kernel Lemma shows that H is f-choosable where $f(v) = d(v)$ for all $v \in V(H)$. Now given any Δ-list-assignment on G, we can color $G - H$ from its lists by minimality of $|G|$, and then extend the coloring to H, which is a contradiction. ■

9. DEGREE-CHOOSABLE GRAPHS

The option to give different vertices lists of different sizes allows us to refine Theorem 8.3. A graph G is *degree-choosable* if it is f-choosable where $f(v) = d(v)$ for all $v \in V(G)$. The degree-choosable graphs were classified by Borodin [7] and independently by Erdős, Rubin, and Taylor [26]. A graph is a *Gallai tree* if each block is a complete graph or an odd cycle.

Theorem 9.1. *A connected graph is degree-choosable if and only if it is not a Gallai tree.*

A minimal counterexample to Brooks' Theorem is regular, and hence degree-choosable if and only if Δ-choosable. So Brooks' Theorem follows immediately from Theorem 9.1, since the only Δ-regular Gallai tree is $K_{\Delta+1}$. We give two proofs of Theorem 9.1. The first uses a structural lemma from Erdős, Rubin, and Taylor [26] known as "Rubin's Block Lemma," but the second does not.

Lemma 9.2. *No Gallai tree is degree-choosable.*

Proof. Assign disjoint lists to the blocks as follows. For each block B, let L_B be a list of size 2 if B is an odd cycle and size k if B is K_{k+1}. The list for each vertex v is the union of the lists for blocks containing it: $L(v) = \cup_{B \ni v} L_B$. Note that $|L(v)| = d(v)$ for all v.

We show that G has no coloring from these lists, by induction on the number of blocks. For the base case, G is complete or an odd cycle, and the proposition clearly holds. Otherwise, let B be an endblock of G, and let $v \in B$ be a cutvertex. If G has an L-coloring, then each vertex in $B - v$ gets colored from L_B, so each color in L_B appears on a neighbor of v. Now $G \setminus (B - v)$ is again a Gallai tree, with lists as specified above; by hypothesis it has no good coloring from its lists. ■

The following lemma has many different proofs [26, 25, 31, 52]. We follow the presentation of Hladký, Král, and Schauz [31]. Although it is known as Rubin's Block [26], the result was implicit in the much earlier work of Gallai [29, 30] and Dirac.

Rubin's Block Lemma. *If G is a 2-connected graph that is not complete and not an odd cycle, then G contains an induced even cycle with at most one chord.*

Proof. *Let G be a 2-connected graph that is neither complete nor an odd cycle. Since G is not complete, it has a minimal cutset S, with $|S| \geq 2$. Choose $u, v \in S$ and let C be a cycle formed from the union of shortest paths P_1 and P_2 joining u and v in two components of $G \setminus S$ (see Figure 11). Now C has at most one chord, the edge uv. If C has even length, then we are done. If C has odd length, then one of the paths joining u and v in C has odd length; call it P. If uv is present, then $P + uv$ is a chordless even cycle. Thus, uv is absent and C is an induced odd cycle of length at least 5. Since G is not an odd cycle, there exists $w \in V(G \setminus C)$.*

Suppose first that no vertex $w \in V(G \setminus C)$ has two neighbors on C. Since G is 2-connected, there is a shortest path R with endpoints on C and interior disjoint from C, and R has length at least 3. Now $V(C) \cup V(R)$ induces two 3-vertices with three vertex disjoint paths between them. Two paths have the same parity, so together they induce a chordless even cycle.

So instead some vertex $w \in V(G \setminus C)$ has two or more neighbors on C; call them $v_1, \ldots, v_k$ (see Figure 11). The v_i split C into paths P_i with each v_i the endpoint of two

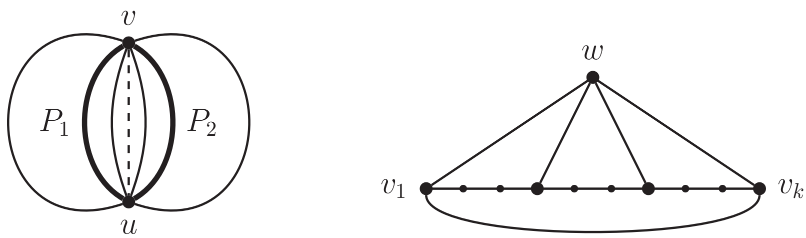

FIGURE 11. The figure on the left shows an induced cycle C formed from P_1 and P_2. The figure on the right shows w, not on C, and its neighbors on C.

paths. If any P_i has even length, then $V(P_i) \cup \{w\}$ induces a chordless even cycle. So each P_i has odd length; since C has odd length, $k \geq 3$. If $k > 3$, then $V(P_1) \cup V(P_2) \cup \{w\}$ induces an even cycle with one chord. If $k = 3$, then some path P_i, say P_3 by symmetry, has length at least 3, since C has length at least 5. Now again $V(P_1) \cup V(P_2) \cup \{w\}$ induces an even cycle with exactly one chord. ■

The following proof is similar to Lovász's proof of Brooks' Theorem, which we saw in Section 2. In the list coloring context, two parts of that proof break down: (i) nonadjacent neighbors of a common vertex v need not have a common color and (ii) we cannot permute colorings on different blocks to agree on a cutvertex. The first problem has an easy solution, but the second is more serious. If in our induced path uvw either u or w is a cutvertex, then when we color u and w first we disconnect the graph of uncolored vertices. So rather than coloring toward a subgraph that we colored first, we instead color toward a subgraph that we can color last—any degree-choosable subgraph will do.

Proof 1 of Theorem 9.1. Lemma 9.2 shows that no Gallai tree is degree-choosable. So now suppose G is not a Gallai tree. By [rubin]Rubin's Block Lemma, G has an induced even cycle with at most one chord; call this subgraph H. We can greedily color the vertices of $G - H$ in order of decreasing distance from H, since each vertex has an uncolored neighbor when we color it. Now we extend the coloring to H. When H is an even cycle, we can use Lemma 4.1. So assume instead that H is an even cycle with one chord; label the vertices $v_1, \ldots, v_n$ around the cycle so that $d_H(v_1) = 3$. Since v_1 has 3 colors, we color it with some color not available for v_n. Now we greedily color the vertices in order of increasing index. ■

Many proofs of list coloring theorems can be easily extended to prove their analogues for online list coloring. The classification of degree-choosable graphs illustrates this idea well. The analogue of degree-choosable is *degree-paintable*, and a connected graph is degree-paintable precisely when it is not a Gallai tree. Suppose that H is a connected degree-paintable induced subgraph of G. Let σ be an order of the vertices by decreasing distance from H. If S_k denotes the vertices available on round k, then the algorithm greedily forms a maximal independent set I_k by adding vertices from $(V(G) - V(H)) \cap S_k$ in the order σ. For the vertices in S_k with no neighbor in I_k, the algorithm then plays on H the strategy that shows it is degree-paintable. This produces a valid coloring. To complete the classification of degree-paintable graphs, we need only verify that every even cycle with at most one chord is degree-paintable.

Alon and Tarsi [2] developed a powerful tool, which gives an alternate short proof of Theorem 9.1 from [rubin]Rubin's Block Lemma. A subgraph H of a directed graph D is

Eulerian if in H each vertex v has indegree $d_H^-(v)$ equal to outdegree $d_H^+(v)$. Let EE and EO denote the sets of Eulerian subgraphs of D where the number of edges is even and odd, respectively.

Alon–Tarsi Theorem. *Let D be an orientation of a graph G, and let L be a list assignment such that $|L(v)| \geq 1 + d^+(v)$ for all v. If $|EE| \neq |EO|$, then G is L-colorable.*

Let G be a graph that is not a Gallai Tree, and let H be an induced even cycle with at most one chord in G, as guaranteed by [rubin]Rubin's Block Lemma. Order the vertices outside of H by increasing distance from H (breaking ties arbitrarily); call this order σ. Now orient each edge uv as $u \to v$ if u precedes v in σ. Orient the cycle edges in H consistently, and orient the chord, if it exists, arbitrarily. It is easy to see that each vertex v has indegree at least 1, and hence outdegree at most $d(v) - 1$. So the Alon–Tarsi Theorem implies that G is degree-choosable if $|EE| \neq |EO|$. Every Eulerian subgraph J must be a subgraph of H, for otherwise the vertex of J that comes last in σ has outdegree 0. If H is an even cycle, then $|EO| = 0$ and $|EE| = 2$, since both the empty subgraph and all of H are in EE. If H is a cycle with a chord, then either $|EE| = 3$ and $|EO| = 0$ or else $|EE| = 2$ and $|EO| = 1$. In both cases, the Alon–Tarsi Theorem shows that G is degree-choosable. This is essentially the proof given by Hladký, Král, and Schauz [31].

After our first proof of Theorem 9.1, we outlined how to extend the result to characterize degree-paintable graphs. However, we omitted the tedious proof that an even cycle with at most one chord is degree-paintable. One advantage of the Alon–Tarsi proof of Theorem 9.1 is that it extends easily to paintability. Schauz [56, 57] proved an analogue of the Alon–Tarsi Theorem for online list coloring,[10] so in fact the proof of the paintability result is nearly identical.

We extend this idea [17] to show that G^2 is online $(\Delta^2 - 1)$-choosable unless $\omega(G^2) \geq \Delta^2$; here G^2 is formed from G by adding edge uv for each pair u and v at distance 2 in G. Our approach in that proof is quite similar to the method used above to prove Theorem 9.1 via the Alon–Tarsi Theorem. Applying these techniques to G^2 gives $d^-(v) \geq 2$ for all v. Now however, we seek a subgraph H that is online "degree-1"-choosable, that is, H is online f-choosable, where $f(v) = d(v) - 1$. So the bulk of the work lies in showing that every square graph contains either such an induced subgraph or else a large clique.

We now conclude our digression into online list coloring and the Alon–Tarsi Theorem, and we finish the section with a second proof of Theorem 9.1. Kostochka, Stiebitz, and Wirth [41] gave a short proof of Theorem 9.1, which we reproduce below. Further, they extended the result to hypergraphs.

Proof 2 of Theorem 9.1. Lemma 9.2 shows that no Gallai tree is degree-choosable. So now suppose there exists a graph that is not a Gallai tree, but is also not degree-choosable. Let G be such a graph with as few vertices as possible. Since G is not degree-choosable, no induced subgraph H of G is degree-choosable (if such an H exists, then we color $G - H$ greedily towards H, and extend the coloring to H since it is degree-choosable). Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2, choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees.

[10]Let D be an orientation of a graph G, and let f be a list size assignment such that $f(v) \geq 1 + d^+(v)$ for all v. If $|EE| \neq |EO|$, then G is f-paintable.

Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Now let L be a list assignment on G such that $|L(v)| = d(v)$ for all $v \in V(G)$ and G is not L-colorable. Suppose two vertices in G get different lists. Since G is connected, we have adjacent $v, w \in V(G)$ such that $L(v) - L(w) \neq \emptyset$. Pick $c \in L(v) - L(w)$ and color v with c. Now we can finish by coloring in order of decreasing distance from w in $G - v$. So instead $L(v) = L(w)$ for all $v, w \in V(G)$; in particular, G is regular.

Pick an arbitrary $v \in V(G)$ and consider the Gallai tree $G - v$. Since G is regular and 2-connected, v must be adjacent to all noncutvertices in all endblocks of $G - v$. So, if $G - v$ has at least two endblocks, then $d(v) \geq 2(\Delta(G) - 1)$. Since $2(\Delta(G) - 1) > \Delta(G)$ when $\Delta(G) \geq 3$, we must have $\Delta(G) = 2$. Since G is not L-colorable it is not 2-colorable and hence is an odd cycle, a contradiction. Therefore $G - v$ has only one endblock. As noted above, $G - v$ is a Gallai tree; since $G - v$ has only one endblock (and, hence, only one block), it is complete. Thus G is also complete, again a contradiction. ■

10. FURTHER DIRECTIONS

In this final section, we conclude our survey by discussing two conjectures that strengthen Brooks' Theorem. Determining the chromatic number of a graph is well-known to be NP-hard. However, Brooks' Theorem shows that determining whether a graph has $\chi = \Delta + 1$ is easy. If $\Delta = 2$, look for odd cycles; otherwise, look for a $K_{\Delta+1}$. It is natural to ask how close t must be to Δ so that we can easily check whether $\chi = t$. Emden-Weinert, Hougardy, and Kreuter [24] gave the following lower bound on this threshold.

Theorem 10.1. *For any fixed Δ, deciding whether a graph G with maximum degree Δ has a $(\Delta + 1 - k)$-coloring is NP-complete for any k such that $k^2 + k > \Delta$, when $\Delta + 1 - k \geq 3$.*

Molloy and Reed [47] then proved a matching upper bound, for sufficiently large Δ.

Theorem 10.2. *For any fixed sufficiently large Δ, deciding whether a graph G with maximum degree Δ has a $(\Delta + 1 - k)$-coloring is in P for every k such that $k^2 + k \leq \Delta$.*

Further, they conjectured that the same result holds for all values of Δ. (Section 15.4 of Molloy and Reed [48] has more on this question.) Viewed in this framework, Brooks' Theorem describes the case $k = 1$. Now we consider the case $k = 2$. In 1977, Borodin and Kostochka [8] posed the following conjecture.

Borodin–Kostochka Conjecture. *If G has $\Delta \geq 9$ and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.*

If true, the conjecture is best possible in two ways. First, even when we require $\omega \leq \Delta - 2$, we cannot conclude $\chi \leq \Delta - 2$. For example, form G from a disjoint 5-cycle and $K_{\Delta-4}$ by adding all edges with one endpoint in each graph (this is the *join* of C_5 and $K_{\Delta-4}$). Now $\omega = \Delta - 2$, but $\chi = \Delta - 1$, since every proper coloring uses at least 3 colors on the 5-cycle and cannot reuse any color on the clique.

Second, the lower bound on Δ cannot be reduced, as demonstrated by the following construction, shown in Figure 12. Form G from five disjoint copies of K_3, say $D_1, \ldots, D_5$, by adding edge uv if $u \in D_i$, $v \in D_j$, and $|i - j| \equiv 1 \bmod 5$. This graph is 8-regular

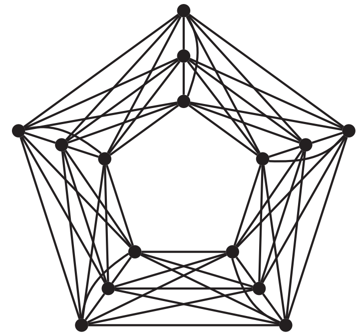

FIGURE 12. A construction showing that the hypothesis $\Delta \geq 9$ in the Borodin–Kostochka Conjecture is necessary and best possible.

with $\omega = 6$. Each color class has size at most 2, so $\chi = \lceil 15/2 \rceil = 8$. Thus $\chi = \Delta$, but $\omega = \Delta - 2$. For $\Delta \leq 8$, various other examples are known [16] where $\chi = \Delta$ and $\omega < \Delta$.

Reed [55] proved the Borodin–Kostochka Conjecture when $\Delta \geq 10^6$ and the present authors [16] proved it for claw-free graphs (those where no vertex has three pairwise nonadjacent neighbors). Although this question remains open, stronger versions of the conjecture are believed true. Already in 1977, Borodin and Kostochka were convinced that the same upper bound holds for the list chromatic number. Recently, we conjectured [17] that the bound holds even for the online list chromatic number. Reed posed the following conjecture, which is along similar lines, but much more far-reaching.

Reed's Conjecture. *Every graph G satisfies* $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

In 1998, Reed proved [54] that there exists a positive ϵ such that $\chi \leq \lceil \omega\epsilon + (\Delta + 1)(1 - \epsilon) \rceil$. The original conjecture is that this upper bound holds when $\epsilon = \frac{1}{2}$. In 2012, King and Reed [35] gave a much shorter proof of the same result. A key ingredient of their proof is the result of King [33] that every graph with $\omega > \frac{2(\Delta+1)}{3}$ has a hitting set (recall that a hitting set is independent and intersects every maximum clique). About the same time, they used the Claw-free Structure Theorem of Chudnovsky and Seymour to prove that Reed's Conjecture holds for all claw-free graphs [36]. Section 21.3 of Molloy and Reed [48] gives further evidence for Reed's Conjecture by showing that the desired upper bound holds for the fractional chromatic number, even without rounding up.

We conclude this section by showing that Reed's Conjecture is best possible, using random graphs. Specifically, we show that if $\epsilon > \frac{1}{2}$, then the bound $\chi \leq \lceil \omega\epsilon + (\Delta + 1)(1 - \epsilon) \rceil$ fails for some graph G. The proof we present is from the end of [54]. Before we give the details of the random construction, we mention in passing that Kostochka [39] also showed this using the explicit examples that Catlin [15] constructed to disprove the Hajós Conjecture. Let $H_t = t \cdot C_5$ (i.e. C_5 where each edge has multiplicity t) and let G_t be the line graph of H_t; Figure 12 shows G_3. Catlin showed that for odd t we have $\chi(G_t) = \frac{5t+1}{2}$, $\Delta(G_t) = 3t - 1$, and $\omega(G_t) = 2t$. So, for any $\epsilon > \frac{1}{2}$, we can choose t large enough to make the bound fail.

Now we give the details of the random construction. We will construct a graph H on n vertices such that $\chi(H) \geq \frac{1}{2}n - n^{3/4}$ and $\omega(H) \leq 8n^{3/4} \log n$. When we form G by joining H to $K_{\Delta+1-n}$, we see that the desired bound fails for G when $\epsilon > \frac{1}{2}$. Let H be a random graph on n vertices, where each edge appears independently with probability p and let $p = 1 - n^{-3/4}$. The expected number of cliques of size k is $\binom{n}{k}p^{\binom{k}{2}}$. So when $k \geq 8n^{3/4}\log n$, the expected number of k-cliques is arbitrarily small for sufficiently large n.

$$\begin{aligned}\binom{n}{k}p^{\binom{k}{2}} &= \binom{n}{k}\left(1-n^{-3/4}\right)^{\binom{k}{2}}\\ &\leq n^k\left(1-n^{-3/4}\right)^{k^2/4}\\ &\leq 2^{k\log n}\left(e^{-n^{-3/4}}\right)^{k^2/4}\\ &\leq e^{k\log n - k^2 n^{-3/4}/4}\\ &\leq e^{-k\log n}.\end{aligned}$$

The expected number of independent sets of size 3 is $\binom{n}{3}(1-p)^3 = \binom{n}{3}(1-(1-n^{-3/4}))^3 \leq \frac{1}{6}n^{3/4}$. By deleting one vertex from each independent set of size 3, we get a graph H on $n - \frac{1}{6}n^{3/4}$ vertices with independence number 2. So $\chi(H) \geq \frac{n}{2} - \frac{1}{12}n^{3/4}$ and $\omega(H) \leq 8n^{3/4}\log n$. Now $\chi(G) \geq (\Delta + 1 - n) + (\frac{1}{2}n - \frac{1}{12}n^{3/4}) = \Delta + 1 - \frac{1}{2}n - \frac{1}{12}n^{3/4}$. Similarly, $\omega(G) \leq (\Delta + 1 - n) + 8n^{3/4}\log n$.

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