Random Graphs III

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Outline of Lecture III

1. **Subgraph containment with adversary:**
   - ▷ Existence of monoχ subgraphs in **coloured random graphs**; properties of the form $G(n,p) \to (H)_r^{e}$ (omit "e"; we are done with vertex colourings)
   - ▷ Existence of monochromatic subgraphs in 'dense' subgraphs of random graphs; statements of the form $G(n,p) \to_\eta H$
2. **Regularity method for sparse graphs:** some of the (partial) successes
3. **Literature:** shall discuss some of the literature

One last thing about the vertex case

Exercise 1^{++}: determine $p_0 = p_0(n)$ such that if $p \gg p_0$, then a.e. $G(n, p)$ is such that

$$G(n, p) \to (K^3, K^4)^v \tag{1}$$

Have you obtained the threshold?

The results for K^3

Theorem 1. *There is a large enough constant* C *such that if* $p \geq C/\sqrt{n}$*, then a.e.* $G(n,p)$ *satisfies*

$$G(n,p) \rightarrow (K^3)_2. \quad (2)$$

Theorem 2. *For any* $\eta > 0$*, there* C *such that if* $p \geq C/\sqrt{n}$ *then a.e.* $G(n,p)$ *satisfies*

$$G(n,p) \rightarrow_{1/2+\eta} K^3. \quad (3)$$

The case $p = 1$

Theorem 3 (Goodman). *The number of monochromatic triangles in any 2-colouring of K^n is $\geq \binom{n}{3} - \frac{1}{8}n(n-1)^2 = (\frac{1}{4} + o(1))\binom{n}{3}$.*

Proof*.* Count the number N of 2-coloured cherries. Say have $b(x)$ blue edges and $r(x)$ red edges at vertex x. Have $N = \sum_x b(x)r(x) \leq n(n-1)^2/4$ (use $b(x) + r(x) = n - 1$). The number of non-monochromatic triangles is $N/2$. □

Clearly, there are 2-colourings with $\leq \frac{1}{4}\binom{n}{3}$ monochromatic triangles!

The case $p = 1$

Theorem 4. *Goodman implies Mantel:* $\mathrm{ex}(n, K^3) \le n^2/4$.

Proof. Suppose $K^3 \not\subset G^n$ and $e(G^n) > n^2/4$, so that $e(G^n) \ge n^2/4+3/4$. Count the number N of pairs (e, T) where $e \in E(G^n)$, $T \in \binom{V(G^n)}{3}$ and $e \subset T$. Say there are t_i triples containing i edges of G. Then, by Theorem 3,

$$\begin{aligned}(n^2/4 + 3/4)(n-2) &\le e(G^n)(n-2) \\ &= N \le t_1 + 2t_2 \le 2(t_1 + t_2) \le n(n-1)^2/4,\end{aligned}$$

which is a contradiction for $n \ge 4$. □

Theorems 1 and 2 from Goodman's counting

Theorem 5. *For every* $\varepsilon > 0$ *there is* C *such that if* $p \geq C/\sqrt{n}$*, then a.e.* $G(n, p)$ *is such that any* 2*-colouring of its edges contains at least* $(1/4 - \varepsilon)p^3\binom{n}{3}$ *monochromatic* K^3*.*

Proof. Exercise 2^+. □

Exercise 3^+: derive Theorem 2 from Theorem 5.

Exercise 4^{++}: show that the threshold for Theorem 1 is indeed $1/\sqrt{n}$.

Exercise 5: show that the threshold for Theorem 2 is indeed $1/\sqrt{n}$.

The Rödl–Ruciński theorem

Definition 6 (2-density and $m_2(H)$). *The 2-density* $d_2(H)$ *of a graph* H *with* $|V(H)| > 2$ *is*

$$\frac{|E(H)| - 1}{|V(H)| - 2}. \tag{4}$$

For $H = K^1$ *and* $2K^1$ *let* $d_2(H) = 0$*; set* $d_2(K^2) = 1/2$*. Let*

$$m_2(H) = \max\{d_2(J)\colon J \subset H,\ |V(J)| > 0\}. \tag{5}$$

Exercise 6: consider, say, $H = K^h$. Show that if $p \ll n^{-1/m_2(H)}$, then a.s. $\#\{H \hookrightarrow G(n,p)\} \ll e(G(n,p))$. On the other hand, if $p \gg n^{-1/m_2(H)}$, then a.s. $\#\{H \hookrightarrow G(n,p)\} \gg e(G(n,p))$.

The Rödl–Ruciński theorem

Theorem 7. *Let H be a graph containing at least a cycle and let $r \geq 2$ be an integer. Then there exist constants c and C such that*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (H)_r) = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(H)} \\ 1 & \text{if } p \geq Cn^{-1/m_2(H)}. \end{cases} \tag{6}$$

▷ In particular, $p_0 = p_0(n) = n^{-1/m_2(H)}$ is a threshold for the property $G(n,p) \to (H)_r$.

The Rödl–Ruciński theorem

Exercise 7: show that there exists a graph G with the property that $G \to (K^h)_r$ but $K^{h+1} \not\subset G$.

Exercise 8: Given a graph G, let $\mathcal{H}_3(G)$ be the 3-uniform hypergraph whose hypervertices are the edges of G and the hyperedges are the edge sets of the triangles in G. Show that, for any integers ℓ and r, there is a graph G satisfying $G \to (K^3)_r$ such that $\mathcal{H}_3(G)$ has girth $\geq \ell$.

Turán type results for subgraphs of random graphs

Generalized Turán number:

$$\mathrm{ex}(G, H) = \max\left\{|E(G')| \colon H \not\subset G' \subset G\right\}. \tag{7}$$

- $\triangleright$ $\mathrm{ex}(n, H) = \mathrm{ex}(K^n, H)$
- $\triangleright$ $\mathrm{ex}(Q^d, C^4) = ?$
- $\triangleright$ $\mathrm{ex}(G, K^h) = ?$ for (n, d, λ)-graphs G

Turán type results for subgraphs of random graphs

Exercise 9: show that, for any G and H, we have

$$e(G, H) \geq \left(1 - \frac{1}{\chi(H) - 1}\right) e(G). \tag{8}$$

▷ Interested in knowing when this is sharp for $G = G(n, p)$ (up to $o(e(G))$)

Turán type results for subgraphs of random graphs

Theorem 8. *Let* H *be a graph with degeneracy* d *and suppose* $np^d \gg 1$*. Then*

$$\mathrm{ex}(G(n,p),H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) p \binom{n}{2}. \qquad (9)$$

Conjecture 9. *For any graph* H*, if* $np^{m_2(H)} \to \infty$ *then* (9) *holds almost surely.*

Example: if $H = K^k$, have $m_2(H) = (k+1)/2$, but Theorem 8 supposes $np^{1/(k-1)} \gg 1$.

Cycles and small cliques

Theorem 10. *Conjecture 9 holds for cycles.*

Theorem 11. *Conjecture 9 holds for* K^4, K^5, *and* K^6.

A sharp result for even cycles

Theorem 12. *Let* $k \geq 2$ *be an integer and let* $p = p(n) = \alpha n^{-1+1/(2k-1)}$ *be such that*

$$2 \leq \alpha \leq n^{1/(2k-1)^2}. \tag{10}$$

Then

$$\mathrm{ex}(G(n,p), C^{2k}) \asymp \frac{(\log \alpha)^{1/(2k-1)}}{\alpha} e(G(n,p)). \tag{11}$$

Szemerédi's regularity lemma

1. Tool for identifying the quasirandom structure of deterministic graphs
2. Works very well for large, dense graphs: n-vertex graphs with $\geq cn^2$ edges, $n \to \infty$
3. Variant for sparse graphs exists (sparse = with $o(n^2)$ edges)
4. Much harder to use

ε-regularity

Definition 13 (ε-regular pair)**.** $G = (V, E)$ *a graph;* U*,* $W \subset V$ *non-empty and disjoint. Say* (U, W) *is* ε*-regular* $($*in* $G)$ *if*

▷ *for all* $U' \subset U$*,* $W' \subset W$ *with* $|U'| \geq \varepsilon|U|$ *and* $|W'| \geq \varepsilon|W|$*, we have*

$$\left|\frac{|E(U', W')|}{|U'||W'|} - \frac{|E(U, W)|}{|U||W|}\right| \leq \varepsilon. \tag{12}$$

ε-regularity

The pair (U, W) is ε-regular if

$$\left| \frac{|E(U', W')|}{|U'||W'|} - \frac{|E(U, W)|}{|U||W|} \right| \le \varepsilon. \tag{13}$$

Equivalently,

$$|E(U', W')| = |U'||W'| \left(\frac{|E(U, W)|}{|U||W|} \pm \varepsilon \right) \tag{14}$$

Clearly, not meaningful if

$$\frac{|E(U, W)|}{|U||W|} \to 0 \tag{15}$$

and ε is fixed. (We think of $G = (V, E)$ with $n = |V| \to \infty$.)

ε-regularity; multiplicative error version

Replace

$$|E(U', W')| = |U'||W'| \left(\frac{|E(U, W)|}{|U||W|} \pm \varepsilon \right) \tag{16}$$

by

$$|E(U', W')| = (1 \pm \varepsilon)|E(U, W)| \frac{|U'||W'|}{|U||W|} \tag{17}$$

Altered condition becomes

▷ *for all* $U' \subset U$, $W' \subset W$ *with* $|U'| \geq \varepsilon|U|$ *and* $|W'| \geq \varepsilon|W|$, *we have*

$$\left| |E(U', W')| - |E(U, W)| \frac{|U'||W'|}{|U||W|} \right| \leq \varepsilon |E(U, W)| \frac{|U'||W'|}{|U||W|}. \tag{18}$$

OK even if $|E(U, W)|/|U||W| \to 0$.

Szemerédi's regularity lemma, sparse version

Any graph with no 'dense patches' admits a Szemerédi partition with the new notion of ε-regularity.

Definition 14 ((η, b)-bounded). *Say* $G = (V, E)$ *is* (η, b)*-bounded if for all* $U \subset V$ *with* $|U| \geq \eta|V|$*, we have*

$$\#\{\textit{edges within } U\} \leq b|E|\binom{|U|}{2}\binom{|V|}{2}^{-1}. \tag{19}$$

[$\triangleright$ Something like one-sided (p, η)-uniformity]

Szemerédi's regularity lemma, sparse version

Theorem 15 (The regularity lemma)**.** *For any* $\varepsilon > 0$, $t_0 \geq 1$, *and* b, *there exist* $\eta > 0$ *and* T_0 *such that any* (η, b)*-bounded graph* G *admits a partition* $V = V_1 \cup \cdots \cup V_t$ *such that*

1. $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$
2. $t_0 \leq t \leq T_0$
3. *at least* $(1 - \varepsilon)\binom{t}{2}$ *pairs* (V_i, V_j) $(i < j)$ *are* ε*-regular.*

Proof*.* Just follow Szemerédi's original proof. □

A counting lemma (simplest version)

Setup. $G = (V_1, V_2, V_3; E)$ tripartite with

1. $|V_i| = m$ for all i
2. (V_i, V_j) ε-regular for all $i < j$
3. $|E(V_i, V_j)| = \rho m^2$ for all $i < j$

Notation: $G = G_3^{(\varepsilon)}(m, \rho)$ [G is an ε-regular triple with density ρ]

▷ Wish to embed K^3 with $V(K^3) = \{x_1, x_2, x_3\}$ such that x_i is placed in V_i.

A counting lemma (simplest version)

Just like random:

Lemma 16 (Counting Lemma; Embedding lemma). $\forall \rho > 0$, $\delta > 0$ $\exists \varepsilon > 0$, m_0: *if* $m \geq m_0$, *then*

$$\left| \#\{K^3 \hookrightarrow G_3^{(\varepsilon)}(m,\rho)\} - \rho^3 m^3 \right| \leq \delta m^3. \qquad (20)$$

Tough: counting Lemma is false if $\rho \longrightarrow 0$

Fact 17. $\forall \varepsilon > 0\ \exists \rho > 0,\ m_0\ \forall m \geq m_0\ \exists G_3^{(\varepsilon)}(m, \rho)$ *with*

$$K^3 \not\subset G_3^{(\varepsilon)}(m, \rho). \tag{21}$$

[cf. Lemma 16]

Change of focus: from *counting* $K^3 \subset G_3^{(\varepsilon)}(m, \rho)$ to *existence of* $K^3 \subset G_3^{(\varepsilon)}(m, \rho)$

Key observation

Counterexamples to the embedding lemma in the sparse setting do exist (Fact 17), but

are extremely rare.

An asymptotic enumeration lemma

Lemma 18. $\forall\beta > 0\ \exists\varepsilon > 0,\ C > 0,\ m_0$: *if* $T = \rho m^2 \geq C m^{3/2}$*, then*

$$\#\{G_3^{(\varepsilon)}(m,\rho) \not\supset K^3\} \leq \beta^T \binom{m^2}{T}^3. \tag{22}$$

Observe that $\rho \geq C/\sqrt{m} \to 0$.

Consequence for random graphs

Easy expectation calculations imply

- if $p \gg 1/\sqrt{n}$, then almost every $G(n,p)$ is such that

$$\left(K^3\text{-free } G_3^{(\varepsilon)}(m,\rho)\right) \not\subset G(n,p), \tag{23}$$

  if (*) $mp \gg \log n$ and $\rho \geq \alpha p$ for some fixed α.

Conclusion. Recovered an 'embedding lemma' in the sparse setting, *for subgraphs of random graphs*.

Corollary 19 (EL for subgraphs of r.gs). *If $p \gg 1/\sqrt{n}$ and $(*)$ holds, then almost every $G(n,p)$ is such that if $G_3^{(\varepsilon)}(m,\rho) \subset G(n,p)$, then*

$$\exists \iota \colon K^3 \hookrightarrow G_3^{(\varepsilon)}(m,\rho) \subset G(n,p). \tag{24}$$

The easy calculation

Recall $T = \rho m^2$, $\rho \geq \alpha p$, and $mp \gg \log n$. Therefore

$$\mathbb{E}(\#\{K^3\text{-free } G_3^{(\varepsilon)}(m,\rho) \hookrightarrow G(n,p)\}) \leq n^{3m} o(1)^T \binom{m^2}{T}^3 p^{3T}$$

$$\leq n^{3m}\left(o(1)\frac{em^2}{T}\right)^{3T} p^{3T} \leq n^{3m}\left(o(1)\frac{em^2 p}{T}\right)^{3T}$$

$$= n^{3m}\left(o(1)\frac{em^2 p}{\rho m^2}\right)^{3T} \leq n^{3m}\left(o(1)\frac{e}{\alpha}\right)^{3T}$$

$$\leq e^{3m\log n}\left(o(1)\frac{e}{\alpha}\right)^{3\alpha m \log n} = o(1).$$

Superexponential bounds

Suppose we wish to prove a statement about all subgraphs of $G(n,p)$.

- Too many such subgraphs: about $2^{p\binom{n}{2}}$
- $G(n,p)$ has no edges with probability $(1-p)^{\binom{n}{2}} \geq \exp\{-2pn^2\}$, if, say, $p \leq 1/2$.
- Concentration inequalities won't do ($2^{p\binom{n}{2}}$ vs e^{-2pn^2}).
- Bounds of the form

$$o(1)^T \binom{\binom{m}{2}}{T} \tag{25}$$

for the cardinality of a family of 'undesirable subgraphs' $\mathcal{U}(m,T)$ do the job. Use of such bounds goes back to Füredi (1994).

An embedding lemma for K^3 (sparse setting)

Scheme:

1. Proved an asymptotic enumeration lemma:

$$\#\{G_3^{(\varepsilon)}(m,\rho) \not\supset K^3\} = o(1)^T \binom{m^2}{T}^3. \qquad (26)$$

   [$T = \rho m^2$]

2. Observed that this implies a.e. $G(n, \omega/\sqrt{n})$ contains no K^3-free $G_3^{(\varepsilon)}(m,\rho)$ [any $\omega = \omega(n) \to \infty$ as $n \to \infty$].

3. Obtained a K^3-embedding lemma for subgraphs of $G(n,p)$, even when $p = \omega/\sqrt{n}$: for all $G_3^{(\varepsilon)}(m,\rho) \subset G(n,p)$, have $K^3 \subset G_3^{(\varepsilon)}(m,\rho)$.

General graphs H?

Let us consider the case $H = K^k$.

Conjecture 20. $\forall k \geq 4,\ \beta > 0\ \exists \varepsilon > 0,\ C > 0,\ m_0$: *if* $T = \rho m^2 \geq Cm^{2-2/(k+1)}$, *then*

$$\#\{G_k^{(\varepsilon)}(m, \rho) \not\supset K^k\} \leq \beta^T \binom{m^2}{T}^{\binom{k}{2}}. \tag{27}$$

▷ Known for $k = 3$, $k = 4$, and $k = 5$

▷ For general H, the conjecture involves $m_2(H)$

General graphs H?

Suppose $|V(H)| > 2$. Then let

$$d_2(H) = \frac{|E(H)| - 1}{|V(H)| - 2}. \tag{28}$$

For $H = K^1$ and $2K^1$ let $d_2(H) = 0$; set $d_2(K^2) = 1/2$. Finally, let

$$m_2(H) = \max\{d_2(J) \colon J \subset H\}. \tag{29}$$

Conjecture 21. $\forall H, \beta > 0 \; \exists \varepsilon > 0, C > 0, m_0$: *if* $T = \rho m^2 \geq C m^{2-1/m_2(H)}$, *then*

$$\#\{G_H^{(\varepsilon)}(m, \rho) \not\supset H\} \leq \beta^T \binom{m^2}{T}^{e(H)}. \tag{30}$$

▷ Known for cycles

The bipartite case

Conjecture 22. *Suppose* H *is a bipartite graph. For any* $\beta > 0$*, there is* C *such that for any* $M = M(n)$ *such that* $M \geq Cn^{2-1/m_2(H)}$*, we have*

$$\#\{G(n, M) \not\supset H\} \leq \beta^M \binom{\binom{n}{2}}{M} \tag{31}$$

for all large enough n*.*

In short: $\#\{G(n, M) \not\supset H\} = o(1)^M \binom{\binom{n}{2}}{M}$

- Known for even cycles [Exercise 10^{++}: still $^{++}$, even after regularity. What is simpler to show is that $\mathrm{ex}(G(n,p), C^{2k}) = o(p\binom{n}{2})$ if $p \gg n^{-1+1/(2k-1)}$.]

Consequences of the conjecture

1. The Rödl–Ruciński theorem on threshold for Ramsey properties of random graphs (1-statement) and the Turán counterpart, with the best possible threshold

2. Łuczak (2000): structural and enumerative consequences for H-free graphs on n vertices and M edges

Consequences of the conjecture

Under the hypothesis that Conjecture 21 holds for a graph H:

Theorem 23. *Conjecture 22 holds: if H is bipartite, then for every $\beta > 0$ there exists $C = C(\beta, H)$ and n_0 such that for $n \geq n_0$ and $M \geq Cn^{2-1/m_2(H)}$ we have*

$$\#\{G(n, M) \not\supset H\} \leq \beta^M \binom{\binom{n}{2}}{M}. \tag{32}$$

Consequences of the conjecture

Under the hypothesis that Conjecture 21 holds for a graph H:

Theorem 24. *Suppose $\chi(H) = h \geq 3$. Then for every $\delta > 0$ there exists $C = C(\delta, H)$ such that, almost surely, a graph chosen uniformly at random from the family of all H-free labelled graphs on n vertices and $M \geq Cn^{2-1/m_2(H)}$ edges can be made $(h-1)$-partite by removing $\leq \delta M$ edges.*

Consequences of the conjecture

Under the hypothesis that Conjecture 21 holds for a graph H:

Theorem 25. *Suppose $\chi(H) = h \geq 3$. Then for every $\varepsilon > 0$ there exist $C = C(\varepsilon, H)$ and $n_0 = n_0(\varepsilon, H)$ such that, for $n \geq n_0$ and $Cn^{2-1/d_2(H)} \leq M \leq n^2/C$, we have*

$$\left(\frac{h-2}{h-1} - \varepsilon\right)^M \leq \mathbb{P}\left(G(n, M) \not\supset H\right) \leq \left(\frac{h-2}{h-1} + \varepsilon\right)^M. \qquad (33)$$

The hereditary nature of regularity

Setup. $B = (U, W; E)$ an ε-regular bipartite graph with $|U| = |W| = m$ and $|E| = \rho m^2$, $\rho > 0$ constant, and an integer d. Sample $N \subset U$ and $N' \subset W$ with $|N| = |N'| = d$ uniformly at random.

Theorem 26. *For any $\beta > 0$, $\rho > 0$, and $\varepsilon' > 0$, if $\varepsilon \leq \varepsilon_0(\beta, \rho, \varepsilon')$, $d \geq d_0(\beta, \rho, \varepsilon')$, and $m \geq m_0(\beta, \rho, \varepsilon')$, then*

$$\mathbb{P}\Big((N, N') \text{ bad}\Big) \leq \beta^d, \tag{34}$$

where (N, N') is bad if $\Big||E(N, N')|d^{-2} - \rho\Big| > \varepsilon'$ or else (N, N') is not ε'-regular.

The hereditary nature of regularity

Exercise 11^{++}: for any k and $\delta > 0$, there is C such that the following holds. If $\chi(G - F) \geq k$ for any $F \subset E(G)$ with $|F| \leq \delta n^2$, then there is $H \subset G$ with $\chi(H) \geq k$ and $|V(H)| \leq C$. Can you guarantee many such 'witnesses' H?

Local characterization for regularity

Setup. $B = (U, W; E)$, a bipartite graph with $|U| = |W| = m$. Consider the properties

(PC) for some constant p, have $m^{-1} \sum_{u \in U} |\deg(u) - pm| = o(m)$ and

$$\frac{1}{m^2} \sum_{u,u' \in U} |\deg(u, u') - p^2 m| = o(m). \tag{35}$$

(R) (U, W) is $o(1)$-regular (classical sense).

Theorem 27. (PC) *and* (R) *are equivalent.*

A proof of Theorem 26

Let a graph $F = (U, E)$ with $|U| = m$ and $|E| \leq \eta\binom{m}{2}$ be given. Suppose we select a d-set N uniformly at random from U. We are then interested in giving an upper bound for $e(F[N])$, the number of edges that the set N will induce in F.

Lemma 28. *For every α and $\beta > 0$, there exist $\eta_0 = \eta_0(\alpha, \beta) > 0$ such that, whenever $0 < \eta \leq \eta_0$, we have*

$$\mathbb{P}\left(e(F[N]) \geq \alpha\binom{d}{2}\right) \leq \beta^d. \tag{36}$$

Proof. Exercise 12^+. □

Exercise 13^{++}: use Lemma 28 and Theorem 27 to prove Theorem 26.

The hereditary nature of sparse regularity

Definition 29 (((ε, p)-lower-regularity). *Suppose $0 < \varepsilon < 1$ and $0 < p \leq 1$. A bipartite graph $B = (U, W; E)$ is (ε, p)-lower-regular if for all $U' \subset U$ and $W \subset W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$, we have*

$$\frac{e(U', W')}{|U'||W|} \geq (1 - \varepsilon)p. \tag{37}$$

The hereditary nature of sparse regularity

Setup. $B = (U, W; E)$ an (ε, p)-lower-regular bipartite graph with $|U| = |W| = m$ and $|E| = pm^2$ and integer d. Sample $N \subset U$ and $N' \subset W$ with $|N| = |N'| = d$ uniformly at random.

Theorem 30. *For all $0 < \beta, \varepsilon' < 1$, there exist $\varepsilon_0 = \varepsilon_0(\beta, \varepsilon') > 0$ and $C = C(\varepsilon')$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < p < 1$, the following holds. Let $G = (U, W; E)$ be an (ε, p)-lower-regular bipartite graph and suppose $d \geq Cp^{-1}$. Then*

$$\mathbb{P}\Big((N, N') \text{ bad}\Big) \leq \beta^d, \tag{38}$$

where (N, N') is bad if (N, N') is not (ε', p)-lower-regular.

The hereditary nature of sparse regularity

Setup. $B = (U, W; E)$ an (ε, p)-lower-regular bipartite graph with $|U| = |W| = m$, $|E| = pm^2$, and $p \geq \alpha q$. Also, suppose we have two other bipartite graphs $A = (U', U; E_A)$ and $C = (W, W'; E_C)$, also (ε, p)-lower-regular.

Corollary 31 (Quite imprecise...). *Suppose* $A \cup B \cup C \subset G(n, q)$ *and* $u' \in U'$ *and* $w' \in W'$ *are 'typical' vertices. If* $pm \gg 1/p$ *then* $B[\Gamma_A(u'), \Gamma_C(w')]$ *is* $(f(\varepsilon), p)$*-lower-regular* $(f(\varepsilon) \to 0$ *as* $\varepsilon \to 0)$*.*

▷ Corollary above may be used in inductive embedding schemes.

Local characterization for sparse regularity

Very imprecisely: a similar statement to Theorem 27 may be proved for subgraphs of random graphs.

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Yoshi and Yoshi, admiring your patience for being still in the room