Random Graphs II

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Outline of Lecture II

1. **Subgraph containment:** small subgraphs ($1 - o(1)$ probability, $1 - e^{-\Omega(\mu)}$ probability, $1 - e^{-\omega\mu}$)
2. **Subgraph containment:** large (and sparse) subgraphs (matchings, long paths, Hamilton cycles, bounded degree subgraphs)
3. **Subgraph containment with adversary:** existence of subgraphs in colourings and 'dense' subgraphs (Ramsey type results and Turán type results) [mostly won't get there today]

Subgraphs in r.gs: small subgraphs

Definition 1 (Density and $m(H)$; balanced graphs)**.** *The density $d(H)$ of a graph H with $|V(H)| > 0$ is*

$$|E(H)|/|V(H)| \quad (1)$$

$[= (1/2) \times$ *average degree*$]$. *We also set*

$$m(H) = \max\{d(J) \colon J \subset H,\ |V(J)| > 0\}. \quad (2)$$

We say that H is balanced if $\max$ in (2) achieved by $J = H$.

$\triangleright$ *Simple*: $\mathbb{E}(\#\{J \hookrightarrow G(n,p)\}) = o(1)$ if $p \ll n^{-1/d(J)}$, where $\#\{J \hookrightarrow G(n,p)\}$ is the number of embeddings of J into $G(n,p)$. This implies that almost no $G(n,p)$ contains J for such a p.

$\triangleright$ Exercise 1: find nice classes of balanced graphs.

Subgraphs in r.gs: small subgraphs

Theorem 2. *The threshold function for the event* $\{H \subset G(n,p)\}$ *is* $p_0 = n^{-1/m(H)}$*.*

Proof*.* We have already seen the 0-statement. Just need to show the 1-statement. Compute the variance and apply the second moment method. For the variance, use $\mathrm{Var}(X) = \sum_{(H',H'')} \mathrm{Cov}(X_{H'}, X_{H''})$, where $X = \sum_{H'} X_{H'}$ and $X_{H'} = [H' \subset G(n,p)]$ and the sum is over all $H \hookrightarrow K^n$. Recall $\mathrm{Cov}(X, X') = 0$ if X and X' independent. We have to estimate $\mathrm{Var}(X) = \sum_{(H',H'')} \mathrm{Cov}(X_{H'}, X_{H''})$, where the sum is over *overlapping* pairs (H', H'') of copies of H. [Exercise 2: complete this proof]. $\square$

Probability of containment

▷ If $p = p_0/\omega$ and $\omega \to \infty$, then $\mathbb{P}(H \subset G(n,p)) \leq 1/\omega'$ for some $\omega' \to \infty$ polynomially related to ω. In fact, $\mathbb{P}(H \subset G(n,p)) \leq \Phi_H = 1/\omega'$, where

$$\Phi_H = \Phi_H(n,p) = \min\{\mathbb{E}(\#\{J \hookrightarrow G(n,p)\}) \colon J \subset H,\ |E(J)| > 0\}. \quad (3)$$

▷ If $p = p_0\omega$ and $\omega \to \infty$, then, writing $X = \#\{H \hookrightarrow G(n,p)\}$, we have $\mathbb{P}(X = 0) \leq \mathrm{Var}(X)/\mathbb{E}(X)^2 = 1/\omega'$ for some $\omega' \to \infty$ polynomially related to ω. In fact, we have $\mathrm{Var}(X)/\mathbb{E}(X)^2 = O(1/\Phi_H) = 1/\omega'$.

Probability of containment

Recall

$$\Phi_H = \Phi_H(n,p) = \min\{\mathbb{E}(\#\{J \hookrightarrow G(n,p)\}) : J \subset H,\ |E(J)| > 0\}. \qquad (4)$$

We concluded

$$1 - \Phi_H \leq \mathbb{P}(H \not\subset G(n,p)) = O(1/\Phi_H). \qquad (5)$$

Can we do better? [*Application*: Can we approach the problem "$G(n,p) \to (K^3)^v_2$?" with the union bound?]

Theorem 3. *Suppose* $|E(H)| > 0$*. Then, for any* $p = p(n) < 1$*, we have*

$$\exp\left\{-\frac{1}{1-p}\Phi_H\right\} \leq \mathbb{P}(H \not\subset G(n,p)) \leq \exp\left\{-\Theta(\Phi_H)\right\}. \qquad (6)$$

An application

▷ **Therefore, can do better!** *Application*: show that if $p = Cn^{-2/3}$ and C is a large enough constant, then almost every $G(n, p)$ is such that $G(n, p) \to (K^3)^v_2$, that is, any colouring of the vertices of $G(n, p)$ with 2 colours necessarily contains a monochromatic K^3. [Exercise 3: prove this statement. Generalize it from K^3 to arbitrary graphs H and to more than 2 colours.]

The FKG inequality

[We just stick to random graphs] Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two *increasing* graph properties. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two *decreasing* graph properties.

Theorem 4. *The following hold:*

(i) $\mathbb{P}(G(n,p) \in \mathcal{P}_1 \cap \mathcal{P}_2) \geq \mathbb{P}(G(n,p) \in \mathcal{P}_1)\mathbb{P}(G(n,p) \in \mathcal{P}_2)$

(ii) $\mathbb{P}(G(n,p) \in \mathcal{Q}_1 \cap \mathcal{Q}_2) \geq \mathbb{P}(G(n,p) \in \mathcal{Q}_1)\mathbb{P}(G(n,p) \in \mathcal{Q}_2)$

(iii) $\mathbb{P}(G(n,p) \in \mathcal{P}_1 \cap \mathcal{Q}_2) \leq \mathbb{P}(G(n,p) \in \mathcal{P}_1)\mathbb{P}(G(n,p) \in \mathcal{Q}_2)$

$\triangleright$ Remark: (*i*) is equivalent to $\mathbb{P}(G(n,p) \in \mathcal{P}_1 \mid \mathcal{P}_2) \geq \mathbb{P}(G(n,p) \in \mathcal{P}_1)$ and (*iii*) is equivalent to $\mathbb{P}(G(n,p) \in \mathcal{P}_1 \mid \mathcal{Q}_2) \leq \mathbb{P}(G(n,p) \in \mathcal{P}_1)$.

$\triangleright$ Exercise 4: How do the probabilities $\mathbb{P}(G(n,p) \text{ is Hamiltonian})$ and $\mathbb{P}(G(n,p) \text{ is Hamiltonian} \mid G(n,p) \text{ is planar})$ compare?

The FKG inequality

Remark 5. *In fact, in Theorem 4, one may leave out the hypothesis that the $\mathcal{P}_i$ and the $\mathcal{Q}_i$ are closed under isomorphism.*

The FKG inequality

We consider the decreasing events $\{X_{J'} = 0\}$, where J' ranges over all copies of a $J \subset H$ that achieves the minimum in the definition of Φ_H (see (4)): that is, $\Phi_H = \mathbb{E}(\#\{J \hookrightarrow G(n,p)\})$.

FKG implies that

$$\mathbb{P}(J \not\subset G(n,p)) = \mathbb{P}(X_{J'} = 0 \text{ for all } J') \geq \prod_{J'} \mathbb{P}(X_{J'} = 0) = \prod_{J'} (1 - p^{e(J)}). \tag{7}$$

Using $1 - x \geq e^{-x/(1-x)}$, we get that $\mathbb{P}(J \not\subset G(n,p))$ is

$$\geq \exp\left\{-\frac{1}{1-p^{e(J)}}\mathbb{E}(\#\{J \hookrightarrow G(n,p)\})\right\} \geq \exp\left\{-\frac{1}{1-p}\Phi_H\right\}. \tag{8}$$

This proves the lower bound in Theorem 3.

Janson's inequality

[We just stick to random graphs] Let H be fixed. Let $X = \#\{H \hookrightarrow G(n,p)\}$. We have $X = \sum_{H'} X_{H'}$, where the sum ranges over all copies H' of H in K^n and $X_{H'} = [H' \subset G(n,p)]$. Set

$$\Delta^* = \sum_{(H',H'')} \mathbb{E}(X_{H'} X_{H''}), \tag{9}$$

where the sum is over all pairs (H', H'') of copies of H with at least one common edge. Note that this is very similar to

$$\mathsf{Var}(X_H) = \sum_{(H',H'')} \mathbb{E}(X_{H'} X_{H''}) - \mathbb{E}(X_H)\mathbb{E}(X_{H''}). \tag{10}$$

Janson's inequality

Put $\mu = \mathbb{E}(X) = \mathbb{E}(\#\{H \hookrightarrow G(n, p)\})$.

Exercise 5: $\Delta^* = \Theta(\mu^2/\Phi_H)$.

Exercise 6: $\mathrm{Var}(X_H) = \Theta(\mu^2/\Phi_H)$ if p is bounded away from 1 (and $= O(\mu^2/\Phi_H)$ always).

Janson's inequality

Theorem 6. *Let* $\mu = \mathbb{E}(X_H)$. *Then*

$$\mathbb{P}(H \not\subset G(n,p)) \leq \exp\left\{-\frac{\mu^2}{\Delta^*}\right\} = \exp\{-\Theta(\Phi_H)\}. \tag{11}$$

▷ Got the upper bound in Theorem 3.

The $G(n, M)$ model

Let us briefly discuss $\mathbb{P}(H \not\subset G(n, M))$ for small subgraphs H.

- **Threshold:** $n^{2-1/m(H)}$
- Analogue of Theorem 3?
  - Define $\Phi_H = \Phi_H(n, M)$ as $\Phi_H(n, p)$ with $p = M/\binom{n}{2}$.
  - The bounds in Theorem 3 cannot be true for all $M = M(n)$: if $M < e(H)$ and if $M > \text{ex}(n, H)$, then we know $\mathbb{P}(H \not\subset G(n, p))$ quite precisely!

The $G(n, M)$ model

Theorem 7. *If* $M \geq e(H)$, *then*

$$\mathbb{P}(H \not\subset G(n, M)) \leq \exp\{-\Theta(\Phi_H)\}. \quad (12)$$

Theorem 8. *If* H *is such that* $M \geq c\Phi_H$ *for some suitably small constant* $c = c(H) > 0$, *then*

$$\mathbb{P}(H \not\subset G(n, M)) \geq \exp\{-\Theta(\Phi_H)\}. \quad (13)$$

Theorem 9. *If* H *is such that* $M \geq c\Phi_H$ *for some constant* $c > 0$, *and it is not bipartite and* $M \leq c\binom{n}{2}$ *for some constant* $c < 1 - 1/(\chi(H) - 1)$, *then* (13) *also holds*

Exercise 7^+: Prove the above three theorems. Particular interest (and quick): Theorem 7 when $\Phi_H \gg \log n$

The $G(n, M)$ model

Conjecture 10. *Suppose H is a bipartite graph. For any $\beta > 0$, there is C_0 such that for any $M = M(n)$ such that $\Phi_H \geq C_0 M$, we have*

$$\mathbb{P}(H \not\subset G(n, M)) \leq \beta^M \tag{14}$$

for all large enough n.

In short: $\mathbb{P}(H \not\subset G(n, M)) = o(1)^M$

- Known for even cycles [Exercise 8^{++} (now); simpler later, after the notion of sparse regularity]
- Known when M is larger

The $G(n, M)$ model

▷ If true Conjecture 10 would have interesting consequences.

Exercise 9: deduce a fault-tolerance result for $G(n, M)$ with respect to H. Estimate $f(n, \eta, H) = \min |E(\Gamma)|$, where Γ ranges over all graphs with the property $\Gamma \to_\eta H$.

Exercise 10: translate the hypothesis $\Phi_H \geq C_0 M$ to something nicer. Suppose $|V(H)| > 2$. Then let

$$d_2(H) = \frac{|V(E)| - 1}{|V(H)| - 2}. \tag{15}$$

For $H = K^1$ and $2K^1$ let $d_2(H) = 0$; set $d_2(K^2) = 1/2$. Finally, let

$$m_2(H) = \max\{d_2(J) : J \subset H\}. \tag{16}$$

Consider $M_0 = n^{2-1/m_2(H)}$.

Subgraphs in r.gs: large subgraphs

- ▷ Matchings in random bipartite graphs
- ▷ Matchings in random graphs
- ▷ Long paths in random graphs
- ▷ Hamilton cycles in random graphs
- ▷ Bounded degree spanning subgraphs in random graphs

Matchings in random bipartite graphs

Theorem 11. *Let* $p = C(\log n)/n$, *where* $C > 4$ *is some constant. Then a.e. random bipartite graph* $G(n, n; p)$ *contains a perfect matching.*

Remark 12. *For any* $r > 0$, *the probability of failure in Theorem 11 is* $\leq 1/n^r$ *if* $C \geq C(r)$ *and* $n \geq n_0(r)$. *For instance, for* $r = 1$, *it suffices to take* $C > 6$ *(and* $n \geq n_0$*).*

Matchings in random bipartite graphs

Proof of Theorem 11. Let U and W be the vertex classes of $G = G(n, n; p)$. Note that, by Hall's theorem, if there is no perfect matching, then there is a pair (X, Y) with $1 \leq |X| \leq \lceil n/2 \rceil$, $|Y| = n - |X|$, and $e(X, Y) = \emptyset$ and either with $X \subset U$ and $Y \subset W$ or else with $X \subset W$ and $Y \subset U$ (in fact, we may even get $|Y| = n - |X| + 1$). Let us estimate the expected number $\mathbb{E}(Z)$ of such pairs (X, Y). We have

$$\mathbb{E}(Z) = 2 \sum_{1 \leq k \leq \lceil n/2 \rceil} \binom{n}{k} \binom{n}{n-k} (1-p)^{k(n-k)}, \tag{17}$$

which is

$$\leq 2 \sum_{k \geq 1} \left(n^2 e^{-(C/2) \log n} \right)^k = o(1), \tag{18}$$

because $C > 4$. The result follows. □

Matchings in random graphs

Theorem 13. *Let*

$$p = \frac{1}{n}\left(\log n + c_n\right). \tag{19}$$

Then

$$\lim_{n\to\infty,\, n \text{ even}} \mathbb{P}(\nu(G(n,p)) = n/2) = \begin{cases} 0 & \text{if } \lim_n c_n = -\infty, \\ e^{-e^{-c}} & \text{if } \lim_n c_n = c \in \mathbb{R}, \\ 1 & \text{if } \lim_n c_n = \infty. \end{cases} \tag{20}$$

▷ **Heuristic (which may be turned precise):** leading obstructions are the isolated vertices.

Matchings in random graphs

Let $\{\delta \geq 1\}$ denote the event that the minimum degree is at least 1, and let $\textsc{Conn}$ denote the event that the graph is connected.

Theorem 14. *For almost every* $\mathbf{G} = (G_t)_{t=0}^{N}$ *with* n *even, we have*

$$\tau(\mathbf{G}, \textsc{Conn}) = \tau(\mathbf{G}, \nu = n/2) = \tau(\mathbf{G}, \delta \geq 1). \tag{21}$$

Exercise 11: deduce Theorem 13 from Theorem 14.

Long paths in random graphs

Let $\ell(G)$ denote the length of the longest path in G. We shall sketch the proof of the following.

Theorem 15. *For any $\varepsilon > 0$, there is $C = C(\varepsilon)$ such that if $p = C/n$, then $\ell(G(n,p)) \geq (1-\varepsilon)n$ almost surely.*

$\triangleright$ **Following is true:** even if $C = 1 + \varepsilon$ and $\varepsilon > 0$ is a small constant, we have $\ell(G(n,p)) \geq cn$ for some $c = c(\varepsilon) > 0$.

Long paths in random graphs

Definition 16. *The* k*-core of a graph* G *is its unique maximal subgraph with minimum degree at least* k *(possibly empty). Let us write* $\mathrm{core}_k(G)$ *for the* k*-core of* G*.*

Lemma 17. *For any integer* $k \geq 1$ *and any real* $\varepsilon > 0$*, there is* $C = C(k, \varepsilon)$ *such that if* $p = C/n$*, then* $|V(\mathrm{core}_k(G(n,p)))| \geq (1-\varepsilon)n$ *almost surely.*

Proof*.* Exercise 12. $\square$

Recap: expansion and bipartite Pósa

Definition 18 ((b, f)-expansion)**.** *Let* $B = (U, W; E)$ *be a bipartite graph with vertex classes* U *and* W *and edge set* E*. Let positive reals* b *and* f *be given. We say that* B *is* $(b, f; U)$*-expanding if, for every* $X \subset U$ *with* $|X| \leq b$*, we have* $|\Gamma(X)| \geq f|X|$*. If* B *is both* $(b, f; U)$*-expanding and* $(b, f; W)$*-expanding, we say that* B *is* (b, f)*-expanding.*

Lemma 19. *Let* $b \geq 1$ *be an integer. If the bipartite graph* B *is* $(b, 2)$*-expanding, then* B *contains a path* P^{4b} *on* $4b$ *vertices.*

Long paths in random graphs

Proof of Theorem 15 (Sketch). Fix an arbitrary constant $\delta > 0$. Choose $C = pn$ large so that the k-core H of $G = G(n,p)$ has at least $(1-\delta)n$ vertices, where k is some large constant (we shall need $C \gg k$). Split the vertex set of H into two parts U and W maximizing $e(U,W)$. Then every vertex sends at least as many edges to the opposite part as it does to its part. Also, if C is large enough, then $|U|$, $|W| \geq (1/2-\delta)n$. Prove that the induced bipartite graph $G[U,W]$ is $(b,2)$-expanding, for $b = (1/4-\delta)n$ (take $k \gg \sqrt{C}$). Apply the bipartite version of Pósa's lemma (Lemma 19). Take δ small enough with respect to ε. [Exercise 13: fill in the details.] $\square$

Long paths in random graphs

Let $\mathrm{circ}(G)$ be the length of the longest cycles in G.

Exercise 14: show that almost surely $G(n, p)$ has $\mathrm{circ}(G(n, p)) \geq (1-\varepsilon)n$ if $pn \geq C_\varepsilon$.

Hamilton cycles in random graphs

Theorem 20. *Let*

$$p = \frac{1}{n}\left(\log n + \log\log n + c_n\right). \tag{22}$$

Then

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \textit{ is Hamiltonian}) = \begin{cases} 0 & \textit{if } \lim_n c_n = -\infty, \\ e^{-e^{-c}} & \textit{if } \lim_n c_n = c \in \mathbb{R}, \\ 1 & \textit{if } \lim_n c_n = \infty. \end{cases} \tag{23}$$

▷ **Heuristic (which may be turned precise):** leading obstructions are vertices of degree < 2.

Hamilton cycles in random graphs

Let $\{\delta \geq 2\}$ denote the event that the minimum degree is at least 2 and let $\textsc{Ham}$ denote the event that the graph is Hamiltonian.

Theorem 21. *For almost every* $\mathbf{G} = (G_t)_{t=0}^{N}$, *we have*

$$\tau(\mathbf{G}, \textsc{Ham}) = \tau(\mathbf{G}, \delta \geq 2). \qquad (24)$$

Exercise 15: deduce Theorem 20 from Theorem 21.

We shall sketch the proof of a weak version of the results above:

Theorem 22. *If* $p = C(\log n)/n$ *and* C *is large enough, then* $G(n,p)$ *is almost surely Hamiltonian.*

Path rotation

Definition 23 (Path rotation)**.** *Let* $P = x_1 \ldots x_h$ *be an* x_1*-path in* G*: a path beginning at* x_1*, which we think of as rooted at* x_1*. Suppose* $\{x_j, x_h\} \in E(G)$*. The rotation of* P *with pivot* x_j *is the* x_1*-path*

$$P' = x_1 x_2 \ldots x_{j-1} x_j x_h x_{h-1} \ldots x_{j+1}, \qquad (25)$$

obtained by removing the edge $\{x_j, x_{j+1}\}$ *and adding the edge* $\{x_j, x_h\}$*.*

Definition 24 (Left and right rotation)**.** *Let* $P = x_1 \ldots x_h$ *be an* x_1*-path. We shall consider* x_1 *as the left endvertex of* P *and* x_h *as the right endvertex of* P*. A right rotation (resp., left rotation) will be a rotation of* P *considered as an* x_1*-path (resp.,* x_h*-path). Thus, a right rotation preserves the left endvertex and vice-versa.*

Pósa's lemma

Definition 25 (Transform). *A transform of a path P is a path obtained by applying a sequence of rotations to P. A right transform (resp., left transform) of P is a path obtained by applying a sequence of right rotations (resp., left rotations) to P.*

The following subtle lemma is central in Posá's method.

Lemma 26. *Let $P = x_1x_2 \dots x_h$ be a longest x_1-path in a graph G and let U be the set of right endvertices of the right transforms of P. Set*

$$N = \left\{x_i \colon 1 \le i < h, \{x_{i-1}, x_{i+1}\} \cap U \ne \emptyset\right\} \tag{26}$$

and $R = V(P) \setminus (U \cup N)$. Then the graph G contains no U–R edge.

Proof*.* Exercise 16. □

Pósa's lemma

Corollary 27. *Let $u \geq 1$ be an integer. Suppose a graph G is such that*

$$|U \cup \Gamma(U)| \geq 3|U| \tag{27}$$

for all $U \subset V(G)$ with $|U| \leq u$. Then G contains a path P^{3u} on $3u$ vertices.

Proof. Exercise 17. □

Exercise 18: prove Lemma 19.

Pósa's lemma

Lemma 28. *Let* u *and* $h \geq 2$ *be integers. Suppose* G *is such that* (27) *holds for all* $U \subset V(G)$ *with* $|U| < u$*. Suppose further that* $\ell(G) = h$ *and* $\mathrm{circ}(G) \leq h$*. Then there there are* $\geq \binom{u+1}{2}$ *vertex pairs that are not edges of* G *such that the addition of any of them to* G *creates a cycle of length* $h + 1$*.*

Proof*.* Fix a longest path P, and suppose U is the set of right endvertices of the right transforms of P. By Lemma 26 and (27), we have that $|U| \geq u$. Consider $y_1, \ldots, y_u \in U$, and consider the u right transforms naturally associated with these y_i. Let the set of left endvertices of the left transforms of these u paths be $Y_1, \ldots, Y_u$. We again have $|Y_i| \geq u$. All the y_i–Y_i pairs are such that their addition creates a cycle of length $h + 1$. □

Hamilton cycles in random graphs

Proof of Theorem 22 (Sketch). It follows from the $(p, e^{3/2}\sqrt{d})$-bijumbledness of $G = G(n, p)$ that G is expanding enough to guarantee paths of length $n - t$ where $t = \lfloor n/\log\log n \rfloor$, say. (See the proof of Theorem 15.) Now let $q = D(\log n)/n^2$ and consider $G \cup \bigcup_{1 \le i \le t} G(n, q)$: that is, add t independent copies of $G(n, q)$ to G. Let $G_j = G \cup \bigcup_{1 \le i \le j} G(n, q)$ ($0 \le j < t$). The probability that $\ell(G_j) > \ell(G_{j-1})$ fails is at most, say, $1/n^2$ (choosing D large). The final step is to prove that $G \cup \bigcup_{1 \le i \le t} G(n, q)$ is indeed Hamiltonian. [Exercise 19: fill in the details.] □

Hamilton cycles in bipartite random graphs

Problem 20^{++}: prove the analogue of Theorem 22 for the random bipartite graph $G(n, n; p)$.

The analogue of Theorem 21 is also known to hold.

Bounded degree spanning subgraphs of random graphs

A problem of Bollobás: let Q^d be the d-dimensional hypercube.

Problem 29. *For which* p *do we have* $Q^d \subset G(2^d, p)$ *almost surely* $(n = 2^d \to \infty)$*?*

Theorem 30 (Alon and Füredi 1992)**.** *Let* $H = H^n$ *satisfy* $\Delta(H) \leq d$*, and* $n/(d^2+1) \geq N_0$*, where* N_0 *is some suitable absolute constant. Let* $p = p(n)$ *be such that*

$$p^d \lfloor n/(d^2+1) \rfloor \geq 7 \log \lfloor n/(d^2+1) \rfloor. \tag{28}$$

Then $G(n,p)$ *fails to contain* H *with probability* $\leq 2d^2(d^2+1)/n$*.*

Bounded degree spanning subgraphs of random graphs

Corollary 31. *Let* $p > 1/2$*. Then a.e.* $G(2^d, p)$ *contains* Q^d *as a subgraph.*

Proof*.* Exercise 21. □

Corollary 32. *Let* $H = H^n$ *have* $\Delta(H) \le d$*. If* $p = C((\log n)/n)^{1/d}$ *for some large absolute* C*, then a.e.* $G(n, p)$ *contains* H *as a subgraph.*

Proof*.* Exercise 22 (d is not necessarily a constant). □

Remark 33. *Let* $H = H^n$ *be* d*-regular. Then* $\mu = \mathbb{E}(\#\{H \hookrightarrow G(n, p)\}) = n!p^{nd/2}$*. If* $p = n^{-2/d}$*, then* $\mu = o(1)$*. In particular, almost no* $G(2^d, 1/4)$ *contains* Q^d*.*

Bounded degree spanning subgraphs of random graphs

Exercise 23: let L_k be the $k \times k$ square lattice, that is, the graph on the $(i, j) \in [k] \times [k]$ with two such pairs joined by an edge if they differ by 1 in one coordinate. Find p_- such that almost no $G(k^2, p_-)$ contains L_k. Using Theorem 30, find p_+ such that almost every $G(k^2, p_+)$ contains L_k.

Bounded degree spanning subgraphs of random graphs

Riordan (2000) resolved the spanning hypercube and the spanning lattice problem as follows.

Theorem 34. *Let* $p = 1/4 + 6(\log d)/d$*. Then almost every* $G(2^d, p)$ *contains a* Q^d*.*

Theorem 35. *Let* $p \gg 1/k$*. Then almost every* $G(k^2, p)$ *contains* L_k*.*

Open problem 24: the $k \times k$ comb (Kahn). Some partial results known for spanning bounded degree trees.

Bounded degree spanning subgraphs of random graphs

Proof of Theorem 30 (Sketch). Let us just give the steps of the proof.

- ▷ Apply the theorem of Hajnal and Szemerédi to $H^{\leq 2}$: obtain a partition $V(H) = U_1 \cup \cdots \cup U_D$, where $D = d^2 + 1$, with $U_i = \lfloor n/D \rfloor$ or $\lceil n/D \rceil$ for all i and $\Delta(H[U_i, U_j]) \leq 1$ for all $i \neq j$ (that is, we have at most a matching between U_i and U_j).

- ▷ Partition $V(G(n, p))$ as $\bigcup_{1 \leq i \leq D} W_i$ with $|W_i| = |U_i|$ for all i. We embed H into $G = G(n, p)$ by defining bijections $f_i \colon U_i \to W_i$ for $i = 1, \ldots, D$ in turn.

Bounded degree spanning subgraphs of random graphs

Proof of Theorem 30 (Sketch). (Cont'd)

- Take any bijection for $f_1 \colon U_1 \to W_1$. Having suceeded in defining $f_i \colon U_i \to W_i$ for $i < j$ ($1 < j \le D$), we define f_j.
- Generate the edges in the random bipartite graph $G[\bigcup_{1\le i<j} W_i, W_j]$.
- **Crucial observation:** the probability that the required bijection

$$f_j \colon U_j \to W_j \tag{29}$$

  fails to exist is at most the probability that a perfect matching fails to exist in the random bipartite graph $G(U_j, W_j; p^d)$. But this probability is $\le 1/\lfloor n/D \rfloor$ (recall Remark 12). We need this not to fail $D-1$ times.
- Done! [Exercise 25: fill in the details. In particular, why did we square the graph H?]