SOME OF MY FAVOURITE PROBLEMS IN NUMBER THEORY, COMBINATORICS, AND GEOMETRY

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To the memory of my old friend Professor George Svéd.
I heard of his untimely death while writing this paper.

Introduction

I wrote many papers on unsolved problems and I cannot avoid repetition, but I hope to include at least some problems which have not yet been published. I will start with some number theory.

I. Number theory

1. Let $1 \le a_1 < a_2 < \cdots < a_k \le n$ be a sequence of integers for which all the subset sums $\sum_{i=1}^{k} \varepsilon_i a_i$ ($\varepsilon_i = 0$ or 1) are distinct. The powers of 2 have of course this property. Put $f(n) = \max k$. Is it true that

$$f(n) < \frac{\log n}{\log 2} + c_1 \tag{1}$$

for some absolute constant c_1? I offer 500 dollars for a proof or a disproof of (1). The inequality

$$f(n) < \frac{\log n}{\log 2} + \frac{\log \log n}{\log 2} + c_2$$

is almost immediate, since there are 2^k sums of the form $\sum_i \varepsilon_i a_i$ and they must be all distinct and all are $< kn$. In 1954 Leo Moser and I (see **[28]**) by using the second moment method proved

$$f(n) < \frac{\log n}{\log 2} + \frac{\log \log n}{2 \log 2} + c_3,$$

which is the current best upper bound.

Conway and Guy found 24 integers all $\le 2^{22}$ for which all the subset sums are distinct. Perhaps

$$f(2^n) \le n + 2??$$

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Perhaps the following variant of the problem is more suitable for computation: Let $1 \le b_1 < b_2 < \cdots < b_\ell$ be a sequence of integers for which all the subset sums $\sum_{i=1}^{n} \varepsilon_i a_i$ ($\varepsilon_i = 0$ or 1) are different. Is it true that

$$\min b_\ell > 2^{\ell - c} \tag{2}$$

for some absolute constant c? Inequality (2) of course is equivalent to (1). The determination of the exact value of b_ℓ is perhaps hopeless but for small ℓ the value of $\min b_\ell$ can no doubt be determined by computation, and this I think would be of some interest.

2. Covering congruences. This is perhaps my favourite problem. It is really surprising that it has not been asked before. A system of congruences

$$a_i \pmod{n_i}, \qquad n_1 < n_2 < \cdots < n_k \tag{3}$$

is called a *covering system* if every integer satisfies at least one of the congruences in (3). The simplest covering system is 0 (mod 2), 0 (mod 3), 1 (mod 4), 5 (mod 6), 7 (mod 12). The main problem is: Is it true that for every c one can find a covering system all whose moduli are larger than c? I offer 1000 dollars for a proof or disproof.

Choi [**13**] found a covering system with $n_1 = 20$ and a Japanese mathematician whose name I do not remember found such a system with $n_1 = 24$. If the answer to my question is positive: Denote by $f(t)$ the smallest integer k for which there is a covering system

$$a_i \pmod{n_i}, \quad 1 \le i \le k, \quad n_1 = t, \quad k = f(t).$$

It would be of some mild interest to determine $f(t)$ for the few values of $t = n_1$ for which we know that a covering system exists.

Many further unsolved problems can be asked about covering systems. Selfridge and I asked: Is there a covering system all whose moduli are odd? Schinzel asked: Is there a covering system where $n_i \nmid n_j$, *i.e.*, where the moduli form a primitive sequence. A sequence is called *primitive* if no term divides any other. Schinzel [**58**] used such covering systems for the study of irreducibility of polynomials. Herzog and Schönheim asked: Let G be a finite abelian group. Can its elements be partitioned into cosets of distinct sizes?

More generally, let $n_1 < n_2 < \ldots$ be a sequence of integers. Is there a reasonable condition which would imply that there is a covering system whose moduli are all among the n_i? Quite likely there is no such condition. Let us now drop the condition that the set of moduli is finite, but to avoid triviality we insist that in the congruence $a_i \pmod{n_i}$ only the integers greater than or equal to n_i are considered. When if ever can we find such a system?

3. Perhaps it is of some interest to relate the story of how I came to the problem of covering congruences. In 1934 Romanoff [**57**] proved that the lower density of the integers of the form $2^k + p$ (p prime) is positive. This was surprising since the number of sums $2^k + p \le x$ is cx. Romanoff in a letter in 1934 asked me if there were infinitely many odd numbers not of the form $2^k + p$. Using covering congruences I proved in [**27**] that there is an arithmetic progression of odd numbers no term of which is of the form $2^k + p$. Independently Van der Corput also proved that there are infinitely many odd numbers not of the form $2^k + p$. Crocker **[16]** proved that there are infinitely many odd integers not of the form $2^{k_1} + 2^{k_2} + p$, but his proof only gives that the number of integers $\leq x$ not of the form $2^{k_1} + 2^{k_2} + p$ is $> c \log\log x$. This surely can be improved but I am not at all sure if the upper density of the integers not of the form $2^{k_1} + 2^{k_2} + p$ is positive. One could ask the following (probably unattackable) problem. Is it true that there is an r so that every integer is the sum of a prime and r or fewer powers of 2. Gallagher **[47]** proved (improving a result of Linnik) that for every ε there is an r_ε so that the lower density of the integers which are the sum of a prime and r_ε powers of 2 is $1-\varepsilon$. No doubt lower density always could be replaced by density, but the proof that the density of the integers of the form $2^k + p$ exists seems unattackable.

I think that every arithmetic progression contains infinitely many integers of the form $2^{k_1} + 2^{k_2} + p$. Thus covering congruences cannot be used to improve the result of Crocker.

Perhaps the following rather silly conjecture could be added. Is it true that the set of odd integers not of the form $2^k + p$ is the not necessarily disjoint union of an infinite arithmetic progression and perhaps a sequence of density 0?

4. Let $n_1 < n_2 < \dots$ be an arbitrary sequence of integers. Besicovitch proved more than 60 years ago that the set of the multiples of the n_i does not have to have a density. In those prehistoric days this was a great surprise. Davenport and I proved **[19, 20]** that the set of multiples of the $\{n_i\}$ have a logarithmic density and the logarithmic density equals the lower density of the set of multiples of the $\{n_i\}$. Now the following question is perhaps of interest: Exclude one or several residues mod n_i (where only the integers $\geq n_i$ are excluded). Is it true that the logarithmic density of the integers which are not excluded always exists? This question seems difficult even if we only exclude one residue mod n_i for every n_i.

For a more detailed explanation of these problems see the excellent book of Halberstam and Roth, *Sequences*, Springer–Verlag, or the excellent book of Hall and Tenenbaum, *Divisors*, Cambridge University Press.

Tenenbaum and I recently asked the following question: let $n_1 < n_2 < \dots$ be an infinite sequence of positive integers. Is it then true that there always is a positive integer k for which almost all integers have a divisor of the form $n_i + k$? In other words, the set of multiples of the $n_i + k$ $(1 \leq i < \infty)$ has density 1. Very recently Ruzsa found a very ingenious counterexample. Tenenbaum thought that perhaps for every $\varepsilon > 0$ there is a k for which the density of the multiples of the $n_i + k$ has density $> 1 - \varepsilon$.

5. Let $n_1 < n_2 < \dots < n_k$ be a sequence of integers. I would like to choose residues $a_i \pmod{n_i}$ for which the number of integers $\leq x$ not satisfying any of the congruences $a_i \pmod{n_i}$ should be as small as possible. Clearly the density can be made to be $\leq \prod_{i=1}^k (1 - 1/n_i)$ and if $(n_i, n_j) = 1$ then clearly for every choice of the a_i the density equals $\prod_{i=1}^k (1 - 1/n_i)$, but if (n_i, n_j) is not always 1 the density can be both larger and smaller than $\prod_{i=1}^k (1 - 1/n_i)$.

Let us now restrict ourselves to a special case. The n_i are the integers between t and ct. First of all denote by $\alpha_1(c,t)$ the smallest possible value of the density of the integers satisfying none of the congruences $a_i \pmod{m}$ $(t \leq m \leq ct)$, and let $\alpha_2(c,t)$ be the largest possible value of the density of the integers satisfying none of these congruences. It is well known that for every $c > 1$ we have $\alpha_2(c,t) \to 0$ as $t \to \infty$. Since an old theorem of mine **[24]** states that the density of integers which have a divisor in (t, ct) tends to 0 for every c if $t \to \infty$. Thus to get $\alpha_2(c,t) \to 0$ it suffices to take $a_i = 0$ for every i. Now as we already remarked $\alpha_1(c,t)$ can be clearly made at least as large as $1 - \prod_{t \le u \le ct}(1 - 1/u)$. Can it in fact be made much larger? It is easy to see that it can be 1 only if there is a covering congruence the smallest modulus of which is $\ge t$ and the largest modulus of which is ct. On the other hand perhaps there is a c so that for every $\varepsilon > 0$ there is a t for which $\alpha_1(c,t) > 1 - \varepsilon$. I am not at all sure if this is possible and I give 100 dollars for an answer.

6. Some problems in additive number theory. I met Sidon first in 1932 and he posed two very interesting problems. The first problem stated: Let $A = \{a_1 < a_2 < \dots\}$ be an infinite sequence of integers, and denote by $f(n)$ the number of solutions of $n = a_i + a_j$. Sidon asked: Is there a sequence A for which $f(n) > 0$ for all n but for every $\varepsilon > 0$ we have $f(n)/n^\varepsilon \to 0$? I thought for a few minutes and told Sidon: 'A very nice problem. I am sure such a sequence exists and I hope to have an example in a few days.'

I was a bit too optimistic; I did eventually solve the problem but it took 20 years! Using the probability method I proved that there is a sequence A for which

$$c_1 \log n < f(n) < c_2 \log n. \tag{4}$$

I offer 500 dollars for a proof or disproof of my conjecture that there is no sequence A for which $f(n)/\log n \to c$ with $c > 0$ and finite. Also Turán and I conjectured that if $f(n) > 0$ for all $n > n_0$, then $\limsup f(n) = \infty$ and perhaps even

$$\limsup f(n)/\log n > 0.$$

I offer 500 dollars for a proof or disproof of my conjecture with Turán. Also I offer 100 dollars for an explicit construction of a sequence A for which $f(n) > 0$ for all n but $f(n)/n^\varepsilon \to 0$ for every $\varepsilon > 0$, *i.e.*, for a constructive solution to Sidon's original question.

Sidon also asked: Let $1 \le a_1 < a_2 < \dots < a_k \le n$ and assume that $a_i + a_j$ are all distinct. Put $h(n) = \max k$. Determine or estimate $h(n)$ as accurately as possible. The exact determination of $h(n)$ is perhaps hopeless but Chowla, Turán and I proved

$$h(n) = (1 + o(1))n^{1/2}.$$

Perhaps

$$h(n) = n^{1/2} + O(1), \tag{5}$$

but this is perhaps too optimistic. I give 500 dollars for a proof or disproof of the conjecture

$$h(n) = n^{1/2} + o(n^\varepsilon)$$

for any $\varepsilon > 0$. The excellent book of Halberstam and Roth, *Sequences*, contains a great deal more about this problem and the probabilistic method.

Sidon also asked: Let $A = \{a_1 < a_2 < \dots\}$ be an infinite sequence for which all the sums $a_i + a_j$ are distinct. Put

$$h(n) = \sum_{a_\ell < n} 1.$$

What can one say about $h(n)$? The greedy algorithm easily gives that there is a sequence A for which for every n we have

$$h(n) > cn^{1/3}.$$

I proved that for every sequence A

$$\liminf h(n)/n^{1/2} = 0.$$

Ajtai, Komlós and Szemerédi **[2]** in a very ingenious way constructed a *Sidon sequence* $A = \{a_1 < a_2 < \dots\}$ (*i.e.*, a sequence with all $a_i + a_j$ distinct) for which

$$h(n) > c(n \log n)^{1/3}$$

for some $c > 0$. Probably there is a Sidon sequence A for which

$$h(n) > n^{1/2-\varepsilon}, \tag{6}$$

but (6) is far beyond reach. Rényi and I **[42]** proved that for every $\varepsilon > 0$ there is a sequence A with $h(n) > n^{1/2-\varepsilon}$ for which the number of solutions of $a_i + a_j = n$ is bounded. We used the probability method. Does there exist such a sequence with $h(n) > n^{1/2}/(\log n)^c$? A sharpening of our old conjecture with Turán would state: If $a_n < Cn^2$ for all n then $\limsup f(n) = \infty$. In fact, for what functions $g(n) \to \infty$ does $a_n < n^2 g(n)$ imply $\limsup f(n) = \infty$? (500 dollars)

Here is an old conjecture of mine: Let $a_1 < a_2 < \dots$ be an infinite sequence for which all the triple sums $a_i+a_j+a_k$ are distinct. Is it then true that $\limsup a_n/n^3 = \infty$? I offer 500 dollars for a proof or disproof of this.

7. Let $1 \le a_1 < a_2 < \dots < a_h \le n$ be a maximum Sidon sequence. Can one find a Sidon sequence $b_1 < b_2 < \dots < b_r \le n$ for every r and $n > n_0(r)$ so that the differences $a_j - a_i$, $b_v - b_u$ are all distinct, *i.e.*, so that

$$a_j - a_i \ne b_v - b_u \qquad \text{for all } i < j \text{ and } u < v?$$

More generally: Let $a_1 < a_2 < \dots < a_{k_1} \le n$, $b_1 < b_2 < \dots < b_{k_2} \le n$ be two Sidon sequences for which $a_j - a_i \ne b_v - b_u$ for all $i < j$ and $u < v$. How large can

$$\max\left(\binom{k_1}{2} + \binom{k_2}{2}\right)$$

be? I guessed **[31]** that it is less than $\binom{h(n)}{2} + O(1)$. I offer 100 dollars for a proof or disproof. Assume next that $k_1 = k_2$. I am sure that then

$$\binom{k_1}{2} + \binom{k_2}{2} < (1-c)\binom{h(n)}{2}.$$

An old problem of mine states as follows: Let $a_1 < a_2 < \dots < a_k$ be a Sidon sequence. Can one extend it to a larger Sidon sequence

$$a_1 < \dots < a_k < a_{k+1} < \dots < a_\ell, \qquad a_\ell = (1+o(1))\ell^2.$$

In other words, loosely speaking: can one extend every Sidon sequence to a Sidon sequence which is substantially maximal? Many generalizations are possible.

8. One last Ramsey type problem: Let n_k be the smallest integer (if it exists) for which if we colour the proper divisors of n_k by k colours then n_k will be a monochromatic sum of distinct divisors, namely a sum of distinct divisors in a colour class. I am sure that n_k exists for every k but I think it is not even known if n_2 exists. It would be of some interest to determine at least n_2. An old problem of R.L. Graham and myself states: Is it true that if m_k is sufficiently large and we colour the integers $2 \le t \le m_k$ by k colours then

$$1 = \sum \frac{1}{t_i}$$

is always solvable monochromatically? I would like to see a proof that m_2 exists. (Clearly $m_k \ge n_k$.) Perhaps this is really a Turán type problem and not a Ramsey problem. In other words, if m is sufficiently large and $1 < a_1 < a_2 < \cdots < a_\ell \le m$ is a sequence of integers for which $\sum_\ell 1/a_\ell > \delta \log m$ then

$$1 = \sum \frac{\varepsilon_i}{a_i} \qquad (\varepsilon_i = 0 \text{ or } 1)$$

is always solvable. I offer 100 dollars for a proof or disproof. Perhaps it suffices to assume that

$$\sum_{a_i < m} \frac{1}{a_i} > C(\log\log m)^2$$

for some large enough C. For further problems of this kind as well as for related results see my book with R.L. Graham [**34**]. I hope before the year 2000 a second edition will appear.

9. Some problems on $\varphi(n)$ and $\sigma(n)$. Euler's $\varphi(n)$ function is the number of integers $1 \le t < n$ relatively prime to n, and $\sigma(n)$ is the sum of divisors of n.

Is it true that for infinitely many integers $\varphi(n) = \sigma(m)$ holds? If there are infinitely many primes p for which $p+2$ is also a prime then of course $\sigma(p) = \varphi(p+2) = p+1$, but I could not prove that $\varphi(n) = \sigma(m)$ has infinitely many solutions. It is not difficult to prove that for all t we have that $\varphi(n) = t!$ is solvable. Probably for every sufficiently large t the equation $\sigma(m) = t!$ is also solvable.

Let $a_1, \ldots, a_t$ be the longest sequence for which

$$a_1 < \cdots < a_t \le n \quad \text{and} \quad \varphi(a_1) < \cdots < \varphi(a_t). \tag{7}$$

Probably $t = \pi(n)$. Can one even prove $t < (1+o(1))\pi(n)$ or at least $t = o(n)$? This latest conjecture will probably be easy. Similar questions can be posed about $\sigma(n)$.

A more serious problem states as follows: Schoenberg proved about 70 years ago that $\varphi(n)$ has a distribution function. In other words the density of the integers for which $\varphi(n)/n < c$ exists for every c $(0 \le c \le 1)$. I proved that the distribution function is purely singular. Denote the distribution function by $f(x)$. Is it true that for no x can $f(x)$ have a finite positive derivative? (250 dollars for a proof or disproof.) For more details about these and related questions consult the excellent book of Elliot, *Probabilitstic Number Theory*, Springer-Verlag.

One more problem on the φ function. Is there an infinite sequence of integers $a_1 < a_2 < \ldots$ so that $\varphi(n) = a_i$ is solvable but if n_i is the smallest integer for which $\varphi(n_i) = a_i$ then $n_i/a_i \to \infty$? A more significant problem is due to Carmichael. Is it true that there is no integer t for which $\varphi(n) = t$ has exactly one solution? I proved that if there is an integer t_k for which $\varphi(n) = t_k$ has exactly k solutions then there are infinitely many such integers [**29**]. For many more problems see R.K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag.

10. Let $1 \leq a_1 < a_2 < \cdots < a_n$ be n integers. Denote by $f(n)$ the largest integer for which there are at least $f(n)$ distinct integers of the form $\{a_i + a_j, a_i a_j\}$. Szemerédi and I [**44**] proved $f(n) > n^{1+\varepsilon}$ for some $\varepsilon > 0$ and conjectured $f(n) > n^{2-\varepsilon}$ for any $\varepsilon > 0$ and $n > n_0(\varepsilon)$. This conjecture is still open and I offer 100 dollars for a proof or disproof, and 250 dollars for a more exact bound. The fact that $f(n)/n^2 \to 0$ and more is in our paper. Nathanson proved $f(n) > n^{1+1/31}$.

11. A paper by Burr and myself which I think has been undeservedly forgotten is [**11**]. Let A be a sequence of integers and let $P(A)$ denote the integers which can be represented as the sum of distinct terms of A, *e.g.*, if A consists of the powers of 2 then $P(A)$ is the set of all positive integers.

A sequence $A = \{a_1 < a_2 < \dots\}$ is Ramsey r-*complete* if whenever the sequence is partitioned into r classes $A = A_1 \cup \cdots \cup A_r$ every sufficiently large positive integer is a member of $\bigcup_{i=1}^{r} P(A_i)$. It is *entirely* Ramsey r-complete if every positive integer is a member of $\bigcup_{i=1}^{r} P(A_i)$. In our paper we investigate $r = 2$. We prove that there is an entirely Ramsey 2-complete sequence satisfying

$$a_x > \exp\left\{\frac{1}{2}(\log 2)x^{1/3}\right\} \tag{8}$$

for all sufficiently large x. Also there is a $C > 0$ so that no infinite sequence $A = \{a_1 < a_2 < \dots\}$ satisfying

$$a_x > \exp\left\{Cx^{1/2}\right\} \tag{9}$$

for all sufficiently large x is Ramsey 2-complete. Many problems remain. We could do nothing for $r > 2$ (250 dollars for any non-trivial result). Also, could (8) and (9) be improved (100 dollars)?

Burr has a proof that for every k the sequence t^k $(1 \leq t < \infty)$ is Ramsey r-complete.

12. A purely computational problem (this problem cannot be attacked by other means at present). Call a prime p *good* if every even number $2r \leq p - 3$ can be written in the form $q_1 - q_2$ where $q_1 \leq p$, $q_2 \leq p$ are primes. Are there infinitely many good primes?

The first bad prime is 97 I think. Selfridge and Blecksmith have tables of the good primes up to 10^{37} at least, and they are surprisingly numerous.

13. I proved long ago that every $m < n!$ is the distinct sum of $n - 1$ or fewer divisors of $n!$. Let $h(m)$ be the smallest integer, if it exists, for which every integer less than m is the distinct sum of $h(m)$ or fewer divisors of m. Srinivasan called the numbers for which $h(m)$ exists *practical*. It is well known and easy to see that almost all numbers m are not practical. I conjectured that there is a constant $c \geq 1$ for which for infinitely many m we have $h(m) < (\log\log m)^c$. M. Vose proved that $h(n!) < cn^{1/2}$. Perhaps $h(n!) < c(\log n)^{c_2}$. I would be very glad to see a proof of $h(n!) < n^{\varepsilon}$.

A practical number n is called a *champion* if for every $m > n$, we have $h(m) > h(n)$. For instance, 6 and 24 are champions, as $h(6) = 2$, the next practical number is 24, $h(24) = 3$, and for every $m > 24$, we have $h(m) > 3$. It would be of some interest to prove some results about champions. A table of the champions $< 10^6$ would be of some interest. I conjecture that $n!$ is not a champion for $n > n_0$.

The study of champions of various kinds was started by Ramanujan (Highly composite numbers, *Collected Papers of Ramanujan*). See further my paper with Alaouglu on highly composite and similar numbers **[3]**, and many papers of J.L. Nicolas and my joint papers with Nicolas.

The following related problem is perhaps of some mild interest, in particular, for those who are interested in numerical computations. Denote by $g_r(n)$ the smallest integer which is not the distinct sum of r or fewer divisors of n. A number n is an r-champion if for every $t < n$ we have $g_r(n) > g_r(t)$. For $r = 1$ the least common multiple M_m of the integers $\leq m$ is a champions for any m, and these are all the 1-champions. Perhaps the M_m are r-champions too, but there are other r-champions; *e.g.*, 18 is a 2-champion.

14. Let $f_k(n)$ be the largest integer for which you can give $f_k(n)$ integers $a_i \leq n$ for which you cannot find $k+1$ of them which are relatively prime. I conjectured that you get $f_k(n)$ by taking the multiples $\leq n$ of the first k primes. This has been proved for small k by Ahlswede, and Khachatrian disproved it for $k \geq 8$ (see Ahlswede and Khachatrian **[1]** and also a forthcoming paper of theirs). Perhaps if $n \geq (1+\varepsilon)p_k^2$, where p_k is the kth prime, the conjecture remains true.

II. Combinatorics

First I state some of my favourite old problems.

1. Conjecture of Faber, Lovász and myself. Let $G_1, \dots, G_n$ be n edge-disjoint complete graphs on n vertices. We conjectured more than 20 years ago that the chromatic number of $\bigcup_{i=1}^n G_i$ is n. I offer 500 dollars for a proof or disproof. About 3 years ago Jeff Kahn **[54]** proved that the chromatic number of $\bigcup_{i=1}^n G_i$ is less than $(1+o(1))n$. I immediately gave him a consolation prize of 100 dollars. (For a related result, see Kahn and Seymour **[55]**). Hindman proved our conjecture for $n < 10$ many years ago.

It might be of interest to determine the maximum of the chromatic number of $\bigcup_{i=1}^n G_i$ if we ask that $G_i \cap G_j$ $(i \neq j)$ should be triangle-free or should have at most one edge in common, but it is not clear to me if we get a nice answer. Also one could assume that the G_i are edge-disjoint and could try to determine the largest chromatic number of $\bigcup_{i=1}^m G_i$ for $m > n$. Again I am not sure if we can hope for a nice answer.

2. Problems on Δ-systems. A family of sets A_i, $i = 1, 2, \dots$, is called a *strong* Δ-*system* if all the intersections $A_i \cap A_j$ $(i \neq j)$ are identical, *i.e.*, if $A_i \cap A_j = \bigcap_i A_i$. The family is called a *weak* Δ-*system* if we only assume that the size $A_i \cap A_j$ $(i \neq j)$ is always the same.

Rado and I **[40, 41]** investigated the following question: Denote by $f_s(n,k)$ the smallest integer for which every family of sets A_i $(1 \leq i \leq f_s(n,k))$ with $|A_i| = n$ for all i contains k sets which form a strong Δ-system. In particular, we proved

$$2^n < f_s(n,3) \leq 2^n n! \tag{10}$$

Abott and Hanson proved $f_s(n,3) > 10^{n/2}$. Rado and I conjectured

$$f_s(n,3) < c_3^n \tag{11}$$

and no doubt also

$$f_s(n,k) < c_k^n.$$

I offer 1000 dollars for a proof or disproof of (11). Milner, Rado and I [**37**] also considered finite and infinite strong and weak Δ-systems. We could not prove

$$f_w(n,3) = o(n!).$$

Curiously, the infinite problems were not very difficult; conjecture (11) and the corresponding conjecture for $f_w(n,3)$ remained open.

Recently Kostochka proved

$$f_s(n,3) < n!\left(\frac{c\log n}{\log\log n}\right)^{-n}. \tag{12}$$

I gave Kostochka a consolation prize of 100 dollars. Very recently, Axenovich, Fon-der-Flaas and Kostochka proved

$$f_w(n,3) < n!^{1/2+\varepsilon}$$

for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

3. Many years ago Hajnal and I conjectured that if G is an infinite graph whose chromatic number is infinite, then if $a_1 < a_2 < \dots$ are the lengths of the odd cycles of G we have

$$\sum_i \frac{1}{a_i} = \infty$$

and perhaps $a_1 < a_2 < \dots$ has positive upper density. (The lower density can be 0 since there are graphs of arbitrarily large chromatic number and girth.)

We never could get anywhere with this conjecture. About 10 years ago Mihók and I conjectured that G must contain for infinitely many n cycles of length 2^n. More generally it would be of interest to characterize the infinite sequences $A = \{a_1 < a_2 < \dots\}$ for which every graph of infinite chromatic number must contain infinitely many cycles whose length is in A. In particular, assume that the a_i are all odd.

All these problems were unattackable (at least for us). About three years ago Gyárfás and I thought that perhaps every graph whose minimum degree is ≥ 3 must contain a cycle of length 2^k for some $k \geq 2$. We became convinced that the answer almost surely will be negative but we could not find a counterexample. We in fact thought that for every r there must be a G_r every vertex of which has degree $\geq r$ and which contains no cycle of length 2^k for any $k \geq 2$. The problem is wide open (cf. 4 below).

Gyárfás, Komlós and Szemerédi [**49**] proved that if k is large and $a_1 < a_2 < \dots$ are the lengths of the cycles of a $G(n,kn)$, that is, an n-vertex graph with kn edges, then

$$\sum \frac{1}{a_i} > c\log n.$$

The sum is probably minimal for the complete bipartite graphs.

4. Many years ago I asked: Is it true that for every a and b for which the arithmetic progression $a \pmod b$ contains infinitely many even numbers there is a $c(a,b)$ so that every $G(n, c(a,b)n)$ contains a cycle whose length is $\equiv a \pmod b$? Bollobás **[7]** proved this conjecture, but the best value of the constant $c(a,b)$ is not known.

Now perhaps the following question is of interest: Is there a sequence A of density 0 for which there is a constant $c(A)$ so that for $n > n_0(A)$ every $G(n, c(A)n)$ contains a cycle whose length is in A? This question seems very interesting to me and I offer 100 reais or 100 dollars, whichever is worth more, for an answer. I am almost certain that if A is the sequence of the powers of 2 then no such constant exists. What if A is the sequence of squares? I have no guess. Let $f(n)$ be the smallest integer for which every $G(n, f(n))$ contains a cycle of length a power of 2. I think that $f(n)/n \to \infty$ but $f(n) < n(\log n)^c$ for some $c > 0$.

5. Let k be fixed and $n \to \infty$. Is it true that there is an $f(k)$ so that if $G(n)$ has the property that for every m every subgraph of m vertices contains an independent set of size $m/2 - k$ then $G(n)$ is the union of a bipartite graph and a graph of $\leq f(k)$ vertices, *i.e.*, the vertex set of $G(n)$ is the union of three disjoint sets S_1, S_2 and S_3 where S_1 and S_2 are independent and $|S_3| \leq f(k)$. Gyárfás pointed out that even the following special case is perhaps difficult. Assume that for every even m every m vertices of our $G(n)$ induces an independent set of size at least $m/2$. Is it then true that $G(n)$ is the union of a bipartite graph and a bounded set? Perhaps this will be cleared up before this paper appears, or am I too optimistic?

Hajnal, Szemerédi and I proved that for every $\varepsilon > 0$ there is a graph of infinite chromatic number for which every subgraph of m vertices contains an independent set of size $(1-\varepsilon)m/2$ and in fact perhaps $(1-\varepsilon)m/2$ can be replaced by $m/2 - f(m)$ where $f(m)$ tends to infinity arbitrarily slowly. A result of Folkman implies that if G is such that every subgraph of m vertices contains an independent set of size $m/2 - k$ then the chromatic number of G is at most $2k + 2$ (see **[36]**).

6. Many years ago I proved by the probability method that for every k and r there is a graph of girth $\geq r$ and chromatic number $\geq k$. Lovász when he was still in high school found a fairly difficult constructive proof. My proof still had the advantage that not only was the chromatic number of $G(n)$ large but the largest independent set was of size $< \varepsilon n$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon, r, k)$. Nešetřil and Rödl later found a simpler constructive proof which also had this property.

There is a very great difference between a graph of chromatic number $\aleph_0$ and a graph of chromatic number $\geq \aleph_1$. Hajnal and I in fact proved that if G has chromatic number $\aleph_1$ then G must contain a C_4 and more generally G contains the complete bipartite graph $K(n, \aleph_1)$ for every $n < \aleph_0$. Hajnal, Shelah and I **[35]** proved that every graph G of chromatic number $\aleph_1$ must contain for some k_0 every odd cycle of size $\geq k_0$ (for even cycles this was of course contained in our result with Hajnal), but we observed that for every k and every m there is a graph of chromatic number m which contains no odd cycle of length $< k$. Walter Taylor has the following very beautiful problem: Let G be any graph of chromatic number $\aleph_1$. Is it true that for every $m > \aleph_1$ there is a graph G_m of chromatic number m all finite subgraphs of which are contained in G? Hajnal and Komjáth **[51]** have some results in this direction but the general conjecture is still open. If it would have been my problem, I certainly would offer 1000 dollars for a proof or a disproof. (To avoid financial ruin I have to restrict my offers to my problems.)

7. Hajnal, Szemerédi and I [**36**] have the following problems and results. Let $f(n) \to \infty$ arbitrarily slowly. Is it true that there is a graph G of infinite chromatic number such that, for every n, every subgraph of G of n vertices can be made bipartite by the omission of fewer than $f(n)$ edges? I offer 250 dollars for a proof or disproof. Rödl [**56**] proved the corresponding result for hypergraphs in 1982. It would be of interest to prove or disprove the existence of a G of infinite chromatic number for which $f(n) = o(n^\varepsilon)$ or $f(n) < (\log n)^c$ for some $c > 0$.

Let G have chromatic number $\geq \aleph_1$. Then by our result with Hajnal and Shelah [**35**] the graph G has for every n a subgraph of n vertices the largest independent set of which is $< (1-\varepsilon)n/2$. In our paper with Hajnal and Szemerédi we ask: Does there exist a G of chromatic number $\aleph_1$ every subgraph of n vertices of which has an independent set of size $> cn$ for some $c > 0$? If the answer is negative, can cn be replaced by $n^{1-\varepsilon}$? *I.e.*, is there a G of chromatic number $\aleph_1$ every subgraph of n vertices of which contains an independent set of size $> n^{1-\varepsilon}$ for every $\varepsilon > 0$ if $n \geq n_0(\varepsilon)$?

8. A problem of Faudree, Ordman and myself states: Denote by $h(n)$ the largest integer for which if you colour the edges of the complete graph $K(n)$ by two colours we always have a family of $\geq h(n)$ edge-disjoint monochromatic triangles. We conjectured that

$$h(n) = (1 + o(1))\frac{n^2}{12}. \tag{13}$$

If true (13) is easily seen to be best possible. We divide the vertex set into two sets S_1 and S_2 with $\big||S_1| - |S_2|\big| \leq 1$. The edges joining S_1 and S_2 are coloured with the first colour and all the other edges are coloured with the second colour.

Perhaps there is an absolute constant $c > 0$ for which there are more than $(1 + c)n^2/24$ monochromatic triangles all of which have the same color. Jacobson conjectured that the result could be $n^2/20$; he has a simple example which shows that if true this is best possible.

If we drop the condition of edge-disjointness then the number of monochromatic triangles was determined long ago by Goodman [**48**] and others.

9. Some Ramsey type problems. Let G and H be two graphs. Then $r(G, H)$ is the smallest integer n for which if we colour the edges of $K(n)$ by two colours I and II there is either a G all whose edges are coloured I or an H all whose edges are coloured II. For simplicity put $r(t) = r(K(t), K(t))$. It is known that

$$ct2^{t/2} < r(t) < t^{-1/2}\binom{2t-2}{t-1}, \tag{14}$$

for some constant $c > 0$. It would be very desirable to improve (14) and prove that

$$c = \lim_{t\to\infty} r(t)^{1/t}$$

exists and if c exists determine its value. By (14) the value of this limit, if it exicts, is between $\sqrt{2}$ and 4. I offer 100 dollars for the proof of the existence of c and 250 dollars for the value of c. I give 1000 dollars for a proof of the non-existence of c, but this is really a joke as c certainly exists. The proof of the lower bound of (14) is probabilistic. I give 100 dollars for a constructive proof of

$$r(t) > (1 + c)^t$$

for some $c > 0$. All these problems are well known. Now I state a few less well known questions. Harary conjectured and Sidorenko [**59**] proved that for any H_n of size n without isolated vertices

$$r(K_3, H_n) \leq 2n + 1$$

(the *size* is the number of edges).

In a recent paper, Faudree, Rousseau, Schelp and I [**33**] define G to be *Ramsey size linear* if there is an absolute constant C for which

$$r(G, H_n) < Cn \tag{15}$$

holds for any graph H_n of size n. We obtain many results about graphs which satisfy (15) but I have to refer to our paper. Here I only state some of our unsolved problems: $K(4)$ is known *not* to be Ramsey size linear but all its subgraph are Ramsey size linear. Are there other such graphs and in fact are there infinitely many such graphs?

Is $K(3,3)$ Ramsey size linear? For further problems I have to refer to our paper.

10. In a recent paper with Ordman and Zalcstein [**38**] we have among many others the following question: Let $G(n)$ be a chordal graph, *i.e.*, a graph whose every cycle of length greater than 3 has a diagonal. Can we partition the edges of $G(n)$ into $n^2/6+cn$ cliques? We could only prove this with $n^2/4-\varepsilon n^2$ cliques where $\varepsilon > 0$ is very small. For further problem I have to refer to our paper.

11. Fajtlowicz, Staton and I considered the following problem (the main idea was due to Fajtlowicz). Let $F(n)$ be the largest integer for which every graph of n vertices contains a regular induced subgraph of $\geq F(n)$ vertices. Ramsey's theorem states that $G(n)$ contains a trivial subgraph, *i.e.*, a complete or empty subgraph of $c \log n$ vertices. (The exact value of c is not known but $1/2 \leq c \leq 2$; *cf.* 9 above.) We conjectured $F(n)/\log n \to \infty$. This is still open. We observed $F(5) = 3$ (since if $G(5)$ contains no trivial subgraph of 3 vertices then it must be a pentagon). Kohayakawa and I worked out that $F(7) = 4$ but the proof is by an uninteresting case analysis. It would be very interesting to find the smallest integer n for which $F(n) = 5$, *i.e.*, the smallest n for which every $G(n)$ contains a regular induced subgraph of ≥ 5 vertices. Probably this will be much more difficult than the proof of $F(7) = 4$ since in the latter we could use properties of perfect graphs. Bollobás observed that $F(n) < c\sqrt{n}$ for some $c > 0$. Fajtlowicz, McColgan, Reid, and Staton have a forthcoming paper on this problem [**45**].

12. Here is an old conjecture of Erdős–Ko–Rado: Let $|S| = 4n$, $A_\ell \subset S$, $|A_\ell| = 2n$, and assume that for $1 \leq i < j \leq h(n)$ we have $|A_i \cap A_j| \geq 2$. Then

$$\max h(n) = \frac{1}{2}\left(\binom{4n}{2n} - \binom{2n}{n}^2\right). \tag{16}$$

I offer 400 dollars for a proof or disproof of (16). It is easy to see that if (16) is true then it is best possible.

13. Ralph Faudree and I considered the following problem: Let $G(n)$ be a graph of n vertices; let $3 \leq a_1 < a_2 < \cdots < a_k \leq n$ be the lengths of the cycles of our $G(n)$. Now conisder all the graphs of n vertices and consider all the possible sequences, in other words $3 \leq b_1 < b_2 < \cdots < b_k \leq n$ belongs to our set of sequences if there is a graph $G(n)$ which has cycles of length $\{b_i\}$ and no cycle of length t if t is not one of our b_i. Now denote by $f(n)$ the number of our possible sequences. Clearly $f(n) \leq 2^{n-2}$ and for $n \geq 5$ we have $f(n) < 2^{n-2}$. Our first problem is to prove $f(n)/2^n \to 0$. We showed $f(n) > 2^{n/2}$. To see this consider a Hamilton cycle and join one of its points to some of the points to the right of our point at distance $a_1 < \cdots < a_k < n/2$. This graph has all the cycles of length $n - a_i$, which are $> n/2$, and not other cycles of length $> n/2$. This shows $f(n) > 2^{n/2}$. Probably $f(n)/2^{n/2} \to \infty$.

One hopes to be able to prove $f(n)/2^n \to 0$ by showing that many conditions are necessary for a sequence $3 \leq a_1 < \cdots < a_k \leq n$ to be the sequence of cycle lengths of a graph. As far as I know the only such result is due to Faudree, Flandrin, Jacobson, Lehel, and Schelp: If $G(n)$ contains all the odd cycles then it must contain at least n^c even cycle lengths for $c = 1/6$, and they conjecture that $c = 1/3$ is the correct value. Further, if the conjecture is true they have shown this to be best possible. To prove $f(n)/2^n \to 0$ we would need a much more precise result. It would be of interest to determine $\lim f(n)^{1/n} = c$. We know that $2^{1/2} \leq c \leq 2$. The determination of the exact value of $f(n)$ may be hopeless.

14. I did a great deal of work on extremal graph problems. Those who are interested can study the excellent book of Bollobás [**8**] and the excellent survey paper of Simonovits [**60**]. Here I just want to mention a little known conjecture of mine: Let $f(n)$ be the smallest integer for which every $G(n)$ every vertex of which has degree $\geq f(n)$ contains a C_4. Is it true that $f(n+1) \geq f(n)$? If this is too optimistic is it at least true that there is an absolute constant c so that for every $m > n$

$$f(m) > f(n) - c?$$

The same question can of course be asked for other graphs instead of C_4.

Finally let $g(n)$ be the smallest integer for which every subgraph of the n-dimensional cube which has $g(n)$ edges contains a C_4. An old conjecture of mine states that

$$g(n) < \left(\frac{1}{2} + o(1)\right) n 2^{n-1}. \tag{17}$$

I offer 100 dollars for a proof or a disproof of (17). See for this problem and generalizations (C_4 may be replaced by other graphs) the recent papers of Fan Chung [**14**], Conder [**15**] and Brouwer, Dejter and Thomassen [**10**].

15. In the book of Bondy and Murty [**9**] the following old conjecture of mine is stated (Problem 26, p. 250). Let G be an arbitrary n-chromatic graph. Then

$$r(G, G) \geq r(n) = r(K(n), K(n)). \tag{18}$$

Inequality (18) is trivial for $n = 3$. Unfortunately it already fails for $n = 4$. Faudree and McKay [**46**] proved that the Ramsey number of the pentagonal wheel is 17. Probably the conjecture fails for every $n > 4$ but perhaps $r(G, G)$ cannot be much smaller than $r(n)$. In fact, $r(G,G) > (1-\varepsilon)^n r(n)$ should hold for some $0 < \varepsilon < 1$ and perhaps even

$$\lim_{n\to 0} r(G,G)/r(n) > 0.$$

Both conjectures may be unattackable at present.

16. Brendan McKay and I conjectured that if $G(n, c_1 n^2)$ is a graph which contains no trivial subgraph of $c_2 \log n$ vertices then there is a $t > \varepsilon n^2$ ($\varepsilon > 0$ a constant) so that our $G(n)$ has an induced subgraph of i edges for each $i \le t$ edges. I think this is a nice conjecture and I offer 100 dollars for a proof or disproof. The problem has been almost completely forgotten and perhaps it is not difficult. We only proved it for some $t < c(\log n)^2$.

III. Some Problems in Combinatorial Geometry

1. Let $x_1, \dots, x_n$ be n distinct points in the plane. Let $f(n)$ be the number of distinct distances that these points are guaranteed to determine. In other words, if $d(x_i, x_j)$ is the distance between x_i and x_j, there are necessarily at least $f(n)$ distinct numbers among the $d(x_i, x_j)$ $(1 \le i < j \le n)$. I conjectured **[26]** in 1946 that

$$f(n) > cn/(\log n)^{1/2}. \tag{19}$$

The lattice points show that if (19) if true then it is best possible. I offer 500 dollars for a proof or disproof of (19).

Denote by $g(n)$ the maximum number of times the same distance can occur, *i.e.*, $g(n)$ is the maximum number of pairs for which $d(x_i, x_j) = 1$, where the maximum is taken over all configurations $x_1, \dots, x_n$. I conjectured in my 1946 paper that

$$g(n) < n^{1+c/\log\log n}, \tag{20}$$

for some $c > 0$. The lattice points show that (20) if true is best possible. I offer 500 dollars for a proof or disproof of (20). The best results so far are $f(n) > n^{3/4}$ and $g(n) < n^{5/4+\varepsilon}$ for any $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

Let $x_1, \dots, x_n$ be n points in the plane. Let $f(n)$ be the largest integer for which for every x_i there are $\ge f(n)$ points equidistant from x_i. Is it true that for $n > n_0(\varepsilon)$ we have $f(n) = o(n^\varepsilon)$? I offer 500 dollars for a proof but only 100 dollars for a counterexample. Some more unsolved problems: Let $x_1, \dots, x_n$ be n points in the plane. Denote by $f(x_i)$ the number of distinct distances from x_i, and say $f(x_1) \le f(x_2) \le \cdots \le f(x_n)$. I conjectured long ago that

$$f(x_n) > cn/(\log n)^{1/2} \tag{21}$$

and that in fact

$$\sum_{i=1}^{n} f(x_i) > cn^2/(\log n)^{1/2}. \tag{22}$$

Inequalities (21) and (22) are of course stronger than (19). Trivially $f(x_1) = 1$ is possible. It is also easy to see that $f(x_2) > cn^{1/2}$ and in fact N. Saldanha and I observed that $f(x_1)f(x_2) \ge (n-2)/2$ and this is best possible. So far we have no good result for $f(x_3)$. For instance, $f(x_3) > n^{1/2+\varepsilon}$? Determine or estimate the smallest $k = k(n)$ for which $f(x_k) > n^{1-o(1)}$ (it is not known if $f(x_n) > n^{1-o(1)}$, but

I am sure that this is true). Probably much more is true but I have no conjecture. Clearly there are many further open problems. What, for example, is the maximum number of distinct sequences $f(x_1), \ldots, f(x_n)$ which are possible? Perhaps it is more interesting to ask how many distinct numbers there can be among the $f(x_i)$ $(1 \leq i \leq n)$. By our result with Saldanha it is less than $n - cn^{1/2}$ for some $c > 0$. I think that $n - o(n)$ will surely be possible, but perhaps $n - n^{(1-\varepsilon)}$ for arbitrary $\varepsilon > 0$ is not possible. For many nice questions for higher dimensions see, *e.g.*, Avis, Erdős and Pach **[6]** and Erdős and Pach **[39]**.

For related problems and results see the excellent book of Croft, Falconer and Guy, *Unsolved Problems in Geometry*, Springer-Verlag.

2. Let $x_1, \ldots, x_n$ be a convex polygon. I conjectured that the number of distinct distances among the n points is $\geq \lfloor n/2 \rfloor$. We clearly have equality for the regular polygon. Altman **[4, 5]** proved this conjecture. Fishburn determined all cases of equality.

I conjectured that for at least one x_i there are at least $\lfloor n/2 \rfloor$ distinct distances from x_i. This conjecture is still open.

Szemerédi conjectured that if $x_1, \ldots, x_n$ are n points in the plane with no three on a line then there are at least $\lfloor n/2 \rfloor$ distinct distances among the x_i, but he only proved it with $n/3$ (see **[30]**). Many related questions can be asked, *e.g.*, what happens with the number of distinct distances determined by the vertices of convex polyhedra? I have not even a reasonable conjecture. The difficulty of course is that there are no regular polyhedra for $n > 20$.

3. Let $x_1, \ldots, x_n$ be a convex polygon in the plane. Consider the $\binom{n}{2}$ distances $d(x_i, x_j)$ and assume that the distance u_i occurs s_i times. Clearly

$$\sum_i s_i = \binom{n}{2}.$$

I conjectured and Fishburn proved that

$$\sum_i s_i^2 < cn^3. \tag{23}$$

I also conjectured that $\sum_i s_i^2$ is maximal for the regular n-gon for $n > n_0$.

If convexity is not assumed I conjectured

$$\sum_i s_i^2 < n^{3+\varepsilon} \tag{24}$$

for any $\varepsilon > 0$ and $n > n_0(\varepsilon)$. I offer 500 dollars for a proof or disproof of (24).

4. Is it true that every polygon has a vertex which has no four other vertices equidistant from it? I first thought that every convex polygon had a vertex which has no three vertices equidistant from it but Danzer (cf. **[32]**) found a polygon of 9 points every vertex of which has three vertices equidistant from it, and Fishburn and Reeds found a convex polygon of 20 sides whose every vertex has three other vertices at distance 1.

Is the following conjecture true? Let $x_1, \ldots, x_n$ be n points in the plane with the minimum distance among the x_i equal to 1. Let the diameter be minimal under this condition. The points $x_1, \ldots, x_n$ are then asymptotically similar to the triangular lattice, but probably for $n > n_0$ it will never be a subset of the triangular lattice. I do not think this has ever been proved.

Now, I had the following conjecture which seemed obvious to me though I could not prove it. If the diameter of $x_1, \ldots, x_n$ is minimal then there are three of our points x_i, x_j, x_k which form an equilateral triangle of size 1. To my great surprise, Simonovits, Vesztergombi and Sendov all expressed doubts. I now offer 100 dollars for a proof or disproof. In fact I am sure that our set must have a very large intersection with the triangular lattice.

Let $x_1, \ldots, x_n$ be n distinct points in the plane. Assume that if two distances $d(x_i, x_j)$ and $d(x_k, x_\ell)$ differ then they differ by at least 1. Is it then true that the diameter $D(x_1, \ldots, x_n)$ is greater than cn? Perhaps if $n > n_0$ the diameter is in fact $\geq n - 1$. Lothar Piepmeyer has a nice example of 9 points for which the diameter is < 5. Here it is: let first $x = (1 + \sqrt{2})\sqrt{2 - \sqrt{3}}$. Then take 2 equilateral triangles, one of them with side length x, and the second 'around' the first, containing it, with parallel sides, and distances x between corresponding vertices. The 3 remaining points are the centres of the 3 circles determined by the 4 endpoints of the three pairs of parallel sides of the two equilateral triangles.

It is perhaps not uninteresting to try to determine the smallest diameter for each n, but this will alredy be difficult for $n = 9$.

5. An old conjecture of mine states that if $x_1, \ldots, x_n$ are n points with no five on a line then the number of lines containing four of our points is $o(n^2)$. I offer 100 dollars for a proof or disproof. An example of Grünbaum shows that the number of these lines can be $> cn^{3/2}$ for some constant $c > 0$ and perhaps $n^{3/2}$ is the correct upper bound. Sylvester observed that one can give n points in the plane so that the number of lines passing through exactly three of our points is as large as $n^2/6 - cn$ for some constant $c > 0$. This is the so-called Orchard Configuration, named after the Orchard Problem (see Burr, Grünbaum and Sloane [**12**]).

George Purdy and I considered the following related problem. If we no longer insist that no five of the x_i can be on a line then the lattice points in the plane show that we can get cn^2 distinct lines each containing at least four of our points and in fact $c'n^2$ containing exactly four of our points. Denote by $f(n)$ the maximum number of distinct lines which pass through at least 4 of our points. Determine or estimate $f(n)$ as well as you can. Perhaps if there are cn^2 distinct lines each containing more than three points, then there is an $h(n) \to \infty$ such that there is a line containing $h(n)$ distinct points. We cannot prove that $h(n) \geq 5$ but suspect that $h(n) \to \infty$, and perhaps $h(n) > \varepsilon n^{1/2}$ for some $\varepsilon > 0$. It is easy to see that $h(n) < cn^{1/2}$ for some $c > 0$. Clearly several related questions can be asked.

In 1933, while reading the book of Hilbert and Cohn-Vossen, *Anschauliche Geometrie* (*Geometry and Imagination* was the translation), the following problem occurred to me: Let $x_1, \ldots, x_n$ be n points in the plane, not all on a line. Then is there always a line which goes through exactly two of our points? To my disappointment, I could not prove this; a few days later Gallai found a simple proof. L.M. Kelly later informed me that in fact Sylvester conjectured this already in 1893, but the first proof was due to Gallai. Later L.M. Kelly found the simplest known proof. It has been conjectured that for $n > n_0$ there are at least $n/2$ Gallai lines (*i.e.*, lines which go through exactly two of our points). This is still open; the best bound at present, due to Csima and Sawyer [**18**], is $6n/13$.

Suppose that the line with the most points contains only $o(n)$ of our x_i. Denote by $f(n)$ the maximum number of pairs covered by lines containing at least four points. Could it be that the maximum occurs for the lattice? It is not clear to us whether the triangular or the square lattice would be best. This could be decided by a rather messy computation, but trial by computer might help.

Is it true that if the line with most points contains only $o(n)$ of our x_i then the number of lines is $n^2/6 + O(n)$? The Orchard Configuration shows that if true this is best possible.

6. A little more than 10 years ago I asked the following question: Let $x_1, \ldots, x_n$ be n distinct points in the plane. Denote by $f(n)$ the maximum number of distinct unit circles which contain at least three of our points. Trivially $f(n) \leq n(n-1)$ and $f(n) > cn$. I conjectured

$$f(n)/n \to \infty \tag{25}$$

and

$$f(n)/n^2 \to 0. \tag{26}$$

As far as I know (26) is still open but Elekes **[21]** found a very clever proof of $f(n) > cn^{3/2}$. Here is his proof: Let $e_1, \ldots, e_n$ be unit vectors in general position, *i.e.*, all subsums $\sum_{i=1}^{n} \delta_i e_i$ ($\delta_i = 0$ or 1) are distinct. Our points are the $\binom{k}{2}$ points $e_i + e_j$ $(1 \leq i < j \leq n)$. The centres of our unit circles are the $\binom{k}{3}$ points $e_i + e_j + e_k$ $(1 \leq i < j < k \leq n)$. The circle with centre $e_i + e_j + e_k$ contains the three points $e_i + e_j$, $e_i + e_k$, $e_j + e_k$.

The proof of course easily generalizes for r dimensions. You get n points in r-dimensional space with $cn^{1+1/r}$ distinct unit spheres each containing $r+1$ of our points. Perhaps the construction of Elekes is asymptotically best possible.

7. Let $x_1, x_2, \ldots$ be an infinite sequence of points in the plane, or more generally in n-dimensional space. Assume that all the distances $d(x_i, x_j)$ are integers. Anning and I proved that this is possible only if all the points are one a line. Our first proof was rather complicated but later I found a very simple proof **[25]**.

Ulam conjectured that if $x_1, x_2, \ldots$ form a dense set then not all distances can be rational. Can one find for every n a set of n points in general position, *i.e.*, no three on a line and no four on a circle, with all distances integers? The general problem is unsolved. Perhaps Harborth has the strongest results.

8. Let $x_1, \ldots, x_n$ be n points in the plane, not all on a line. Let $L_1, \ldots, L_m$ be the set of all the lines containing at least two of our points. I proved long ago that $m \geq n$ with equality only if $n-1$ of our points are on a line. This easily follows by induction from Gallai–Sylvester (*i.e.*, that there always exists a *Gallai line*, a line that contains exactly two of our points.)

In a forthcoming paper with Purdy, we investigate the following question. Let $f(n)$ be the smallest integer r for which there is a configuration $x_1, \ldots, x_n$ that admits points $y_1, \ldots, y_r$, with all the y_i different from the x_j, and so that all the lines $L_1, \ldots, L_m$ go through at least one y_i. It is easy to see that $f(n) \leq n-1$. To see this just take $n-2$ points on a line and two points off this line as the $x_1, \ldots, x_n$. In this configuration one can easily find $n-1$ points y_j as required. At first we thought that $f(n) = n-1$, but Dean Hickerson found a nice example which shows that $f(n) \leq n-2$. A result of Beck, Szemerédi and Trotter gives $f(n) > cn$ for an absolute constant $c > 0$. Now, a conjecture of Dirac states as follows: Let $x_1, \ldots, x_n$

be n points not all on a line and join every two of them. Then for at least one x_i there are $n/2-c$ distinct lines through x_i ($c>0$ an absolute constant). If this conjecture of Dirac holds then $f(n)\geq n/2-c$, but perhaps even $f(n)=n-2$. The only motivation for our conjecture is that we cannot improve Hickerson's construction which gives $f(n)\leq n-2$. I give 50 dollars for a proof or disproof of the conjecture that $f(n)=n-2$.

9. Here is a problem of mine which is more than 60 years old and has been perhaps undeservedly forgotten. Let S be a unit square. Inscribe n squares with no common interior point. Denote by $e_1,\dots,e_n$ the side lengths of these squares. Put

$$f(n)=\max\sum_{i=1}^{n}e_i.$$

From Cauchy–Schwarz we trivially get that $f(k^2)=k$. Is it true that $f(k^2+1)=k$? It is not hard to see that $f(2)=1$, and perhaps $f(5)=2$ has been proved. As far as I know the general case is open. It is easy to see that $f(n+2)>f(n)$ for all n. For which n is $f(n+1)=f(n)$?

10. Finally I should state the Erdős–Klein–Szekeres problem: Let $f(n)$ be the smallest integer r such that if r points are in the plane with no three on a line then one can find n of them which form the vertices of a convex n-gon. One has $f(4)=5$ (Klein), $f(5)=9$ (Turán and Makai), and Szekeres conjectured $f(n)=2^{n-2}+1$. It is known that $f(n)>2^{n-2}$ and that $f(n)\leq\binom{2n-4}{n-2}$. For more details see Erdős–Szekeres [**43**] and also *The Art of Counting*, in particular the introduction by Szekeres.

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