# The Width of Random Subsets of Boolean Lattices*


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**Abstract**

Suppose we toss an independent coin with probability of success $p$ for each subset of $[n] = \{1, \ldots, n\}$, and form the random hypergraph $\mathcal{P}(n,p)$ by taking as hyperedges the subsets with successful coin tosses. We investigate the cardinality of the largest Sperner family contained in $\mathcal{P}(n,p)$. We obtain a sharp result for the range of $p = p(n)$ in which this Sperner family has cardinality comparable to the cardinality of $\mathcal{P}(n,p)$.


## 1 Results

In this note we investigate random non-uniform hypergraphs, *i.e.*, random families of subsets of an $n$-element set, or, equivalently, a random subset of the Boolean lattice $\mathcal{P}(n) = 2^{[n]}$, where, as usual, $[n] = \{1, \ldots, n\}$. Such random hypergraphs were probably first investigated by Rényi [3] who, answering a question of Erdős, determined the minimum cardinality of a family of random subsets of $[n]$ that guarantees, with high probability, two distinct sets $A$ and $B$ with $A \subset B$. A family containing no such pair of sets, *i.e.*, inducing no relation in the Boolean lattice, is usually called a *Sperner family*. The size of the largest Sperner family in a set-system $\mathcal{Q} \subset \mathcal{P}(n)$ is simply the *width* of $\mathcal{Q}$ when $\mathcal{Q}$ is viewed as a poset, and we shall denote it here by $\alpha(\mathcal{Q})$. For any given $0 < p = p(n) < 1$, let the random set-system $\mathcal{P}(n,p)$ be defined by picking each element of $\mathcal{P}(n)$ independently at random with probability $p$.


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Denote by $\Phi$ the distribution function of the normal distribution, *i.e.*,

$$\Phi(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} \exp(-x^2/2)\, dx.$$

For two functions $f(n)$ and $g(n)$, write $f \sim g$ if $\lim_{n\to\infty} f(n)/g(n) = 1$ and $f \asymp g$ if

$$0 < \liminf_{n\to\infty} f(n)/g(n) \le \limsup_{n\to\infty} f(n)/g(n) < \infty.$$

Our main result in this note may be formulated as follows. In the sequel, we use the term *almost surely* to mean 'with probability tending to 1 as $n \to \infty$'.

**Theorem 1** *Let $b > 0$ be a constant and $p = n^{-b\sqrt{n}}$. Then we almost surely have*

$$\frac{\alpha(\mathcal{P}(n,p))}{|\mathcal{P}(n,p)|} \sim \Phi(2b) - \Phi(-2b).$$

In fact, it will follow from the proof of Theorem 1 that the following structural result holds.

**Theorem 2** *Almost surely, one may obtain a Sperner family of cardinality $\sim \alpha(\mathcal{P}(n,p))$ from $\mathcal{P}(n,p)$ by taking all members of $\mathcal{P}(n,p)$ whose cardinalities are in the interval $\{\lfloor n/2 - b\sqrt{n}\rfloor, \dots, \lfloor n/2 + b\sqrt{n}\rfloor\}$, and then removing from this family the elements that are contained in some other element of the family.*

An immediate corollary to the results above is that, for $\varepsilon > 0$ and $p = n^{-\varepsilon\sqrt{n}}$, we almost surely have

$$\frac{\alpha(\mathcal{P}(n,p))}{|\mathcal{P}(n,p)|} \sim \varepsilon\left(1 + \mathcal{O}\left(\varepsilon^2\right)\right)\sqrt{\frac{8}{\pi}}.$$

The proof of Theorem 1 is given in the next section. For results concerning the *length* of $\mathcal{P}(n,p)$, *i.e.*, the cardinality of a longest chain, the reader is referred to [1]. The threshold function for small sublattices in $\mathcal{P}(n,p)$ are investigated in [2], which may be thought of as a modern sequel to Rényi's pioneering work [3].

# 2 Proofs

## 2.1 Proof of Theorem 1

Before giving the proof of Theorem 1, we state some asymptotics involving binomial coefficients. To simplify the notation, we shall often omit the $\lfloor\ \rfloor$ and $\lceil\ \rceil$ signs whenever they are not crucial. Moreover, we may and shall assume whenever needed that $n$ is larger than any fixed absolute constant.

Let $k$, $\ell$ be integers with $|k|$, $|\ell| = \mathcal{O}(\sqrt{n})$ and $k+\ell > 0$. Then,

$$\begin{aligned}\binom{n}{\frac{n}{2}-k, \frac{n}{2}-\ell, k+\ell} &= \binom{n-k-\ell}{\frac{n}{2}-k}\binom{n}{k+\ell} \\ &\asymp \frac{2^{n-k-\ell}}{\sqrt{n}} \times \frac{n^{k+\ell}}{(k+\ell)!} \\ &\asymp \frac{2^n}{\sqrt{n(k+\ell)}}\left(\frac{en}{2(k+\ell)}\right)^{k+\ell}. \end{aligned} \tag{1}$$

For later reference, notice that the multinomial coefficient on the left-hand side of (1) counts the number of pairs $(A, B)$ with $A \subset B \subset [n]$, $|A| = n/2 - k$, and $|B| = |A| + k + \ell$.

For integers $k$ and $m$ with $k = \mathcal{O}(\sqrt{m})$ and large enough $m$, the following rough bound holds:

$$\binom{m}{k} \geq \frac{1}{m}\left(\frac{em}{k}\right)^k. \tag{2}$$

Let an integer $i$ with $|i| \leq (\log n)\sqrt{n}$ and a real number $b$ with $0 < b = \mathcal{O}(1)$ be given. Then, for large $n$, we have

$$\begin{aligned}\binom{n}{\lfloor n/2\rfloor + i}\Bigg/\binom{n}{\lfloor n/2\rfloor + i + \lfloor b\sqrt{n}\rfloor} &\leq \left(1 + \frac{2|i| + b\sqrt{n}}{n/2 - |i| - b\sqrt{n} + 1}\right)^{b\sqrt{n}} \\ &\leq \exp\left(\frac{5\log n}{\sqrt{n}} b\sqrt{n}\right) = n^{5b}. \end{aligned} \tag{3}$$

For a set-system $\mathcal{Q} \subseteq \mathcal{P}(n)$ and an integer $i$ with $|i| \leq n/2$, denote by $\mathcal{Q}_i$ the $(\lfloor n/2\rfloor - i)$-th *layer* of $\mathcal{Q}$, *i.e.*,

$$\mathcal{Q}_i = \{x : x \in \mathcal{Q} \text{ and } |x| = \lfloor n/2\rfloor - i\}.$$

For a set of integers $S \subseteq \{-\lceil n/2\rceil, \ldots, \lceil n/2\rceil\}$, let $\mathcal{Q}_S = \bigcup_{i\in S} \mathcal{Q}_i$. The set $\mathcal{Q}_S$ will sometimes be viewed as a graph, where $\{x, y\}$ is an edge in $\mathcal{Q}_S$ if and only if $x \subset y$ or $y \subset x$. Instead of $\mathcal{P}(n,p)_S$ we write $\mathcal{P}_S(n,p)$, and instead of $\mathcal{Q}_{\{i,j\}}$ we simply write $\mathcal{Q}_{i,j}$.

Suppose $0 < p < 1$ and $0 < \varepsilon < 1$ are given, and let $X$ be binomially distributed with parameters $p$ and $N$. Then, the standard Chernoff bound for large deviations states that

$$P[|X - Np| \geq \varepsilon Np] \leq 2\exp\left(-\frac{1}{3}\varepsilon^2 Np\right). \tag{4}$$

This implies the following very rough bound on $|\mathcal{P}_i(n,p)|$.

(*) *Suppose* $p \geq (2/3)^n$. *Then, with probability at least* $1 - \exp\{-(5/4)^n\}$, *for every integer* $i$ *with* $|i| \leq (\log n)\sqrt{n}$, *we have*

$$|\mathcal{P}_i(n,p)| = \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor n/2\rfloor - i}p.$$

Let now a positive integer $n$, reals $0 < \varepsilon(n) < 1$, $0 < p(n) < 1$, and $c = c(n) > 0$ be given. Let $j$ be an integer with $c\sqrt{n} \le |j| \le (\log n)\sqrt{n}$. Define $i = i(j)$ by putting

$$i = j - \operatorname{sgn}(j)\lfloor 2c\sqrt{n}\rfloor, \tag{5}$$

where $\operatorname{sgn}(j)$ denotes the sign of $j$. Note that, then, we have $|j-i| = \lfloor 2c\sqrt{n}\rfloor$ and, moreover, $|i| \le |j|$. In what follows, we shall often be interested in the bipartite graph $\mathcal{Q}_{i,j}$ with $i$ and $j$ as above. We now introduce some important definitions for what follows.

We shall say that $\mathcal{Q} \subseteq \mathcal{P}(n)$ is *$(c, \varepsilon, p, j)$-uniform* if properties $(a)$ and $(b)$ below are fulfilled.

$(a)$ The number of vertices of $\mathcal{Q}_{i,j}$ that have degree larger than

$$(1+\varepsilon)\binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor}p$$

is at most $2\exp(-\varepsilon^{-1})|\mathcal{Q}_j|$.

$(b)$ The number of vertices in $\mathcal{Q}_j$ that have $\mathcal{Q}_{i,j}$-degree smaller than

$$(1-\varepsilon)\binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor}p$$

is at most $\exp(-\varepsilon^{-1})|\mathcal{Q}_j|$.

The set-system $\mathcal{Q} \subseteq \mathcal{P}(n)$ is called *$(c, \varepsilon, p)$-uniform* if the following properties are fulfilled:

$(i)$ For all integers $j$ with $|j| \le (\log n)\sqrt{n}$, we have

$$|\mathcal{Q}_j| = (1+\Theta(1/n))\binom{n}{\lfloor n/2\rfloor - j}p.$$

$(ii)$ Let $S = \{-n/2, \dots, -(\log n)\sqrt{n}\} \cup \{(\log n)\sqrt{n}, \dots, n/2\}$. Then $|\mathcal{Q}_S| \le 2^n p/n$.

$(iii)$ For all integers $j$ with $c\sqrt{n} \le |j| \le (\log n)\sqrt{n}$, the set-system $\mathcal{Q}$ is $(c, \varepsilon, p, j)$-uniform.

Now we state three lemmas that together imply Theorem 1.

**Lemma 3** *Let $c = c(n) > 0$ with $c = \mathcal{O}(1)$ be given. Let $p = n^{-c\sqrt{n}}(4c/\mathrm{e})^{2c\sqrt{n}}$ and put $S = \{-c\sqrt{n}, \dots, c\sqrt{n}\}$. Then, almost surely, the number of edges in $\mathcal{P}_S(n,p)$ is at most $2^n p/\sqrt{n}$.*

**Lemma 4** *For $c = c(n) > 0$ with $c = \mathcal{O}(1)$, define $\varepsilon = 2^{-c\sqrt{n}/2}$, and $p = n^{-c\sqrt{n}}(9c/\mathrm{e})^{2c\sqrt{n}}$. Then, almost surely, the random set-system $\mathcal{P}(n,p)$ is $(c, \varepsilon, p)$-uniform.*

**Lemma 5** *Let $c = c(n) > 0$ with $c = \mathcal{O}(1)$ be given. Define $p = n^{-c\sqrt{n}}(9c/\mathrm{e})^{2c\sqrt{n}}$ and let $0 < \varepsilon \leq 1/n$ be given. Assume that $\mathcal{Q}$ is $(c, \varepsilon, p)$-uniform. Then the width of $\mathcal{Q}$ is at most*

$$\left( \sum_{|j|<c\sqrt{n}} \binom{n}{\lfloor n/2 \rfloor - j} + \mathcal{O}\left(\frac{2^n}{n}\right) \right) p.$$

Before proving Lemmas 3, 4, and 5 we deduce Theorem 1 from those lemmas.

*Proof of Theorem 1.* Let $b > 0$ be given. Define $c = c(n)$ by

$$p = n^{-b\sqrt{n}} = n^{-c\sqrt{n}} \left(\frac{4c}{\mathrm{e}}\right)^{2c\sqrt{n}}.$$

Then $c = b + o(1)$. Let $S$ be as in Lemma 3. Then removing from $\mathcal{P}_S(n,p)$ all elements that are contained in some other element of $\mathcal{P}_S(n,p)$ yields a Sperner subfamily of $\mathcal{P}(n,p)$. By Lemma 3 and assertion (*) this subfamily is almost surely of cardinality at least

$$\begin{aligned} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \sum_{|i| \leq c\sqrt{n}} \binom{n}{\lfloor n/2 \rfloor - i} p - \frac{2^n}{\sqrt{n}} p \quad &\sim \quad (\Phi(2c) - \Phi(-2c)) 2^n p \\ &\sim \quad (\Phi(2b) - \Phi(-2b)) 2^n p. \end{aligned}$$

This establishes the lower bound in Theorem 1. To prove the upper bound, now define $c = c(n)$ by

$$p = n^{-b\sqrt{n}} = n^{-c\sqrt{n}} \left(\frac{9c}{\mathrm{e}}\right)^{2c\sqrt{n}}.$$

Again, $c = b + o(1)$. By Lemmas 4 and 5, the width of $\mathcal{P}(n,p)$ is almost surely at most

$$\sim \left(\Phi(2c) - \Phi(-2c) + \mathcal{O}\left(\frac{1}{n}\right)\right) 2^n p \sim (\Phi(2b) - \Phi(-2b))\, 2^n p,$$

and the theorem follows. □

## 2.2 Proofs of the lemmas

It now remains to prove Lemmas 3, 4, and 5.

*Proof of Lemma 3.* Denote by $\sum_{k,\ell}$ the sum over all pairs of integers $k$, $\ell$ with $|k|, |\ell| \leq c\sqrt{n}$ and $k + \ell > 0$. For $\mathcal{Q} \subseteq \mathcal{P}(n)$, denote by $X_S(\mathcal{Q})$ the number of edges in $\mathcal{Q}_S$. Then, by (1), we have

$$
\begin{aligned}
E[X_S(\mathcal{P}(n,p))] &\asymp \sum_{k,\ell} \frac{2^n}{\sqrt{n(k+\ell)}} \left(\frac{\mathrm{e}n}{2(k+\ell)}\right)^{k+\ell} p^2 \\
&= \mathcal{O}\left(2^n n^{-3/4} \left(\frac{\mathrm{e}}{4c}\sqrt{n}\right)^{2c\sqrt{n}} p^2\right) \\
&= \mathcal{O}\left(2^n n^{-3/4} p\right).
\end{aligned}
$$

By Markov's inequality, almost surely we have $X_S \leq 2^n p/\sqrt{n}$, as required. □

*Proof of Lemma 4.* Assertion (*) immediately gives that condition (*i*) holds almost surely. One may also easily check that condition (*ii*) is almost surely satisfied. To prove that condition (*iii*) holds almost surely, it is enough to show that, for any $j$ with $c\sqrt{n} \leq |j| \leq (\log n)\sqrt{n}$, the set-system $\mathcal{P}(n,p)$ is $(c,\varepsilon,p,j)$-uniform with probability at least, say, $1-1/n$.

Thus, in the sequel, we fix $j$ with $c\sqrt{n} \leq |j| \leq (\log n)\sqrt{n}$ and bound from above the probability that $\mathcal{P}(n,p)$ is not $(c,\varepsilon,p,j)$-uniform. By symmetry, we may assume that $j > 0$ and thus $i = i(j)$ in (5) is $i = j - \lfloor 2c\sqrt{n} \rfloor$. In particular, a set in $\mathcal{P}_j(n)$ contains $\binom{n/2+j}{\lfloor 2c\sqrt{n}\rfloor}$ elements from $\mathcal{P}_i(n)$.

For each $x \in \mathcal{P}_j(n)$, let $Y_x$ be the random variable whose value is the the number of neighbours of $x$ in $\mathcal{P}_i(n,p)$ if $x \in \mathcal{P}_j(n,p)$, and 0 otherwise. For all $x \in \mathcal{P}_j(n)$, the random variable $Y_x$ conditioned on $x \in \mathcal{P}_j(n,p)$ is binomially distributed with parameters $p$ and

$$
N = \binom{n/2+j}{\lfloor 2c\sqrt{n}\rfloor}.
$$

Furthermore, by (2), if $n$ is large enough,

$$
Np \geq \frac{1}{n}\left(\frac{\mathrm{e}n}{4c\sqrt{n}}\right)^{2c\sqrt{n}} n^{-c\sqrt{n}} \left(\frac{9c}{\mathrm{e}}\right)^{2c\sqrt{n}} \geq 2^{2c\sqrt{n}} = \varepsilon^{-4}. \tag{6}
$$

By (4), if $X$ is binomially distributed with parameters $p$ and $N$,

$$
P[|X - Np| \geq \varepsilon Np] \leq 2\exp\left(-\frac{1}{3}\varepsilon^2 Np\right) \leq 2\exp\left(-\frac{1}{3}\varepsilon^{-2}\right).
$$

Denote by $Z$ the number of elements in $\mathcal{P}_j(n,p)$ with degree larger than $(1+\varepsilon)Np$ or smaller than $(1-\varepsilon)Np$. By linearity of expectation,

$$
E[Z] = \mathcal{O}\left(\binom{n}{\lfloor n/2\rfloor - j} p \cdot \exp\left(-\frac{1}{3}\varepsilon^{-2}\right)\right).
$$

By Markov's inequality and assertion (*), the probability that $Z > \exp\left(-\varepsilon^{-1}\right)|\mathcal{P}_j(n,p)|$ is, very crudely, at most $1/2n$. We have thus dealt with condition $(b)$ and one part of condition $(a)$.

As to the remaining part of condition $(a)$, we notice that the degree of a vertex from $\mathcal{P}_i(n,p)$ in $\mathcal{P}_{i,j}(n,p)$ is binomially distributed with parameters $p$ and

$$\binom{n/2-i}{\lfloor 2c\sqrt{n}\rfloor} \le \binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor} = N,$$

where we made use of the fact that $|i| \le |j| = j$. Hence the expected number of elements in $\mathcal{P}_i(n,p)$ with degree larger than $(1+\varepsilon)Np$ is, by (3), at most

$$2\exp\left(-\frac{1}{3}\varepsilon^{-2}\right)\binom{n}{\lfloor n/2\rfloor+i}p \le 2\exp\left(-\frac{1}{3}\varepsilon^{-2}\right)\binom{n}{\lfloor n/2\rfloor+j}pn^{10c},$$

provided $n$ is large enough. Again, by Markov's inequality and assertion (*), with probability at most $1/2n$, more than $\exp\left(-\varepsilon^{-1}\right)|\mathcal{P}_j(n,p)|$ elements in $\mathcal{P}_i(n,p)$ have degree larger than $(1+\varepsilon)Np$ in $\mathcal{P}_{i,j}(n,p)$, and the lemma follows. $\square$

In the proof of Lemma 5 we need the following auxiliary result.

**Lemma 6** *Let $0 < c(n) = \mathcal{O}(1)$ be given. Define $p = n^{-c\sqrt{n}}(9c/\mathrm{e})^{2c\sqrt{n}}$. Let $0 < \varepsilon \le 1/n$ be given, and let $\mathcal{Q}$ be $(c,\varepsilon,p,j)$-uniform for some $j$ with $c\sqrt{n} \le |j| \le (\log n)\sqrt{n}$. Define $i = i(j)$ as in (5). Then there is a matching in $\mathcal{Q}_{i,j}$ that covers all but at most $4\varepsilon|\mathcal{Q}_j|$ elements from $\mathcal{Q}_j$, as long as $n$ is sufficiently large.*

*Proof of Lemma 6.* Remove all vertices of $\mathcal{Q}_{i,j}$ of degree larger than $(1+\varepsilon)\binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor}p$. Call the resulting set-system $\mathcal{R}_{i,j}$. By $(a)$, the number of edges of $\mathcal{Q}_{i,j}$ incident to the vertices that have been removed may be bounded from above, very crudely, by

$$2\exp(-\varepsilon^{-1})|\mathcal{Q}_j|\binom{n}{\lfloor 2c\sqrt{n}\rfloor} \le 2\exp(-\varepsilon^{-1})|\mathcal{Q}_j|n^{2c\sqrt{n}} \le \varepsilon|\mathcal{Q}_j|p,$$

provided $n$ is large enough.

Using condition $(b)$, we see that the number of edges $e(\mathcal{R}_{i,j})$ in $\mathcal{R}_{i,j}$ is at least

$$\left(1-\exp(-\varepsilon^{-1})\right)(1-\varepsilon)|\mathcal{Q}_j|\binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor}p - \varepsilon|\mathcal{Q}_j|p \ge (1-2\varepsilon)|\mathcal{Q}_j|\binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor}p.$$

Moreover, $\mathcal{R}_{i,j}$ has maximum degree at most $\Delta(\mathcal{R}_{i,j}) \le (1+\varepsilon)\binom{n/2+|j|}{\lfloor 2c\sqrt{n}\rfloor}p$. Since the edge-set of the bipartite graph $\mathcal{R}_{i,j}$ may be partitioned into $\Delta(\mathcal{R}_{i,j})$ matchings, the graph $\mathcal{R}_{i,j}$ contains a matching covering at least

$$\frac{e(\mathcal{R}_{i,j})}{\Delta(\mathcal{R}_{i,j})} \ge \frac{1-2\varepsilon}{1+\varepsilon}|\mathcal{Q}_j| \ge (1-4\varepsilon)|\mathcal{Q}_j|$$

elements from $\mathcal{Q}_j$, provided $n$ is large enough. $\square$

*Proof of Lemma 5.* For each $j$ with $c\sqrt{n} \leq |j| \leq (\log n)\sqrt{n}$, fix a matching in $\mathcal{Q}_{i,j}$ that covers all but $\leq 4\varepsilon|\mathcal{Q}_j|$ elements from $\mathcal{Q}_j$, as given by Lemma 6. Let $\mathcal{R}$ be the spanning subgraph of $\mathcal{Q}$ having as edges the union of all these matchings. A component in the graph $\mathcal{R}$ corresponds to a chain in $\mathcal{Q}$, when $\mathcal{Q}$ is considered as a poset. Hence the width of $\mathcal{Q}$ is at most the number of components of $\mathcal{R}$.

The graph $\mathcal{R}$ is a disjoint union of paths. Therefore the number of components of $\mathcal{R}$ is just the number of vertices in $\mathcal{R}$ minus the number of edges in $\mathcal{R}$. By properties $(i)$, $(ii)$, and Lemma 6, the number of components of $\mathcal{R}$ is, therefore, almost surely at most

$$\sum_{|j|\geq(\log n)\sqrt{n}} |\mathcal{Q}_j| + \sum_{c\sqrt{n}\leq|j|<(\log n)\sqrt{n}} 4\varepsilon|\mathcal{Q}_j| + \sum_{|j|<c\sqrt{n}} |\mathcal{Q}_j|$$
$$\leq \left(\frac{2^n}{n} + 4\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\varepsilon 2^n + \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\sum_{|j|<c\sqrt{n}}\binom{n}{\lfloor n/2\rfloor - j}\right)p$$

and the lemma follows as $\varepsilon \leq 1/n$. $\square$

# 3 Concluding remarks

It would be interesting to investigate $\alpha(\mathcal{P}(n,p))$ for the range of $p$ in which $\alpha(\mathcal{P}(n,p))$ is comparable to

$$|\mathcal{P}_0(n,p)| = \left|\mathcal{P}(n,p) \cap \binom{[n]}{\lfloor n/2\rfloor}\right|.$$

In particular, one may ask what the threshold functions for the following properties are:

(1) $\alpha(\mathcal{P}(n,p)) = |\mathcal{P}_0(n,p)|$,
(2) $\alpha(\mathcal{P}(n,p)) \sim |\mathcal{P}_0(n,p)|$.

Many similar questions remain open.

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