RAMSEY-TYPE RESULTS FOR ORIENTED TREES

YOSHIHARU KOHAYAKAWA†, TOMASZ ŁUCZAK‡
AND
VOJTĚCH RÖDL*

Abstract. For a graph G and a digraph $\vec{H}$, we write $G \to \vec{H}$ (respectively, $G \overset{a}{\to} \vec{H}$) if every orientation (respectively, acyclic orientation) of the edges of G results in an *induced copy* of $\vec{H}$. In this note we study how small the graphs G such that $G \to \vec{H}$ or such that $G \overset{a}{\to} \vec{H}$ may be, if $\vec{H}$ is a given oriented tree $\vec{T}^n$ on n vertices. We show that there is a graph on $O(n^4 \log n)$ vertices and $O(n^6 (\log n)^2)$ edges such that $G \to \vec{T}^n$ for every n-vertex oriented tree $\vec{T}^n$. We also prove that there exists a graph G with $O(n^2 \log n)$ vertices and $O(n^3 (\log n)^2)$ edges such that $G \overset{a}{\to} \vec{T}^n$ for any such tree $\vec{T}^n$. This last result turns out to be nearly best possible as it is shown that any graph G with $G \overset{a}{\to} \vec{P}^n$, where $\vec{P}^n$ is the directed path of order n, has more than $n^2/2$ vertices and more than $\lfloor n/3 \rfloor^3$ edges if $n \geq 3$.

1991 Mathematics subject classifications: Primary 05C55, 05C20, 05C05; Secondary 05C80

Key words: Ramsey theory, trees, directed graphs, orientations of graphs, random graphs

1. Introduction. In this note we are concerned with some Ramsey-theoretical properties of orientations of graphs. We shall use the term *directed graph* or *digraph* to mean a graph in which the edges have been given a fixed orientation; in particular, between two distinct vertices x and y of a digraph $\vec{H}$ we may have at most one of the directed edges $\overrightarrow{xy}$ and $\overrightarrow{yx}$. An *orientation* $\vec{G}$ of a graph G is a digraph whose underlying graph is G.

In analogy to the arrow notation of Rado, for a given graph G and an acyclic digraph $\vec{H}$, we write $G \to \vec{H}$ if every orientation of the edges of G leads to an *induced* copy of $\vec{H}$. Similarly, $G \overset{a}{\to} \vec{H}$ will mean that every *acyclic* orientation of G contains an induced copy of $\vec{H}$. For any given acyclic digraph $\vec{H}$, the existence of a graph G satisfying $G \overset{a}{\to} \vec{H}$ follows from a result of Rödl [5], and the fact that there is a graph G such that $G \to \vec{H}$ was shown by Cochand and Duchet [2]. For convenience, for any class $\mathcal{D}$ of digraphs, let us write $G \to \mathcal{D}$ (respectively, $G \overset{a}{\to} \mathcal{D}$) if any orientation of G (respectively, any acyclic orientation of G) contains an induced copy of every $\vec{H} \in \mathcal{D}$.

In this note we study how small the graphs G such that $G \to \vec{\mathcal{T}}^n$ or such that $G \overset{a}{\to} \vec{\mathcal{T}}^n$ may be, where $\vec{\mathcal{T}}^n$ is the class of all oriented trees on n vertices. The existence results mentioned above give rather poor upper bounds for the order $|G| = |V(G)|$ and size $e(G) = |E(G)|$

† Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 20570, 01452–990 São Paulo, SP, Brazil. Research partially supported by FAPESP (Proc. 93/0603–1) and by CNPq (Proc. 300334/93–1 and ProTeM–CC–II Project ProComb).

‡ Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322. On leave from the Mathematical Institute of the Polish Academy of Sciences, and Adam Mickiewicz University, Poznań, Poland. Research partially supported by KBN grant 2 1087 91 01.

* Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322. Research partially supported by NSF grant DMS–9401559.

of such graphs G. For instance, Cochand and Duchet's method in [2] gives that, for some absolute constant c and every integer $n \geq 1$, there is a graph $G = G_n$ of order at most $2^{2^{cn}}$ such that $G \rightarrow \vec{\mathcal{T}}^n$. Moreover, it is conjectured in [2] that in this result the double exponential upper bound for the order of G_n may be replaced by an exponential upper bound. As observed by Brightwell and Kohayakawa [1], the existence of a graph G_n with $G_n \rightarrow \vec{\mathcal{T}}^n$ and $|G_n| \leq n^{(2+o(1))n}$ follows by considering graphs with high girth and high chromatic number. Our first result states that much more is in fact true.

THEOREM 1. *There exists an absolute constant C such that for every integer $n \geq 1$ there is a graph G with at most $Cn^4 \log n$ vertices and at most $Cn^6(\log n)^2$ edges such that $G \rightarrow \vec{\mathcal{T}}^n$.*

We now turn to the relation $\xrightarrow{\mathrm{a}}$, and start by observing that the property $G \xrightarrow{\mathrm{a}} \vec{H}$ is, in some sense, a little easier to handle and hence we may expect stronger results. One such result is due to Rödl and Winkler [6], which implies that, for every graph $H = H^n$ of order n, there is a graph G of order $O(n^3(\log n)^2)$ such that $G \xrightarrow{\mathrm{a}} \vec{H}$ for any acyclic orientation $\vec{H}$ of H. A related result was obtained by Brightwell and Kohayakawa [1], who proved the existence of a graph G with at most $\lfloor (4/9)n2^{n/2} \rfloor$ vertices such that $G \xrightarrow{\mathrm{a}} \mathcal{D}^n$, where $\mathcal{D}^n$ is the class of all acyclic digraphs of order n. Let us also mention that the two results above are, in a precise sense, essentially best possible (see [1, 6] for details.) However, for the particular case of oriented trees, they can be considerably improved.

THEOREM 2. *There exists an absolute constant D such that for every integer $n \geq 1$ there is a graph G with at most $Dn^2 \log n$ vertices and at most $Dn^3(\log n)^2$ edges such that $G \xrightarrow{\mathrm{a}} \vec{\mathcal{T}}^n$.*

One may ask how good the bounds given in Theorems 1 and 2 are. Unfortunately, we do not know whether the powers of n in the bounds $O(n^4 \log n)$ and $O(n^6(\log n)^2)$ in Theorem 1 may be reduced to smaller powers. Nevertheless, the following result implies that the bounds given in Theorem 2 cannot be improved except possibly for a logarithmic factor. Let $\vec{P}^n$ be the directed path of order n.

THEOREM 3. *Suppose a graph G is such that $G \xrightarrow{\mathrm{a}} \vec{P}^n$ for some integer $n \geq 3$. Then G has more than $n^2/2$ vertices and more than $\lfloor n/3 \rfloor^3$ edges.*

2. Proof of Theorem 1. Suppose G is a graph on $N \geq 3$ vertices with the following two properties:

(a) every subset $S \subseteq V(G)$ of vertices of G with $s = |S| \geq s_0 = \lfloor 10\mathrm{e}^2 (N \log N)^{1/2} \rfloor$ induces at least $(1/4\mathrm{e}^2)s^2(\log N)^{1/2}N^{-1/2}$ edges;

(b) for every $S \subseteq V(G)$ and every $v \in V(G) \setminus S$, the number of vertices $x \in V(G) \setminus (S \cup \{v\})$ that are adjacent to v and to some vertex in S is at most $|S| \log N$.

We shall prove that then $G\to\vec{\mathcal{T}}^n$ for $n=\lfloor(1/5\mathrm{e}^2)(N/\log N)^{1/4}\rfloor$. More specifically, we shall show that if r is a positive integer, then

(†) for every $U\subseteq V(G)$ with $|U|\geq 5\mathrm{e}^2r^2(N\log N)^{1/2}$ the subgraph $G[U]$ of G induced by U in G is such that $G[U]\to\vec{\mathcal{T}}^r$.

We shall prove (†) by induction on r. The statement is trivial for $r=1$. Let us then assume that $r\geq 1$ and that (†) holds for this value of r. Let us prove (†) for $r+1$. Thus, assume that $U\subseteq V(G)$ with $|U|\geq 5\mathrm{e}^2(r+1)^2(N\log N)^{1/2}$ is given, set $H=G[U]$, and fix an arbitrary orientation $\vec{H}$ of H and an arbitrary oriented $(r+1)$-vertex tree $\vec{T}^{r+1}$. We aim at showing that $\vec{H}$ contains an induced copy of $\vec{T}^{r+1}$. We shall assume that there is a pendant vertex $u\in V(\vec{T}^{r+1})$ that is an out-neighbour of some $v\in V(\vec{T}^{r+1})$. An obvious alteration to the argument below deals with the case in which every pendant vertex of $\vec{T}^{r+1}$ is a source. Let $W\subseteq U=V(H)$ be the set of vertices $x\in V(\vec{H})$ with $\vec{H}$-out-degree $d^+_{\vec{H}}(x)\leq 2(r-1)\log N$.

We now check that $w=|W|<s_0r$. If $w\geq s_0r\geq s_0$, then condition (a) implies that $|E(G[W])|\geq(1/4\mathrm{e}^2)w^2(\log N)^{1/2}N^{-1/2}$ and, since $|E(G[W])|\leq 2w(r-1)\log N$, we conclude that $w\leq 8\mathrm{e}^2r(N\log N)^{1/2}<s_0r$, which is a contradiction. Thus indeed $|W|<s_0r$.

We now consider $U'=U\setminus W$. As $|W|<s_0r\leq 10\mathrm{e}^2r(N\log N)^{1/2}$, we have $|U'|\geq 5\mathrm{e}^2r^2(N\log N)^{1/2}$. Therefore, by the induction hypothesis, $\vec{H}[U']$ contains an induced copy of $\vec{T}^r=\vec{T}^{r+1}-u$; say $\vec{H}[S']$ is isomorphic to $\vec{T}^r$, where $S'\subseteq U'$. Let $v'\in S'$ be the vertex in S' that corresponds to $v\in V(\vec{T}^r)$ through this isomorphism. Set $S=S'\setminus\{v'\}$. Since $v'\notin W$, we have $d^+_{\vec{H}}(v')>2(r-1)\log N\geq(r-1)(1+\log N)$, and hence from (b) the vertex v' has an $\vec{H}$-out-neighbour $u'\in V(\vec{H})=U$ that does not belong to the neighbourhood of S. Thus $\vec{H}[S\cup\{u'\}]$ is an induced copy of $\vec{T}^{r+1}$ in $\vec{H}$ and the induction step is complete. Thus assertion (†) holds.

To finish the proof of Theorem 1, it is enough to show that, for any large enough N, there is a graph G for which both (a) and (b) above hold. We consider the usual binomial random graph $G(N,p)$, whose vertex set is $[N]=\{1,\dots,N\}$ and each pair ij $(1\leq i<j\leq N)$ is present as an edge in $G(N,p)$ independently with probability $p=p(N)$. We shall take $p=p(N)=\mathrm{e}^{-2}((\log N)/N)^{1/2}$. We shall show that then (a) and (b) hold for $G=G(N,p)$ *almost surely*, that is with probability tending to 1 as $N\to\infty$.

Let us consider (a) first. Fix $S\subseteq V(G)$ with $s=|S|\geq s_0$. The number of edges $e(G[S])$ induced by S in $G=G(N,p)$ is binomially distributed with parameters $\binom{s}{2}$ and p. As is well known (see *e.g.* §5 in McDiarmid [4]), a binomial random variable X with parameters m and p is such that, for every $0<\varepsilon<1$,

$$\mathbb{P}(X<(1-\varepsilon)mp)\leq\exp\left\{-\frac{1}{2}\varepsilon^2mp\right\}.$$

Thus the probability that S induces fewer than $(1/4\mathrm{e}^2)s^2((\log N)/N)^{1/2}=s^2p/4$ edges in G is at most $\exp\{-(1/16+o(1))s^2p\}$. Thus the probability that (a) fails for $G=G(N,p)$

is at most

$$\sum_{s\ge s_0}\binom{N}{s}\exp\left\{-\left(\frac{1}{16}+o(1)\right)s^2p\right\}\le\sum_{s\ge s_0}\left(\frac{\mathrm{e}N}{s}\exp\left\{-\left(\frac{1}{16}+o(1)\right)sp\right\}\right)^s$$
$$\le N\left(\frac{\mathrm{e}N}{s_0}\exp\left\{-\left(\frac{1}{16}+o(1)\right)s_0p\right\}\right)^{s_0}.$$

Since $s_0p\ge 9\log N$ for large enough N, property (a) fails for $G=G(N,p)$ with probability at most

$$N\left\{O\left(\left(\frac{N}{\log N}\right)^{1/2}\mathrm{e}^{-(9/16+o(1))\log N}\right)\right\}^{N^{1/2}}=N\left\{O\left(N^{-1/16+o(1)}\right)\right\}^{N^{1/2}}=o(1),$$

as required.

Let us now turn to (b). For convenience, write $\Gamma(x)$ for the neighbourhood of $x\in V(G)$ in $G=G(N,p)$, and as customary put $\Gamma(X)=\bigcup_{x\in X}\Gamma(x)$ for any $X\subseteq V(G)$. Moreover, let $\tilde\Gamma(X)=\Gamma(X)\setminus X$ for any $X\subseteq V(G)$. We now fix $S\subseteq V(G)$, $v\in V(G)\setminus S$, and $x\in V(G)\setminus(S\cup\{v\})$. Then clearly $\mathbb{P}(x\in\tilde\Gamma(S)\cap\Gamma(v))\le p^2s$. Thus $\mathbb{P}\{|\tilde\Gamma(S)\cap\Gamma(v)|\ge t\}\le\binom{N}{t}(p^2s)^t$ for any $t\ge 0$. For $t=\lceil s\log N\rceil$, where $s=|S|$, we have

$$\mathbb{P}\{|\tilde\Gamma(S)\cap\Gamma(v)|\ge s\log N\}\le\left(\frac{\mathrm{e}Np^2s}{t}\right)^t\le\left(\frac{\mathrm{e}Np^2}{\log N}\right)^{s\log N}\le N^{-3s}.$$

Thus the probability that (b) fails is at most

$$\sum_{s\ge 1}\binom{N}{s}N^{1-3s}\le N\sum_{s\ge 1}\left(\frac{\mathrm{e}}{sN^2}\right)^s=o(1).\quad\square$$

3. Proof of Theorem 2. As in the previous section, let $G(N,p)$ be the random graph with vertex set $[N]=\{1,2,\dots,N\}$ in which each pair of vertices is connected by an edge with probability $p=p(N)$, independently for each pair. We claim that almost surely every acyclic orientation of $G(N,p)$ contains every oriented tree on $n=\lfloor(N/150\log N)^{1/2}\rfloor$ vertices as an induced subgraph if $p=p(N)=1/n$. Clearly, this claim implies Theorem 2.

In the sequel, we denote by $\vec G_{\mathrm{incr}}=\vec G_{\mathrm{incr}}(N,p)$ the acyclic orientation of $G=G(N,p)$ where each edge $ij\in E(G)$ $(1\le i<j\le N)$ is directed from the smaller vertex i to the larger vertex j. In order to prove our claim, we shall give an algorithm, Algorithm Find, that, for every given oriented n-vertex tree $\vec T^n$, finds an induced copy of $\vec T^n$ in $\vec G_{\mathrm{incr}}=\vec G_{\mathrm{incr}}(N,p)$ with probability at least $1-N^{-2N}$. Let us quickly check that the existence of such an algorithm proves our claim. Observe first that the number of acyclic orientations of a graph of order N is at most $N!$, and that the number of oriented n-vertex trees is at

most $2^{n-1}n^{n-2} \leq 2^n n^n$. Thus it follows from the existence of Algorithm FIND that the probability that $G(N,p) \xrightarrow{\text{a}} \vec{T}^n$ fails is smaller than $N!2^n n^n N^{-2N} = o(1)$, as required.

Let us now sketch the idea behind Algorithm FIND. Let an n-vertex oriented tree $\vec{T}^n$ be given. Our algorithm searches for an induced copy of $\vec{T}^n$ in $\vec{G}_{\text{incr}}(N,p)$ calling a procedure, Procedure ONE-ROUND, several times until it either finds such a copy, or else it decides it has failed and it halts. (It is the probability that this latter case occurs that we have to show is no greater than N^{-2N}.) Procedure ONE-ROUND receives as input a set $R = \{r_1, \ldots, r_m\} \subseteq [N]$ $(1 \leq m \leq N)$ of *available* vertices of $\vec{G}_{\text{incr}} = \vec{G}_{\text{incr}}(N,p)$. The first time Procedure ONE-ROUND is executed we have $m = N$, that is all vertices of $\vec{G}_{\text{incr}}$ are available, while in the subsequent calls of Procedure ONE-ROUND the set of available vertices is increasingly smaller. Algorithm FIND tries to embed $\vec{T}^n$ into $\vec{G}_{\text{incr}}(N,p)$ in a natural way: each of its calls of ONE-ROUND constitutes a 'greedy' type attempt of embedding $\vec{T}^n$ into $\vec{G}_{\text{incr}}[R]$, the subgraph of $\vec{G}_{\text{incr}} = \vec{G}_{\text{incr}}(N,p)$ induced by the current set R of available vertices. Then, either ONE-ROUND succeeds and returns an induced copy of $\vec{T}^n$ in $\vec{G}_{\text{incr}}[R]$, or else it fails, constructing only a "non-extendable" copy of a subtree $\vec{T}^{n'}$ of $\vec{T}^n$ $(n' < n)$. In the latter case, Procedure ONE-ROUND removes the vertices of this copy from R, returns an updated set of available vertices $R' \subseteq R$, and reports *failure*. The key observation for the probabilistic analysis of Algorithm FIND is the fact that, every time ONE-ROUND is executed and it searches for $\vec{T}^n$ in $\vec{G}_{\text{incr}}[R]$, $R = \{r_1, \ldots, r_m\}$, all pairs $r_i r_j$ $(1 \leq i < j \leq m)$ are not yet "exposed", *i.e.* not yet known or not known to belong to $E(G(N,p))$, so that we may treat the graph induced by R in $\vec{G}_{\text{incr}}(N,p)$ as a random graph on R.

Now let us make the description of Algorithm FIND a little more precise. Fix an arbitrary n-vertex oriented tree $\vec{T}^n$. Let $v_1, \ldots, v_n$ be any ordering of the vertices of $\vec{T}^n$ such that every v_i $(2 \leq i \leq n)$ has precisely one neighbour, say w_i, in the set $\{v_1, \ldots, v_{i-1}\}$. We say that v_i is a *right* vertex if it is an out-neighbour of w_i, and otherwise we say that v_i is a *left* vertex. As mentioned above, Algorithm FIND looks for an induced copy of $\vec{T}^n$ in $\vec{G}_{\text{incr}}(N,p)$ calling Procedure ONE-ROUND several times until it either finds such a copy, or else it gives up and halts. Procedure ONE-ROUND receives as input a set $R = \{r_1, \ldots, r_m\} \subseteq [N]$, $r_1 < \cdots < r_m$ $(m \geq 1)$, of *available* vertices of $\vec{G}_{\text{incr}} = \vec{G}_{\text{incr}}(N,p)$ with all pairs $r_i r_j$ $(1 \leq i < j \leq m)$ not yet known or not known to belong to $E(G(N,p))$. ONE-ROUND either returns an induced copy of $\vec{T}^n$ in $\vec{G}_{\text{incr}}[R]$, or else it returns an updated set of available vertices $R' \subseteq R$ and reports *failure*.

Procedure ONE-ROUND searches for a copy of $\vec{T}^n$ in $\vec{G}_{\text{incr}}[R]$ by trying to map suitably the vertices $v_1, v_2, \ldots$ of $\vec{T}^n$ into R in this order, one by one. Our goal is to find $r_{i_1}, \ldots, r_{i_n}$ such that $v_j \mapsto r_{i_j}$ $(1 \leq j \leq n)$ is an isomorphism of $\vec{T}^n$ onto $\vec{G}_{\text{incr}}[r_{i_1}, \ldots, r_{i_n}]$. More specifically, for each right (respectively left) vertex v_j $(2 \leq j \leq n)$, we would like to have $r_{i_j} > r_{i_\ell}$ (respectively $r_{i_j} < r_{i_\ell}$) for each $1 \leq \ell < j$, and, furthermore, the only neighbour of r_{i_j} among the vertices $r_{i_1}, \ldots, r_{i_{j-1}}$ should be the one corresponding to $w_j \in V(\vec{T}^n)$ in $\vec{G}_{\text{incr}}[r_{i_1}, \ldots, r_{i_{j-1}}]$.

Procedure ONE-ROUND starts by setting $r_{i_1} = r_{\lfloor m/2 \rfloor}$, and by labelling $r_{\lfloor m/2 \rfloor}$ *selected*. Let us assume that v_2 is a right (respectively left) vertex. ONE-ROUND then checks whether the vertex $r_{\lfloor m/2 \rfloor+1}$ (respectively $r_{\lfloor m/2 \rfloor-1}$) is adjacent to r_{i_1} in $\vec{G}_{\mathrm{incr}}$. If not, it labels it *checked*, and examines $r_{\lfloor m/2 \rfloor+2}$ (respectively $r_{\lfloor m/2 \rfloor} - 2$). The procedure carries on in this manner until a neighbour of r_{i_1} is found. ONE-ROUND lets this neighbour be r_{i_2}, and labels it *selected*. In general, let us suppose that the vertices $r_{i_1}, \ldots, r_{i_{j-1}}$ $(2 \le j \le n)$ have already been chosen, and that v_j is a right (respectively left) vertex. Then the procedure examines one by one the vertices that lie on the right (respectively left) of the vertex $\max\{r_{i_1}, \ldots, r_{i_{j-1}}\}$ (respectively $\min\{r_{i_1}, \ldots, r_{i_{j-1}}\}$) until it finds a vertex that is adjacent to precisely the vertex that corresponds to w_j. ONE-ROUND then lets this vertex be r_{i_j}, labels it *selected*, and labels as *checked* all the other vertices that were examined but could not be chosen as r_{i_j}. In case ONE-ROUND fails to define r_{i_j}, it returns $R' = R \setminus S$, where $S \subseteq R$ is the set of vertices that have been labelled *selected*, and reports *failure*.

Clearly, if ONE-ROUND fails, then, while executing this procedure, we must have run out of vertices either on the left or on the right of $r_{\lfloor m/2 \rfloor}$. In either case, at least half of all available m vertices must have been labelled *selected* or *checked*. Since the probability that a vertex we examine is labelled *checked* is at most $1 - p(1-p)^{n-1}$, the probability that our procedure fails is, if $n = o(m)$, smaller than

$$2\sum_{k=1}^{n-1} \binom{m}{k} (1 - p(1-p)^n)^{\lfloor m/2 \rfloor - k} \le 2n \binom{m}{n} \exp\left\{ -\left(\frac{1}{2} + o(1)\right) mp(1-p)^n \right\}$$

$$\le 2n \left(\frac{\mathrm{e}m}{n} \exp\left\{ -\left(\frac{1}{2} + o(1)\right) \frac{m}{n} p(1-p)^n \right\} \right)^n .$$

Algorithm FIND proceeds as follows. It first puts $R = \{r_1, \ldots, r_N\} = [N]$ and calls Procedure ONE-ROUND with this input. If $\vec{T}^n$ is found, it returns this copy and halts. If ONE-ROUND reports *failure*, it calls ONE-ROUND with the set R' returned by the previous call of ONE-ROUND as the input set R for this call. FIND proceeds in this manner until it either finds an induced copy of $\vec{T}^n$ in $\vec{G}_{\mathrm{incr}}$, or else the set of available vertices R' returned by some call of ONE-ROUND is such that $|R'| \le N/2$.

The key observation concerning ONE-ROUND is the following: if a call of ONE-ROUND fails, then the set $R' \subseteq R$ that it returns has at least $|R| - n + 1$ elements, and moreover no pair ij $(i,\, j \in R')$ has been checked for adjacency in $G(N,p)$. Thus the several calls of ONE-ROUND examine *independent* random graphs.

Let us now note that, since FIND only calls ONE-ROUND with sets R with $m = |R| > N/2$, each such call fails with probability at most

$$2n \left(\frac{\mathrm{e}N}{n} \exp\left\{ -\left(\frac{1}{4} + o(1)\right) \frac{N}{n} p\mathrm{e}^{-2pn} \right\} \right)^n \le 2n \left(\frac{\mathrm{e}N}{n} \exp\left\{ -\frac{149}{4\mathrm{e}^2} \log N \right\} \right)^n$$

$$\le 2n \left(\frac{\mathrm{e}N}{n} \mathrm{e}^{-5\log N} \right)^n \le N^{-(4.5+o(1))n}.$$

Moreover, if FIND fails, ONE-ROUND is called at least $N/2n$ times. From the independence property among the calls of ONE-ROUND observed above, we deduce that FIND fails to find an induced copy of $\vec{T}^n$ in $\vec{G}_{\mathrm{incr}} = \vec{G}_{\mathrm{incr}}(N,p)$ with probability at most $\{N^{-(4.5+o(1))n}\}^{N/2n} \le N^{-2N}$, as required. □

4. Proof of Theorem 3. Let first an integer $n \ge 3$ be fixed. We shall show that every graph G on at most $n^2/2$ vertices can be acyclically oriented in such a way that the resulting graph contains no induced copy of $\vec{P}^n$. We use induction on n. For $n = 3$ the assertion can be verified in a straightforward manner. Let G be a graph on at most $n^2/2$ vertices, $n \ge 4$, and assume that the assertion holds for smaller values of n. If G contains no induced path of order n, there is nothing to prove. Thus, let us assume that a subset $S \subseteq V(G)$ of vertices of G induces in G a path of order n. Note that the subgraph $H = G - S$ has fewer than $(n-1)^2/2$ vertices, and so by the induction hypothesis there exists an acyclic orientation $\vec{H}$ of H in which no directed path of order $n-1$ is induced. We are now ready to define a suitable orientation $\vec{G}$ of G. First, direct all edges of the n-vertex path induced by S in such a way that no directed path of length two is created. Secondly, direct all edges between sinks of $G[S]$ and $V(H)$ towards S, and direct all edges between sources of $G[S]$ and $V(H)$ towards $V(H)$. Finally direct the edges in H as in $\vec{H}$. Then, clearly, this orientation of $\vec{G}$ of G contains no induced directed path of order n, as required.

Let us now note that the second bound in Theorem 3 is trivial for $n \le 5$, and hence let us assume that $n \ge 6$. Let G be a graph satisfying $G \xrightarrow{\mathrm{a}} \vec{P}^n$. For $0 \le r \le n-3$, let H_r be the subgraph of G induced by all the vertices of degree at least r in G. Note that H_r must contain more than $(n-r)^2/2$ vertices. Indeed, otherwise the edges of H_r could be oriented in such a way that no induced directed path of order $n-r$ appears, and, furthermore, since the subgraph $\bar{H}_r = G - H_r$ has chromatic number at most r, its edges can be acyclically oriented in such a way that no path of order larger than r is created. Consequently, directing the edges between H_r and $\bar{H}_r$ from H_r to $\bar{H}_r$ we would obtain an orientation of G without an induced $\vec{P}^n$. Thus, the number of edges of G must be larger than

$$\max_{1 \le r \le n-3} \left\{ \frac{r}{2} \cdot \frac{(n-r)^2}{2} \right\} \ge \lfloor n/3 \rfloor^3 . \quad \square$$

5. Final remarks. We note that there are oriented trees $\vec{T}^n$ for which the order and size of the smallest graph G with $G \to \vec{T}^n$ can be easily determined up to a constant factor. For example, if $\vec{S}_n$ is the *directed star*, *i.e.* the directed tree on $n+1$ vertices in which one vertex has outdegree n, and $K_{2n,2n}$ is the complete bipartite graph with $2n$ vertices in each class, then $K_{2n,2n} \to \vec{S}_n$. On the other hand, obviously, if $G \xrightarrow{\mathrm{a}} \vec{S}_n$ then G must have at least $n+1$ vertices and at least $n^2/2$ edges. Similar results may also be obtained for few other rather special classes of directed trees.

Finally, let us mention an example of an oriented tree $\vec{Z}_n$ for which there exist very sparse graphs G with $G \to \vec{Z}_n$. Let $\vec{Z}_n$ be the oriented tree whose underlying graph is the path of length n and the orientation is such that $\vec{Z}_n$ contains no directed paths of length two. Then the following result holds.

THEOREM 4. *There exists an absolute constant A such that for every integer $n \geq 1$ there is a graph G with at most An edges such that $G \to \vec{Z}_n$.*

Sketch of the proof. Haxell, Kohayakawa and Łuczak [3] showed that there exists an absolute constant A' such that, for every integer $n \geq 3$, there is a graph G_n with at most $A'n$ edges such that each two-colouring of the edges of G_n leads to a monochromatic induced cycle of length n. From their proof it follows that in the case when n is even G_n can be chosen to be bipartite. Thus, let G_{2n} be a bipartite graph with bipartition (W_1, W_2) and at most $2A'n$ edges such that every two-colouring of the edges of G_{2n} results in an induced monochromatic cycle of length $2n$. Take any orientation $\vec{G}_{2n}$ of G_{2n}, and colour the edges oriented from W_1 to W_2 *blue* and the edges oriented from W_2 to W_1 *red*. Now the fact that there is a monochromatic induced $2n$-cycle in G_{2n} implies that there exists an induced cycle of length $2n$ in $\vec{G}_{2n}$ containing no oriented paths of length two, and so $G_{2n} \to \vec{Z}_{2n-2}$, and hence obviously $G_{2n} \to \vec{Z}_n$ if $n \geq 2$. □

REFERENCES

[1] Brightwell, G.R. and Kohayakawa, Y., *Ramsey properties of orientation of graphs*, Random Structures & Algorithms, **4** (1993), pp. 413–428.

[2] Cochand, M. and Duchet, P., *A few remarks on orientations of graphs and Ramsey theory*, in "Irregularities of Partitions," Algorithms and Combinatorics 8 (Halász, G. and Sós, V.T., eds), Springer-Verlag, Berlin, 1989, pp. 39–46.

[3] Haxell, P.E., Kohayakawa, Y. and Łuczak, T., *The induced size-Ramsey number of cycles*, Combinatorics, Probability & Computing (to appear).

[4] McDiarmid, C.J.H., *On the method of bounded differences*, in "Surveys in Combinatorics 1989," London Mathematical Society Lecture Notes Series 141 (Siemons, J., ed.), Cambridge University Press, Cambridge, 1989, pp. 148–188.

[5] Rödl, V., *A generalization of Ramsey theorem*, in "Graphs, Hypergraphs and Block Systems," Zielona Góra, 1976, pp. 211–220.

[6] Rödl, V. and Winkler, P., *A Ramsey-type theorem for orderings of a graph*, SIAM J. Discrete Math., **2** (1989), pp. 402–406.