TURÁN'S EXTREMAL PROBLEM IN RANDOM GRAPHS: FORBIDDING ODD CYCLES

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Abstract. For $0 < \gamma \le 1$ and graphs G and H, we write $G \to_\gamma H$ if any γ-proportion of the edges of G span at least one copy of H in G. As customary, we write C^k for a cycle of length k. We show that, for every fixed integer $\ell \ge 1$ and real $\eta > 0$, there exists a real constant $C = C(\ell, \eta)$, such that almost every random graph $G_{n,p}$ with $p = p(n) \ge Cn^{-1+1/2\ell}$ satisfies $G_{n,p} \to_{1/2+\eta} C^{2\ell+1}$. In particular, for any fixed $\ell \ge 1$ and $\eta > 0$, this result implies the existence of very sparse graphs G with $G \to_{1/2+\eta} C^{2\ell+1}$.

§0. Introduction

In this note we are interested in a Turán type problem concerning extremal subgraphs of random graphs, which may be formulated as follows. Given a graph H, we ask for the maximal size of an H-free subgraph of $G_p = G_{n,p}$, the usual binomial random graph on n vertices with edge probability p. Let us make this precise by introducing some definitions and notation.

Let a graph G be given. As customary, write $|G|$ for the *order* $|V(G)|$ of G and $e(G)$ for the *size* $|E(G)|$ of G. If H is a graph, we say that G is *H-free* if G does not contain a subgraph isomorphic to H. Also, we let $\mathrm{ex}(G,H)$ be the maximal size of an H-free subgraph of G, that is $\mathrm{ex}(G,H) = \max\{e(J)\colon H \not\subset J \subset G\}$. For instance, if $G = K^n$, the complete graph on n vertices, then $\mathrm{ex}(K^n, H) = \mathrm{ex}(n,H)$, the usual Turán number of H. Here we are concerned with the case in which $G = G_p = G_{n,p}$, the random graph on $V(G_p) = \{1, \dots, n\}$ where each edge ij ($1 \le i < j \le n$) belongs to G_p with probability p, independently of all other edges. (See Bollobás [**3**] for details concerning random graphs.)

With the notation as above, our Turán type extremal problem for random graphs asks for the investigation of the random variable $Z_H = Z_H(G_p) = \mathrm{ex}(G_p, H)$ for any given $0 < p = p(n) \le 1$.

A celebrated result of Erdős and Stone [**4**] (see also Bollobás [**2**]) states that the asymptotic behaviour of $\mathrm{ex}(n,H) = \mathrm{ex}(K^n, H)$ is determined by the chromatic number of H, provided only that $\chi(H) \ge 3$. Indeed, for any H we have

$$\lim_{n\to\infty} \frac{\mathrm{ex}(K^n, H)}{e(K^n)} = 1 - \frac{1}{\chi(H) - 1}\,.$$

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Now, clearly, for every family of graphs $\{G^n\}_{n\geq 1}$ with each G^n of order n and with at least one edge, we have

$$\liminf_{n\to\infty} \frac{\mathrm{ex}(G^n,H)}{e(G^n)} \geq 1 - \frac{1}{\chi(H)-1} \,. \tag{1}$$

However, if $e(G^n) \to \infty$ as $n \to \infty$ and, for all large n, the number of copies of H in G^n is much larger than $e(G^n)$ and these copies are distributed in G^n in a "uniform" way, then, in fact, one may expect that equality holds in (1) with lim inf replaced by lim. We conjecture that this is indeed the case when the G^n are the random graphs $G_{n,p}$ of sufficiently large density to make the expected number of copies of H in $G_{n,p}$ larger than the expected size of $G_{n,p}$ by a factor that tends to ∞ arbitrarily slowly. Strong evidence in support of this conjecture is provided by a recent, beautiful result of Rödl and Ruciński **[16]**, who proved that every r-colouring of the edges of such a dense random graph $G_{n,p}$ leads to a monochromatic copy of H almost surely, *i.e.* with probability tending to 1 as $n \to \infty$ (For earlier related results see **[11, 14, 15]**.)

The case $H = K^3$ of our conjecture is already well understood. One of the results in Frankl and Rödl **[5]** says that $\mathrm{ex}(G_p, K^3) = (1/2 + o(1))e(G_p)$ almost surely if $p = p(n) \geq n^{-1/2+\varepsilon}$ and $\varepsilon > 0$ is any fixed constant. More recently, Babai, Simonovits, and Spencer **[1]** showed that if $p = 1/2$, then $\mathrm{ex}(G_p, K^3)$ is almost surely the maximal size of a bipartite subgraph of G_p. Moreover, it is shown in **[1]** that almost surely all the K^3-free subgraphs of G_p of maximal size are in fact bipartite. In this note we investigate $\mathrm{ex}(G_p, C^{2\ell+1})$ for $\ell \geq 1$, where as usual we write $C^{2\ell+1}$ for a cycle of length $2\ell + 1$. Our main result is as follows.

Theorem 1. *Let an integer $\ell \geq 1$ and a real number $0 < \eta \leq 1/2$ be fixed. Then there is a constant $C = C(\ell, \eta) > 0$ such that, if $p = p(n) \geq Cn^{-1+1/2\ell}$, then almost every $G_p \in \mathcal{G}(n,p)$ satisfies $\mathrm{ex}(G_p, C^{2\ell+1}) \leq (1/2 + \eta)e(G_p)$.*

Clearly, by (1), we always have $\mathrm{ex}(G_p, C^{2\ell+1}) \geq 1/2$. Moreover, the condition on $p = p(n)$ in Theorem 1 cannot be substantially weakened. To see this, just note that, for $p = p(n) = \delta n^{-1+1/2\ell}$, almost every G_p has $O\{(\delta^{2\ell}/2\ell)e(G_p)\}$ copies of $C^{2\ell+1}$. Thus $\mathrm{ex}(G_p, C^{2\ell+1}) \geq \{1 - O(\delta^{2\ell}/\ell)\}e(G_p)$ almost surely for such p, and hence the lower bound for p in Theorem 1 is, as claimed, essentially best possible: given any $0 < \varepsilon \leq 1$, there is a $\delta = \delta(\varepsilon) > 0$ such that for $p = p(n)$ as above we have $\mathrm{ex}(G_p, C^{2\ell+1}) \geq (1-\varepsilon)e(G_p)$ almost surely. We mention that the general approach in the proof of Theorem 1 was inspired by Rödl and Ruciński **[16]** (cf. Section 2.1).

The result of Frankl and Rödl mentioned above has as a corollary a purely deterministic result that in fact settled a problem of Erdős and Nešetřil. For convenience, for any two graphs G and H and any given $0 < \gamma \leq 1$, let us write $G \to_\gamma H$ if $\mathrm{ex}(G, H) < \gamma e(G)$. Then the deterministic result of Frankl and Rödl **[5]** says that, for any $0 < \eta \leq 1/2$, there is a K^4-free graph $G = G_\eta$ such that $G \to_{1/2+\eta} K^3$. Here, as a corollary to Theorem 1, we show that, for any $0 < \eta \leq 1/2$ and $\ell \geq 1$, there are 'very sparse' graphs $G = G_{\eta,\ell}$ satisfying $G \to_{1/2+\eta} C^{2\ell+1}$. For instance, G may be required to have girth $g(G) = 2\ell+1$, and to contain no two $(2\ell+1)$-cycles that share more than two vertices (cf. Corollary 11).

Finally, we mention that our conjecture above for $H = C^4$ follows from Füredi **[6]** in a rather stronger form. The case in which H is a general even cycle $C^{2\ell}$ ($\ell \geq 2$)

will be dealt with in [**8**], where a completely different approach from the one here will be used.

This note is organised as follows. In §1 we give a few preliminary definitions and lemmas. The proof of Theorem 1, which is based on a somewhat surprising deterministic lemma on subgraphs of pseudo-random graphs, Lemma 6, is the subject of §2. The statement of Lemma 6 and the proof of Theorem 1 are given in Section 2.1, and the proof of Lemma 6 is given in Section 2.2. In §3 we give the deterministic consequences of Theorem 1.

§1. Preliminary Lemmas

We start with a variant of the powerful lemma of Szemerédi [**17**] concerning regular partitions of graphs. Let a graph $G = G^n$ of order $|G| = n$ be fixed. For U, $W \subset V = V(G)$ with $U \cap W = \emptyset$, we write $E(U,W) = E_G(U,W)$ for the set of edges of G that have one endvertex in U and the other in W. We set $e(U,W) = e_G(U,W) = |E(U,W)|$. The following notion will be needed in the sequel. Suppose $0 < \eta \leq 1$ and $0 < p \leq 1$. We say that G is *η-uniform* with *density p* if, for all U, $W \subset V$ with $U \cap W = \emptyset$ and $|U|$, $|W| \geq \eta n$, we have

$$\big|e_G(U,W) - p|U||W|\big| \leq \eta p|U||W|.$$

Now let $H \subset G$ be a spanning subgraph of G. For U, $W \subset V$ with $U \cap W = \emptyset$, let

$$d_{H,G}(U,W) = \begin{cases} e_H(U,W)/e_G(U,W) & \text{if } e_G(U,W) > 0 \\ 0 & \text{if } e_G(U,W) = 0. \end{cases}$$

Suppose $\varepsilon > 0$, U, $W \subset V$, and $U \cap W = \emptyset$. We say that the pair (U,W) is *(ε,H,G)-regular* if for all $U' \subset U$, $W' \subset W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$, we have

$$|d_{H,G}(U',W') - d_{H,G}(U,W)| \leq \varepsilon.$$

We say that a partition $P = (V_i)_0^k$ of $V = V(G)$ is *(ε,k)-equitable* if $|V_0| \leq \varepsilon n$, and $|V_1| = \ldots = |V_k|$. Also, we say that V_0 is the *exceptional* class of P. When the value of ε is not relevant, we refer to an (ε,k)-equitable partition as a *k-equitable* partition. Similarly, P is an *equitable* partition of V if it is a k-equitable partition for some k. Finally, we say that an (ε,k)-equitable partition $P = (V_i)_0^k$ of V is *(ε,H,G)-regular* if at most $\varepsilon\binom{k}{2}$ pairs (V_i,V_j) with $1 \leq i < j \leq k$ are not (ε,H,G)-regular. We can now state an extension of Szemerédi's lemma to subgraphs of η-uniform graphs, observed independently by Rödl [**13**] and Kohayakawa [**10**].

Lemma 2. *For given $\varepsilon > 0$ and $k_0 \geq 1$, there are constants $\eta = \eta(\varepsilon,k_0) > 0$ and $K_0 = K_0(\varepsilon,k_0) \geq k_0$ that depend only on ε and k_0 such that, if G is an η-uniform graph and $H \subset G$ is a spanning subgraph of G, then there is an (ε,H,G)-regular (ε,k)-equitable partition of $V = V(G)$ with $k_0 \leq k \leq K_0$.* $\square$

We now state two lemmas concerning random graphs. Here and below, if $n \geq 1$ and $0 \leq p = p(n) \leq 1$, we let $\mathcal{G}(n,p)$ be the probability space of the random graphs $G_p = G_{n,p}$. The following is immediate from standard estimates for tails of the binomial distribution.

Lemma 3. *Let $0 < \eta \leq 1$ be given, and consider the random graph $G_p = G_{n,p} \in \mathcal{G}(n,p)$ with $0 < p = p(n) < 1$. Put $d = d(n) = np(n)$. Then, there is a constant $d_0 = d_0(\eta)$ such that, if $d \geq d_0$, almost every G_p is η-uniform with density p.* $\square$

We now introduce another uniformity condition for graphs. Let $G = G^n$ be a graph of order n, and suppose $A > 0$ and $p > 0$. Let $d = pn$. We say that G is (p, A)-*uniform* if, for all sets U, $W \subset V(G)$ with $U \cap W = \emptyset$ and $1 \leq |U| \leq |W| \leq d|U|$, we have

$$\big|e_G(U, W) - p|U||W|\big| \leq A\{d|U||W|\}^{1/2}.$$

The following lemma is proved in **[7]**.

Lemma 4. *Let $d = d(n) > 0$ be given, and put $p = p(n) = d/n$. Then a.e. $G_p = G_{n,p} \in \mathcal{G}(n,p)$ is $(p, 20)$-uniform.* $\square$

Let us remark that when dealing with (p, A)-uniform graphs of order n, we shall always have $d = pn$. Our last preliminary lemma concerns the size of induced subgraphs of (p, A)-uniform graphs. In this lemma and in the sequel, we write $O_1(x)$ for a term y that satisfies $|y| \leq x$.

Lemma 5. *Suppose given A and $p > 0$, let $G = G^n$ be a (p, A)-uniform graph, and suppose $U \subset V(G)$. Then the size $e(G[U])$ of the graph $G[U]$ induced by U in G satisfies*

$$e(G[U]) = p\binom{|U|}{2} + O_1(Ad^{1/2}|U|).$$

Sketch of proof. It suffices to notice that

$$e(G[U]) = \binom{u}{2}\left\{\left\lfloor\frac{u}{2}\right\rfloor\left\lceil\frac{u}{2}\right\rceil\right\}^{-1} \operatorname*{Ave}_S e_G(S, U \setminus S),$$

where $u = |U|$ and $\mathrm{Ave}_S\, e_G(S, U \setminus S)$ denotes the average of $e_G(S, U \setminus S)$ when S runs over all $S \subset U$ with $|S| = \lfloor u/2 \rfloor$. $\square$

§2. The Main Result

2.1. Proof of the main result. Our aim in this section is to give the proof of Theorem 1. However, to state the key lemma in our argument, Lemma 6, we need to introduce some definitions and notation. Given a graph H and an integer $\ell \geq 1$, let us define the graph $J = J(H) = J_\ell(H)$ on $V = V(H)$ by joining two distinct vertices x, $y \in V$ in J if and only if there is an x–y path of length 2ℓ in H. Now suppose real constants A, $C > 0$ and $0 < \gamma_0 \leq 1$ are fixed. Moreover, suppose $G = G^n$ is a (p, A)-uniform graph, where $p = p(n) \geq Cn^{-1+1/2\ell}$, and let $H \subset G$ be a subgraph of G of size $e(H) \geq \gamma_0 e(G)$. The main lemma in the proof of Theorem 1, Lemma 6, states that $e(J)$ is nearly as large as $(e(H)/e(G))\binom{n}{2}$ provided C and n are sufficiently large with respect to ℓ, A, and γ_0.

Lemma 6. *Let an integer $\ell \geq 1$, and reals $0 < \delta \leq 1$, $0 < \gamma_0 \leq 1$, and $A > 0$ be fixed. Then there is a constant $C_0 = C_0(\ell, \delta, \gamma_0, A) > 0$ for which the following holds. Let $G = G^n$ be a (p, A)-uniform graph with $p = p(n) \geq C_0 n^{-1+1/2\ell}$, and* *let $H \subset G$ be a subgraph of G with $e(H) \geq \gamma_0 e(G)$. Then, provided n is sufficiently large, we have $e(J_\ell(H)) \geq (1-\delta)(e(H)/e(G))\binom{n}{2}$.*

The proof of Lemma 6 is given in Section 2.2. In that section we also observe that this result is in fact best possible, and make a few remarks on the structure of the extremal subgraphs H for this lemma.

Let us now turn to the proof of Theorem 1. However, as this proof is somewhat technical in parts, we first present its main idea informally here to give the reader some feeling about the way we shall follow. Thus let $\ell \geq 1$ and $0 < \eta \leq 1/2$ be given, and let $p = p(n) \geq Cn^{-1+1/2\ell}$ where C is a suitably large constant to be chosen later. Let us add the technical condition that $p = p(n) = o(1)$ as $n \to \infty$. We view the random graph $G_p = G_{n,p}$ as a union of two independent random graphs $G_{p_1}^{(1)}$ and $G_{p_1}^{(2)}$, both with edge-probability p_1. Thus, as we are assuming that $p = o(1)$, we have in particular that $p_1 \sim p/2$. We decompose $G_{p_1}^{(1)}$ even further, writing it as a union of k independent random graphs $G_{p_2}^{(i)}$ $(1 \leq i \leq k)$. Here k is a fixed large constant to be chosen later, and hence, since $p = o(1)$, we have $p_2 \sim p_1/k \sim p/2k$.

Now suppose F is a subset of edges of G_p with cardinality $(1/2+\eta)e(G_p)$ but spanning no $(2\ell+1)$-cycle. Assume that $G_{p_1}^{(1)}$ and $G_{p_1}^{(2)}$ contain $\gamma_1 e(G_{p_1}^{(1)})$ and $\gamma_2 e(G_{p_1}^{(2)})$ edges of F respectively. Note that $\gamma_1 + \gamma_2 \sim 1 + 2\eta$. Moreover, amongst the $G_{p_2}^{(i)}$ $(1 \leq i \leq k)$, assume that $G_{p_2}^{(j)}$ contains the maximum number of edges from F: we assume that it contains $\gamma^* e(G_{p_2}^{(j)})$ of them, where clearly $\gamma^* \gtrsim \gamma_1$. Now, assuming that C is large enough, we use Lemma 6 to deduce that the edges of F contained in $G_{p_2}^{(j)}$ create at least about $\gamma^*\binom{n}{2}$ pairs of vertices of $G_{p_1}^{(2)}$ that may not form edges in F, as $G_p[F]$ contains no $(2\ell+1)$-cycle. Let us call such pairs of vertices *prohibited*. Now, with very large probability, say $\tilde{P}$, not many fewer than $\gamma^* e(G_{p_1}^{(2)})$ edges of $G_{p_1}^{(2)}$ join prohibited pairs, and hence not many more than $(1-\gamma^*)e(G_{p_1}^{(2)})$ edges of $G_{p_1}^{(2)}$ belong to F. However, the number of such edges is

$$\gamma_2 e(G_{p_1}^{(2)}) \sim (1+2\eta-\gamma_1)e(G_{p_1}^{(2)}) \gtrsim (1+2\eta-\gamma^*)e(G_{p_1}^{(2)}).$$

This shows that actually considerably fewer than $\gamma^* e(G_{p_1}^{(2)})$ edges of $G_{p_1}^{(2)}$ may join prohibited pairs of vertices. Thus the probability that there is a set $F \subset E(G_p)$ as above is at most $1-\tilde{P}$. It turns out that in fact $\tilde{P}$ is so large that $1-\tilde{P}$ can kill the number of possibilities for the set $F \cap E(G_{p_2}^{(j)}) \subset E(G_{p_2}^{(j)})$, provided we pick k large enough. Thus we conclude that almost surely such a set F does not exist, and the theorem follows in this case, namely, when $p = p(n) \geq Cn^{-1+1/2\ell}$ for some suitably large constant C and $p = o(1)$.

We still have to deal with the case in which $p = p(n)$ does not tend to 0 as $n \to \infty$. Here we only sketch an argument based on the original regularity lemma of Szemerédi. Recall $\ell \geq 1$ and $0 < \eta \leq 1/2$ are given. Suppose $0 < \rho \leq 1$ and $A > 0$ are fixed constants and let $G = G^n$ be a (ρ, A)-uniform graph of order n. Then a standard application of Szemerédi's lemma gives that $\mathrm{ex}(G, C^{2\ell+1}) \leq (1/2+\eta)e(G)$ as long as $n \geq n_0$ for some constant $n_0 = n_0(\ell, \eta, \rho, A)$. This implies that there is a function $\rho_0 = \rho_0(n)$ tending to 0 as $n \to \infty$ such that the above statement remains true even if ρ is allowed to vary with n: it suffices that $\rho = \rho(n) \geq \rho_0(n)$. Since almost every random graph $G_p = G_{n,p}$ is $(p, 20)$-uniform, Theorem 1 follows for $p = p(n) \geq \rho_0(n)$.

Let us now give the proof with all the details.

Proof of Theorem 1. Let $A=20$, $k=\lceil 54\eta^{-3}\rceil$, $\varepsilon=\eta/12$, $\delta=\eta/4$, $\gamma_0=\eta$, and let $C=C(\ell,\eta)=2kC_0>0$, where $C_0=C_0(\ell,\delta,\gamma_0,A)>0$ is as given in Lemma 6. We shall show that this choice of C will do in Theorem 1. Thus we fix $p=p(n)\geq Cn^{-1+1/2\ell}$ and investigate $C^{2\ell+1}$-free subgraphs of $G_p\in\mathcal{G}(n,p)$ of large size. As always, we may assume throughout that n is as large as we please. Also, owing to the remarks above, it suffices to consider the case in which $p=p(n)=o(1)$.

We shall 'generate' G_p in $k+1$ 'rounds' as follows. Let $0<p_1=p_1(n)<1$, $0<p_2=p_2(n)<1$ be such that $1-p=(1-p_1)^2$, $1-p_1=(1-p_2)^k$, and consider $k+1$ independent random graphs $G_{p_2}^{(j)}\in\mathcal{G}(n,p_2)$ $(1\leq j\leq k)$, $G_{p_1}^{(2)}\in\mathcal{G}(n,p_1)$. It is clear that $\left\{\bigcup_{1\leq j\leq k}G_{p_2}^{(j)}\right\}\cup G_{p_1}^{(2)}$ is a random element of the space $\mathcal{G}(n,p)$. Below we work with $\mathbf{G}=(G_{p_2}^{(1)},\dots,G_{p_2}^{(k)};G_{p_1}^{(2)})\in\Omega=\left\{\prod_{1\leq j\leq k}\mathcal{G}(n,p_2)\right\}\times\mathcal{G}(n,p_1)$, and always let $G_{p_1}^{(1)}=\bigcup_{1\leq j\leq k}G_{p_2}^{(j)}$ and $G_p=G_{p_1}^{(1)}\cup G_{p_1}^{(2)}$. The 'bad' event $\mathcal{B}$ whose probability we need to show approaches 0 as $n\to\infty$ is that there should be a set $F\subset E(G_p)$ with $|F|>(1/2+\eta)e(G_p)$ and $G_p[F]\not\supset C^{2\ell+1}$.

Recall that we write $O_1(x)$ for a term y that satisfies $|y|\leq x$. Let $\Omega'\subset\Omega$ be the set of $\mathbf{G}=(G_{p_2}^{(1)},\dots,G_{p_2}^{(k)};G_{p_1}^{(2)})\in\Omega$ for which the $G_{p_2}^{(j)}$ $(1\leq j\leq k)$ are (p_2,A)-uniform, $e(G_{p_2}^{(j)})=(1+O_1(\varepsilon))(p/2k)\binom{n}{2}$ $(1\leq j\leq k)$, $e(G_{p_1}^{(i)})=(1+O_1(\varepsilon))(p/2)\binom{n}{2}$ $(i\in\{1,2\})$, and finally $e(G_p)=(1+O_1(\varepsilon))p\binom{n}{2}$. Since $\mathbb{P}(\Omega')=1-o(1)$ as $n\to\infty$, below we always assume that our $\mathbf{G}$ are in Ω'. In particular, we concentrate on $\mathcal{B}'=\mathcal{B}\cap\Omega'$.

Now, for each $\mathbf{G}\in\mathcal{B}'$, fix $F=F(\mathbf{G})\subset E(G_p)$ with $|F|>(1/2+\eta)e(G_p)$ and $G_p[F]\not\supset C^{2\ell+1}$. Moreover, let $F_i=F_i(\mathbf{G})=F\cap E(G_{p_1}^{(i)})$, $f_i=f_i(\mathbf{G})=|F_i|$, and $\gamma_i=\gamma_i(\mathbf{G})=f_i/e(G_{p_1}^{(i)})$ for $i\in\{1,2\}$, and similarly $F^{(j)}=F^{(j)}(\mathbf{G})=F\cap E(G_{p_2}^{(j)})$, $f^{(j)}=f^{(j)}(\mathbf{G})=|F^{(j)}|$, and $\gamma^{(j)}=\gamma^{(j)}(\mathbf{G})=f^{(j)}/e(G_{p_2}^{(j)})$ for $1\leq j\leq k$. Let us now fix $\mathbf{G}\in\mathcal{B}'$.

Note that clearly $|F|=|F(\mathbf{G})|\leq f_1+f_2$, and hence $1/2+\eta<|F|/e(G_p)\leq\{e(G_{p_1}^{(1)})/e(G_p)\}\gamma_1+\{e(G_{p_1}^{(2)})/e(G_p)\}\gamma_2\leq(1+3\varepsilon)(\gamma_1+\gamma_2)/2$. Thus $\gamma_1+\gamma_2\geq(1+2\eta)/(1+3\varepsilon)$. Furthermore, we have that $f_1\leq f^{(1)}+\dots+f^{(k)}$, and hence $\gamma_1\leq((1+3\varepsilon)/k)\sum_{1\leq j\leq k}\gamma^{(j)}$. Hence, putting $\gamma^*=\gamma^*(\mathbf{G})=\max_{1\leq j\leq k}\gamma^{(j)}$, we have $(1+3\varepsilon)(\gamma^*+\gamma_2)\geq(1+3\varepsilon)(\gamma_1+\gamma_2)\geq1+2\eta$, and therefore $\gamma^*+\gamma_2\geq(1+2\eta)/(1+3\varepsilon)\geq1+\eta$. Note that in particular $\gamma^*\geq\eta$. Now, for $1\leq j\leq k$, let $\mathcal{B}'_j=\{\mathbf{G}\in\mathcal{B}'\colon\gamma^{(j)}(\mathbf{G})=\gamma^*(\mathbf{G})\}$. Clearly $\mathcal{B}'=\bigcup_{1\leq j\leq k}\mathcal{B}'_j$, and hence it suffices to show that $\mathbb{P}(\mathcal{B}'_j)=o(1)$ as $n\to\infty$ for all fixed $1\leq j\leq k$. Thus let us now fix $1\leq j\leq k$, and consider $\mathbf{G}\in\mathcal{B}'_j$. We have

$$\begin{aligned}\mathbb{P}(\mathcal{B}'_j)&=\sum_{G_0}\mathbb{P}(\mathcal{B}'_j\cap\{\mathbf{G}\colon G_{p_2}^{(j)}=G_0\})\\&=\sum_{G_0}\mathbb{P}(\mathcal{B}'_j\mid G_{p_2}^{(j)}=G_0)\mathbb{P}(G_{p_2}^{(j)}=G_0)\leq\max_{G_0}\mathbb{P}(\mathcal{B}'_j\mid G_{p_2}^{(j)}=G_0),\end{aligned}\tag{2}$$

where G_0 ranges over all (p_2,A)-uniform graphs on $V(G_p)$ with size $e(G_0)=(1+O_1(\varepsilon))(p/2k)\binom{n}{2}$. We now *fix* one such G_0 and argue that $\mathbb{P}(\mathcal{B}'_j\mid G_{p_2}^{(j)}=G_0)$ is small. For $F_0\subset E(G_0)$, let $P(j,G_0,F_0)=\mathbb{P}\{\mathbf{G}\in\mathcal{B}'_j\text{ and }F^{(j)}(\mathbf{G})=F_0\mid G_{p_2}^{(j)}=G_0\}$.

Then

$$\mathbb{P}(\mathcal{B}'_j \mid G^{(j)}_{p_2} = G_0) = \sum_{F_0 \subset E(G_0)} P(j, G_0, F_0) \leq 2^{(1+\varepsilon)(p/2k)\binom{n}{2}} \max_{F_0 \subset E(G_0)} P(j, G_0, F_0). \tag{3}$$

Let us now *fix* $F_0 \subset E(G_0)$ with $P(j, G_0, F_0) > 0$. In particular, $\gamma^* = |F_0|/e(G_0) \geq \eta = \gamma_0$. It now follows from Lemma 6 that $e(J_\ell(G_0[F_0])) \geq (1-\delta)\gamma^* \binom{n}{2}$.

Now, for each $\mathbf{G} \in \Omega$, let us put $F' = F'(\mathbf{G}) = E(G^{(2)}_{p_1}) \cap E(J_\ell(G_0[F_0]))$, and let $f' = f'(\mathbf{G}) = |F'|$. Note that then, clearly, $f' = f'(\mathbf{G})$ $(\mathbf{G} \in \Omega)$ has binomial distribution $\mathrm{Bi}(e(J_\ell(G_0[F_0])), p_1)$ with parameters $e(J_\ell(G_0[F_0]))$ and p_1. Now suppose $\mathbf{G} \in \mathcal{B}'$, $G^{(j)}_{p_2} = G_0$, and $F^{(j)} = F_0$. Then, since $F = F(\mathbf{G})$ does not span a $(2\ell+1)$-cycle, we have $F'(\mathbf{G}) \cap F_2(\mathbf{G}) = \emptyset$. Since $F'(\mathbf{G}) \cup F_2(\mathbf{G}) \subset E(G^{(2)}_{p_1})$, we have $f'(\mathbf{G}) + \gamma_2(\mathbf{G}) e(G^{(2)}_{p_1}) = f'(\mathbf{G}) + f_2(\mathbf{G}) \leq e(G^{(2)}_{p_1})$, and hence $f'(\mathbf{G}) \leq (1-\gamma_2(\mathbf{G}))e(G^{(2)}_{p_1})$. Therefore

$$\begin{aligned} P(j, G_0, F_0) &\leq \mathbb{P}\{\mathbf{G} \in \Omega' \text{ and } f'(\mathbf{G}) \leq (1-\gamma_2(\mathbf{G}))e(G^{(2)}_{p_1})\} \\ &\leq \mathbb{P}\left\{ f'(\mathbf{G}) \leq (1-\gamma_2(\mathbf{G}))(1+\varepsilon)\frac{p}{2}\binom{n}{2} \right\}. \end{aligned} \tag{4}$$

We now bound this last probability using that $f' = f'(\mathbf{G}) \sim \mathrm{Bi}(e(J_\ell(G_0[F_0])), p_1)$. Clearly, $\mathbb{E}(f') \leq p_1\binom{n}{2} \leq p\binom{n}{2}$. Moreover, recalling that $\gamma^* + \gamma_2 \geq 1 + \eta$, we have

$$\begin{aligned} \mathbb{E}(f') - (1-\gamma_2)(1+\varepsilon)\frac{p}{2}\binom{n}{2} &\geq \big\{(1-\delta)\gamma^* - (1+\varepsilon)(1-\gamma_2)\big\}\frac{p}{2}\binom{n}{2} \\ &\geq (\eta - \delta - \varepsilon)\frac{p}{2}\binom{n}{2} \geq \frac{\eta}{3}p\binom{n}{2} \geq \frac{\eta}{3}\mathbb{E}(f'), \end{aligned}$$

and in particular $\mathbb{E}(f') \geq (\eta p/3)\binom{n}{2}$. Hence, by Hoeffding's inequality **[9]** (see also inequality (5.6) in McDiarmid **[12]**), $f' \leq (1-\gamma_2)(1+\varepsilon)\frac{p}{2}\binom{n}{2}$ occurs with probability no greater than

$$\mathbb{P}\left\{ f' \leq \left(1 - \frac{\eta}{3}\right)\mathbb{E}(f') \right\} \leq \exp\left\{ -\frac{\eta^3}{54}p\binom{n}{2} \right\}.$$

Thus, by (3) and (4), we have

$$\mathbb{P}(\mathcal{B}'_j \mid G^{(j)}_{p_2} = G_0) \leq 2^{(1+\varepsilon)(p/2k)\binom{n}{2}} \exp\left\{ -\frac{\eta^3}{54}p\binom{n}{2} \right\} \leq \exp\left\{ -\frac{C}{352}\eta^3 n^{1+1/2\ell} \right\}.$$

Thus, by (2), we have $\mathbb{P}(\mathcal{B}'_j) = o(1)$ as $n \to \infty$, as required. $\square$

Remark. The idea of looking at $\mathcal{G}(n,p)$ as $\Omega = \left\{ \prod_{1 \leq j \leq k} \mathcal{G}(n, p_2) \right\} \times \mathcal{G}(n, p_1)$ in the above proof was inspired by Rödl and Ruciński **[16]**, where a similar, slightly simpler, decomposition is used.

An immediate corollary of Theorem 1 is as follows.

Corollary 7. *Let an integer $\ell \geq 1$ be fixed, and suppose $\omega = \omega(n) \to \infty$ as $n \to \infty$. Then if $0 < p = p(n) = \omega n^{-1+1/2\ell} \leq 1$ we almost surely have $\mathrm{ex}(G_p, C^{2\ell+1}) = (1/2 + o(1))e(G_p)$.* $\square$

Remark. A simple alteration to the above proof of Theorem 1 gives in fact that, for any fixed $\ell \geq 1$ and $0 < \eta \leq 1/2$, there is a constant $C' = C'(\ell, \eta) > 0$ such that, if $p = p(n) \geq C' n^{-1+1/2\ell}$, then almost every $G_p \in \mathcal{G}(n,p)$ is such that any subgraph $H \subset G_p$ with $e(H) \geq (1/2 + \eta)e(G_p)$ contains at least $c\eta e(G_p)$ copies of $C^{2\ell+1}$, where $c > 0$ is an absolute constant. This result is, however, rather weak for large p, as in this case one would expect the number of such copies to be much greater than $e(G_p)$. This problem will be addressed elsewhere, as will the problem of investigating the structure of the extremal subgraphs for Theorem 1.

2.2. Proof of the main lemma. The main ingredients of the proof of Lemma 6 are a simple deterministic extremal result on edge-weighted graphs, Lemma 8, and the variant of Szemerédi's lemma, Lemma 2, given in §1. Let a graph $H = H^k$ of order k be given, and suppose $\bar{\gamma} = (\gamma_e)_{e \in E(H)}$ is a family of *weights* $0 \leq \gamma_e \leq 1$ on the edges $e \in E(H)$ of H. Given vertices x, $y \in H$, not necessarily distinct, let $Z_{x,y} = \Gamma_H(x) \cap \Gamma_H(y)$, and put

$$w(x,y) = w_{H,\bar{\gamma}}(x,y) = \begin{cases} 0 & \text{if } Z_{x,y} = \emptyset \\ \max\{\gamma_{zy} \colon z \in Z_{x,y}\} & \text{if } Z_{x,y} \neq \emptyset. \end{cases}$$

Also, for $x \in H$, let $d^{\bar{\gamma}}(x) = d^{H,\bar{\gamma}}(x) = \sum_{y \in \Gamma_H(x)} \gamma_{xy}$ be the $\bar{\gamma}$-*degree* of x. For simplicity, put also $\bar{\gamma}(H) = \sum_{e \in E(H)} \gamma_e$. Assume now that $\mathbf{x} = (x_1, \ldots, x_k)$ is an ordering of the vertices of H. Then we let $w(H, \bar{\gamma}, \mathbf{x}) = \sum_{1 \leq i \leq j \leq k} w(x_i, x_j)$. We shall in fact be interested in certain special orderings of $V(H)$. Let us say that an ordering $\mathbf{x} = (x_1, \ldots, x_k)$ of the vertices of H *respects* $\bar{\gamma}$ if, for any $1 \leq i \leq k$, the vertex x_i is a vertex of minimal $\bar{\gamma}_i$-degree in $H_i = H[x_i, \ldots, x_k]$, the subgraph of H induced by $\{x_i, \ldots, x_k\}$, where $\bar{\gamma}_i = (\gamma_e)_{e \in E(H_i)}$. We may now give the first ingredient of the proof of Lemma 6.

Lemma 8. *Let $H = H^k$ be a graph of order k, and let $\bar{\gamma} = (\gamma_e)_{e \in E(H)}$ with $0 \leq \gamma_e \leq 1$ $(e \in E(H))$ be given. Let $\mathbf{x} = (x_1, \ldots, x_k)$ be an ordering of the vertices of H that respects $\bar{\gamma}$. Then $w(H, \bar{\gamma}, \mathbf{x}) \geq \bar{\gamma}(H)$.*

Proof. We prove this lemma by induction on k. If $k = 1$, there is nothing to prove. Thus assume $k \geq 2$, and suppose the lemma holds for smaller values of k. Consider $H_2 = H[x_2, \ldots, x_k]$ and $\bar{\gamma}_2 = (\gamma_e)_{e \in E(H_2)}$. Clearly, $\mathbf{x}_2 = (x_2, \ldots, x_k)$ is an ordering of the vertices of H_2 that respects $\bar{\gamma}_2$, and hence by induction we have $w(H_2, \bar{\gamma}_2, \mathbf{x}_2) \geq \bar{\gamma}_2(H_2)$. If $\Gamma_H(x_1) = \emptyset$, we have $w(H, \bar{\gamma}, \mathbf{x}) = w(H_2, \bar{\gamma}_2, \mathbf{x}_2) \geq \bar{\gamma}_2(H_2) = \bar{\gamma}(H)$, as required. Thus assume $\Gamma_H(x_1) \neq \emptyset$, and pick $z \in \Gamma_H(x_1)$. From our assumption on the ordering $\mathbf{x}$, we have $d^{H,\bar{\gamma}}(z) \geq d^{H,\bar{\gamma}}(x_1)$. Therefore

$$\begin{aligned} w(H, \bar{\gamma}, \mathbf{x}) &\geq w(H_2, \bar{\gamma}_2, \mathbf{x}_2) + \sum_{y \in \Gamma_H(z)} w(x_1, y) \geq \bar{\gamma}_2(H_2) + \sum_{y \in \Gamma_H(z)} \gamma_{zy} \\ &= \bar{\gamma}_2(H_2) + d^{H,\bar{\gamma}}(z) \geq \bar{\gamma}_2(H_2) + d^{H,\bar{\gamma}}(x_1) = \bar{\gamma}(H), \end{aligned}$$

as required. $\square$

Our next lemma, Lemma 9, is the key observation that enables us to use Lemma 2 together with Lemma 8 to prove Lemma 6. To describe the context in which Lemma 9 applies, let an integer $\ell \geq 1$, and reals A, C, ε, $\mu > 0$, and $\gamma \geq \rho > 0$ be fixed. Assume that $G = G^n$ is a (p, A)-uniform graph with $p \geq Cn^{-1+1/2\ell}$, and let $H \subset G$ be a spanning subgraph of G. Let three pairwise disjoint sets V_1, V_2, $V_3 \subset V = V(G)$ with $|V_i| = m \geq \mu n$ $(i \in \{1, 2, 3\})$ be given. Furthermore, assume (V_1, V_2) and (V_2, V_3) are (ε, H, G)-regular pairs with $d_{1,2} = d_{H,G}(V_1, V_2) \geq \rho$ and $d_{2,3} = d_{H,G}(V_2, V_3) \geq \gamma$. Roughly speaking, Lemma 9 states that the graph $J = J(H) = J_\ell(H)$ is such that $e_{J_\ell}(V_1, V_3)$ is nearly as large as γm^2, provided only that C and n are sufficiently large and ε is sufficiently small with respect to ℓ, A, μ, γ, and ρ.

In the proof of Lemma 9, we shall need the following simple definitions. Let J be a bipartite graph with a fixed bipartition, say $V(J) = X \cup Y$. Then we shall say that J is a (b, f)-*expander*, and that it is (b, f)-*expanding*, if for all $U \subset V(J)$ with $|U| \leq b$ and moreover $U \subset X$ or $U \subset Y$ we have $|\Gamma_J(U)| \geq f|U|$. Also, if G is a graph and U, $W \subset V(G)$ are two disjoint sets of vertices, we shall write $G[U, W]$ for the bipartite subgraph of G with vertex classes U, W and with edge set $E(U, W) = E_G(U, W)$.

Lemma 9. *Let an integer $\ell \geq 1$, and reals A, $C > 0$, $0 < \delta \leq 1$, $0 < \mu \leq 1$, and $0 < \rho \leq \gamma \leq 1$ be fixed. Let the graphs $H \subset G$, and $J_\ell = J_\ell(H)$ be as above. Then, if $C \geq C_0 = 32(A/\delta\rho\mu)^2$ and $0 < \varepsilon \leq \varepsilon_0 = \delta\rho/24$, we have $e(J_\ell[V_1, V_3]) \geq (1 - \delta)\gamma m^2$ provided n is sufficiently large.*

Proof. We assume throughout that n is large, and in particular that G is η-uniform for $\eta = \varepsilon\mu$. We prove Lemma 9 by establishing three claims.

(1) *There are sets $\bar{V}_i \subset V_i$ with $|\bar{V}_i| \geq (1 - 2\varepsilon)m$ $(i \in \{1, 2, 3\})$ such that $H_{1,2} = H[\bar{V}_1, \bar{V}_2]$ and $H_{2,3} = H[\bar{V}_2, \bar{V}_3]$ have minimal degree $\delta(H_{1,2}) \geq (1 - \delta/4)d_{1,2}pm$ and $\delta(H_{2,3}) \geq (1 - \delta/4)d_{2,3}pm$.*

To prove (1), one may use the argument in the proof of Lemma 2 of [**7**]. Here we only sketch an argument. To find the $\bar{V}_i$ $(i \in \{1, 2, 3\})$, we successively delete the vertices $v \in V_1 \cup V_2 \cup V_3$ that violate the condition we seek. Then one may easily show that, owing to the (ε, H, G)-regularity of (V_1, V_2) and (V_2, V_3), this process finishes with the required sets $\bar{V}_i$ $(i \in \{1, 2, 3\})$.

Our next claim follows from (1) above and the (p, A)-uniformity of G.

(2) *For $(i, j) = (1, 2)$ and $(i, j) = (2, 3)$, the bipartite graph $H_{i,j} = H[\bar{V}_i, \bar{V}_j]$ is $((1 - \delta/2)d_{i,j}m/f, f)$-expanding for any $0 < f \leq (\delta\rho\mu/4A)^2 d$.*

To prove (2), fix $(i, j) \in \{(1, 2), (2, 3)\}$, let $\sigma \in \{i, j\}$, and suppose $0 < f \leq (\delta\rho\mu/4A)^2 d$. Pick $U \subset \bar{V}_\sigma$ with $u = |U| \leq (1 - \delta/2)d_{i,j}m/f$, and put $W = \Gamma_{H_{i,j}}(U)$. Assume for a contradiction that $w = |W| < fu$. Then, by (1) and the (p, A)-uniformity of G, we have

$$\begin{aligned}\left(1 - \frac{\delta}{4}\right) d_{i,j}pmu &\leq e_{H_{i,j}}(U, W) \leq e_G(U, W) \\ &\leq puw + A(duw)^{1/2} \leq \left(1 - \frac{\delta}{2}\right) d_{i,j}pmu + A(duw)^{1/2}.\end{aligned}$$

Thus $(\delta/4)d_{i,j}pmu \leq A(duw)^{1/2}$, and hence $w \geq (\delta\rho\mu/4A)^2 du \geq fu$, which is a contradiction. Thus $|\Gamma_{H_{i,j}}(U)| \geq f|U|$, as required.

Our third and last claim is as follows.

(3) *For all $x \in \bar{V}_1$, we have $|\Gamma_{J_\ell}(x) \cap \bar{V}_3| \geq (1 - 3\delta/4)\gamma m$.*

To check (3), note that $d = pn = Cn^{1/2\ell}$, and hence that $H_{1,2}$ and $H_{2,3}$ are $((1 - \delta/2)d_{i,j}m/f, f)$-expanding for $0 < f \leq 2n^{1/2\ell}$. Now fix $x \in \bar{V}_1$, and let $X_0 = \{x\}$. Let $X_1, X_2, \dots, X_{2\ell-1}$ be such that $n^{i/2\ell} \leq |X_i| \leq 2n^{i/2\ell}$ for $1 \leq i \leq 2\ell - 1$, and $X_1 \subset \Gamma_{H_{1,2}}(X_0)$, $X_2 \subset \Gamma_{H_{1,2}}(X_1) \setminus X_0, \dots,$ $X_{2\ell-1} \subset \Gamma_{H_{1,2}}(X_{2\ell-2}) \setminus (X_1 \cup \dots \cup X_{2\ell-3})$. Finally, let $X_{2\ell} = \Gamma_{H_{2,3}}(X_{2\ell-1})$. From the expansion property (2) of the $H_{i,j}$, the result follows.

Lemma 9 follows from (1) and (3) above, as $e(J_\ell[V_1, V_3]) \geq (1 - 3\delta/4)\gamma m|\bar{V}_1| \geq (1 - \delta)\gamma m^2$. $\Box$

We are now ready to prove Lemma 6, the main lemma in the proof of Theorem 1.

Proof of Lemma 6. Let an integer $\ell \geq 1$, and reals $0 < \delta \leq 1$, $0 < \gamma_0 \leq 1$, and $A > 0$ be fixed. We may clearly assume that $\delta \leq 15\gamma_0$. Let $\varepsilon = \gamma_0\delta^2/2200$ and $k_0 = \lceil 1/\varepsilon \rceil$, and let $0 < \eta = \eta(\varepsilon, k_0) \leq 1$ and $K_0 = K_0(\varepsilon, k_0) \geq k_0$ be as given in Lemma 2. We may and shall assume that $\eta \leq \varepsilon/2K_0$. Let $\rho = \gamma_0\delta/15 \leq 1$ and $C_0 = C_0(\ell, \delta, \gamma_0, A) = 10^7(AK_0/\delta^2\gamma_0)^2$. We shall show that this C_0 will do in our lemma. Thus, let $H \subset G$ be a subgraph of a (p, A)-uniform graph $G = G^n$ with $e(H) \geq \gamma_0 e(G)$, where $p = p(n) \geq C_0 n^{-1+1/2\ell}$. Let $\gamma = e(H)/e(G)$. We now investigate $J_\ell = J_\ell(H)$. We may clearly assume that H is a spanning subgraph of G. Furthermore, we may assume throughout that n is larger than some suitably chosen large constant, and in particular that $G = G^n$ is η-uniform with density p (cf. Lemma 5). We now apply Lemma 2 to $H \subset G$ to obtain an ε-equitable (ε, H, G)-regular partition $P = (V_i)_0^k$ of $V = V(G)$ with $k_0 \leq k \leq K_0$. For convenience, let $m = |V_i|$ $(1 \leq i \leq k)$.

We now define a graph H^* on $[k]$ by letting $ij \in E(H^*)$ $(1 \leq i < j \leq k)$ if and only if (V_i, V_j) is an (ε, H, G)-regular pair with $d_{H,G}(V_i, V_j) \geq \rho$. Moreover, for $e = ij \in E(H^*)$, let $\gamma_e = d_{H,G}(V_i, V_j)$, and let $\bar{\gamma} = (\gamma_e)_{e \in E(H^*)}$. We may and shall assume that the natural ordering $\mathbf{x} = (1, \dots, k)$ of the vertices of H^* respects $\bar{\gamma}$. Finally, let us define a spanning subgraph $H' \subset H$ of H by letting $xy \in E(H)$ be an edge of H' if and only if $x \in V_i$ and $y \in V_j$ for some $1 \leq i < j \leq k$ with $ij \in E(H^*)$.

(1) *We have $e(H') \geq (1 - \delta/6)\gamma p\binom{n}{2}$.*

We first estimate $|E(H) \setminus E(H')|$. From the (p, A)-uniformity of G it follows that, for any $W \subset V(G)$ with $|W| \geq \eta n$, we have $e(G[W]) \leq (1 + \eta)p\binom{|W|}{2}$ if n is large enough. Hence $e(H[V_0]) \leq \varepsilon^2 pn^2$. Moreover, writing $d = pn$ as usual, we have that $e_G(V_0, V \setminus V_0)$ is at most

$$p|V_0||V \setminus V_0| + A\{d|V_0||V \setminus V_0|\}^{1/2} \leq \varepsilon pn^2 + And^{1/2} \leq 2\varepsilon pn^2$$

for large enough n. Now note that $\sum e_G(V_i, V_j)$ with the sum over all $1 \leq i < j \leq k$ such that (V_i, V_j) is *not* (ε, H, G)-regular is at most $\varepsilon\binom{k}{2}(1 + \eta)pm^2 \leq \varepsilon pn^2$. Also, $\sum e_H(V_i, V_j)$ with the sum extended over all $1 \leq i < j \leq k$ such that $d_{H,G}(V_i, V_j) \leq \rho$ is at most $\rho\binom{k}{2}(1+\eta)pm^2 \leq \rho pn^2$. Finally, for large enough n, we have that $\sum_{1 \leq i \leq k} e(G[V_i]) \leq k(1+\eta)p\binom{m}{2} \leq pn^2/k$. Therefore $|E(H) \setminus E(H')| \leq (4\varepsilon + \rho + 1/k)pn^2 \leq (5\varepsilon + \rho)pn^2$ if n is sufficiently large. On the other hand, we have $e(H) = \gamma e(G) \geq \gamma(1-\eta)p\binom{n}{2}$. Thus

$$e(H') \geq \gamma(1-\eta)p\binom{n}{2} - 2(6\varepsilon+\rho)p\binom{n}{2}$$
$$= \left(1-\eta-12\frac{\varepsilon}{\gamma}-2\frac{\rho}{\gamma}\right)\gamma p\binom{n}{2} \geq \left(1-\frac{\delta}{6}\right)\gamma p\binom{n}{2}.$$

We now look at $(H^*, \bar{\gamma})$. As in Lemma 8, let $\bar{\gamma}(H^*) = \sum_{e\in E(H^*)} \gamma_e$.

(2) *We have* $\bar{\gamma}(H^*) \geq (1-\delta/2)\gamma k^2/2$.

To prove (2), note that

$$\left(1-\frac{\delta}{6}\right)\gamma p\binom{n}{2} \leq e(H') = \sum_{e=ij\in E(H^*)} \gamma_e e_G(V_i, V_j)$$
$$\leq (1+\eta)pm^2\bar{\gamma}(H^*) \leq (1+\eta)p\frac{n^2}{k^2}\bar{\gamma}(H^*).$$

Therefore $\bar{\gamma}(H^*) \geq \{(1-\delta/6)/(1+\eta)\}(1-1/n)\gamma k^2/2 \geq (1-\delta/2)\gamma k^2/2$.

We may now complete the proof of our lemma.

(3) *We have* $e(J_\ell(H)) \geq (1-\delta)\gamma\binom{n}{2}$.

Let $1 \leq i < j \leq k$. By Lemma 9, applied with $\delta/6$, we have that $e(J_\ell[V_i, V_j]) \geq (1-\delta/6)w(i,j)m^2$, where $w(i,j) = w_{H^*,\bar{\gamma}}(i,j)$ is as defined before Lemma 8. Thus, by (2) and Lemma 8, we have

$$e(J_\ell(H)) \geq \sum_{1\leq i<j\leq k} e(J_\ell[V_i,V_j]) \geq \left(1-\frac{\delta}{6}\right)m^2\Big\{w(H^*,\bar{\gamma},\mathbf{x}) - \sum_{1\leq i\leq k} w(i,i)\Big\}$$
$$\geq \left(1-\frac{\delta}{6}\right)m^2\bar{\gamma}(H^*) - km^2 \geq \frac{1}{2}\left(1-\frac{2\delta}{3}\right)\gamma k^2m^2 - \frac{n^2}{k}$$
$$\geq \frac{1}{2}(1-\varepsilon)^2\left(1-\frac{2\delta}{3}\right)\gamma n^2 - \varepsilon n^2 \geq (1-\delta)\gamma\binom{n}{2},$$

as required. $\square$

Remark. Lemma 6 is essentially best possible in the sense that the lower bound for $e(J_\ell(H))$ cannot be substantially improved. In fact, any graph G on n vertices contains many subgraphs H for which $e(J_\ell(H))$ is not greater than $(e(H)/e(G))\binom{n}{2}$. To mention just two examples of quite different nature, first fix any $1 \leq k \leq n$ and let $H = H^k$ be a k-vertex subgraph of G of maximum size, and note that then $e(H) \geq e(G)\binom{k}{2}\binom{n}{2}^{-1}$ whereas trivially $e(J_\ell(H)) \leq \binom{k}{2}$. Now suppose $1 \leq k \leq n/2$ and let $H = H^{2k}$ be a balanced bipartite subgraph of G of order $2k$ and maximum size. Then $e(H) \geq 2e(G)k^2/n^2$ whilst $e(J_\ell(H)) \leq k^2$. From these two examples, we also see that the 'extremal' or 'near-extremal' subgraphs H for Lemma 6 may have quite a varied structure.

§3. Deterministic Consequences

In this section we give a few results concerning the existence, for any given $0 < \varepsilon \leq 1/2$ and $\ell \geq 1$, of very sparse graphs G that satisfy $G \to_{1/2+\varepsilon} C^{2\ell+1}$.

If H is a graph of order $|H| \geq 3$ and size $e(H) \geq 1$, its *2-density* is $d_2(H) = (e(H)-1)/(|H|-2)$. In particular, the 2-density of the $(2\ell+1)$-cycle $C^{2\ell+1}$ $(\ell \geq 1)$ is $d_2(C^{2\ell+1}) = 2\ell/(2\ell-1)$. For integers $k \geq 3$ and $\ell \geq 1$, let $\mathcal{H}_{k,\ell}$ be the family of all graphs H with $3 \leq |H| \leq k$, $e(H) \geq 1$, and $d_2(H) > d_2(C^{2\ell+1})$. Also, let $\mathrm{Forb}(\mathcal{H}_{k,\ell})$ be the collection of all graphs G that are H-free for all $H \in \mathcal{H}_{k,\ell}$.

Theorem 10. *Let $0 < \varepsilon \leq 1/2$ and integers $k \geq 3$ and $\ell \geq 1$ be fixed. Then there exists a graph $G = G_{\varepsilon,k,\ell} \in \mathrm{Forb}(\mathcal{H}_{k,\ell})$ such that $G \to_{1/2+\varepsilon} C^{2\ell+1}$.*

Proof. Put $\eta = \varepsilon/2$ and let $C = C(\ell, \eta/2)$ be as given in Theorem 1. Let $p = p(n) = Cn^{-1+1/2\ell}$, and consider $G_p \in \mathcal{G}(n,p)$. Theorem 1 then tells us that $G_p \to_{1/2+\eta} C^{2\ell+1}$ almost surely, and it is a standard matter to check that a.e. G_p contains, very generously, at most $(\varepsilon/2)e(G_p)$ copies of elements of $\mathcal{H}_{k,\ell}$ as subgraphs. Fix a G_p satisfying both conditions above, and let $G \subset G_p$ be a spanning subgraph of G_p obtained from G_p by the removal of at least one edge from each subgraph of G_p that is a copy of an element of $\mathcal{H}_{k,\ell}$, and such that $e(G) \geq (1-\varepsilon/2)e(G_p)$. Then clearly $G \in \mathrm{Forb}(\mathcal{H}_{k,\ell})$. Moreover, we have that $G \to_{1/2+\varepsilon} C^{2\ell+1}$. To see this, suppose $J \subset G$ is a subgraph of G with $e(J) \geq (1/2+\varepsilon)e(G)$. Then $e(J) \geq (1/2)(1+2\varepsilon)(1-\varepsilon/2)e(G_p) \geq (1/2)(1+\varepsilon)e(G_p)$. As $G_p \to_{(1+\varepsilon)/2} C^{2\ell+1}$, the graph J must contain a $(2\ell+1)$-cycle and we are done. □

We now single out a corollary to Theorem 10. In Corollary 11 below, the property that G belongs to $\mathrm{Forb}(\mathcal{H}_{k,\ell})$ in Theorem 10 is replaced by a collection of simpler and more concrete conditions. For instance, one of these conditions is that the girth $g(G)$ of G should be at least $2\ell+1$. To state another condition that appears in Corollary 11, we need to introduce a definition.

Let G be a graph and $\ell \geq 1$ a positive integer. Suppose $C_1, \dots, C_h$ $(h \geq 2)$ are distinct $(2\ell+1)$-cycles of G, and $e_1 \in E(C_1), \dots, e_{h-1} \in E(C_{h-1})$ are $h-1$ edges of G such that $E(C_i) \cap \bigcup_{1 \leq j < i} E(C_j) = \{e_{i-1}\}$ $(2 \leq i \leq h)$. Then we say that $(C_1, \dots, C_h)$ is a *$(h, C^{2\ell+1})$-path* in G. Now assume that $(C_1, \dots, C_h)$ is a $(h, C^{2\ell+1})$-path in G and that the edge $e \in E(G)$ joins a vertex in $V(C_1) \setminus \bigcup_{1<j\leq h} V(C_j)$ to a vertex in $V(C_h) \setminus \bigcup_{1 \leq i < h} V(C_i)$. We then say that $(C_1, \dots, C_h; e)$ is a *$(h, C^{2\ell+1})$-cycle* in G.

It is immediate to check that if $(C_1, \dots, C_h)$ is a $(h, C^{2\ell+1})$-path, then $H = \bigcup_{1 \leq j \leq h} C_j$ has 2-density $d_2(H) \geq d_2(C^{2\ell+1})$. Also, if $(C_1, \dots, C_h; e)$ is a $(h, C^{2\ell+1})$-cycle, then $H' = H + e$ has 2-density $d_2(H') > d_2(C^{2\ell+1})$.

Corollary 11. *For any $0 < \varepsilon \leq 1/2$ and integers $k \geq 2$ and $\ell \geq 1$, there is a graph $G = G_{\varepsilon,k,\ell}$ such that (i) $g(G) = 2\ell+1$, (ii) no two $(2\ell+1)$-cycles in G share more than two vertices, and if two such cycles do share two vertices then they also share an edge, (iii) G contains no $(h, C^{2\ell+1})$-cycles for any $2 \leq h \leq k$, and (iv) $G \to_{1/2+\varepsilon} C^{2\ell+1}$.* □

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