PARTITIONING BY MONOCHROMATIC TREES

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Abstract. Any r-edge-coloured n-vertex complete graph K^n contains at most r monochromatic trees, all of different colours, whose vertex sets partition the vertex set of K^n, provided $n \geq 3r^4 r!(1-1/r)^{3(1-r)} \log r$. This comes close to proving, for large n, a conjecture of Erdős, Gyárfás, and Pyber, which states that $r-1$ trees suffice for all n.

§1. Introduction

The *tree partition number* of r-edge-coloured complete graphs is defined to be the minimum k such that whenever the edges of a complete graph K^n are coloured with r colours, the vertices of K^n can be covered by at most k vertex-disjoint monochromatic trees. The *cycle partition number* is defined similarly. Erdős, Gyárfás, and Pyber **[1]** proved that the cycle partition number (and hence the tree partition number) is at most $cr^2 \log r$ for some constant c. They conjectured in **[1]** that the cycle partition number is r, and that the tree partition number is $r-1$. Here we prove that the tree partition number is at most r provided n is sufficiently large with respect to r. Our main theorem, the proof of which is postponed until the next section, is as follows.

Theorem 1. *Let $r \geq 1$ and $n \geq 3r^4 r!(1-1/r)^{3(1-r)} \log r$ be integers, and suppose the edges of K^n are coloured with r colours. Then K^n contains $t \leq r$ monochromatic trees $T_1, \dots, T_t$ of radius at most 2, each of a different colour, such that their vertex sets $V(T_i)$ $(1 \leq i \leq t)$ partition the vertex set of K^n.*

Note that the lower bound for n above is, for large r, about $3\mathrm{e}^3 r^4 r! \log r$. As shown in **[1]**, the conjecture for tree partitions is best possible, if true, when $r-1$ is a prime power. To see this, let $n = (r-1)^2$, and let K^n be a complete graph whose vertices are labelled with the points of an affine plane of order $r-1$. Let a colouring of the edges of K^n be given by assigning colour i to the edge xy if the line through x and y is in the ith parallel class. Then each monochromatic component of this colouring is a complete subgraph of order $r-1$, and so at least $r-1$ monochromatic trees would be required to partition the set of vertices. This example can be extended to an example for any $n \geq (r-1)^2(r-2)$, by replacing each vertex x of $K^{(r-1)^2}$ with a set of vertices X of size $\lfloor n/(r-1)^2 \rfloor$ or $\lceil n/(r-1)^2 \rceil$.

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Then edge zw for $z \in Z$, $w \in W$ is given colour i if $Z \neq W$ and the line through Z and W is in the ith parallel class, and colour 1 if $Z = W$. Then the monochromatic components of this colouring each have at most $(r-1)\lceil n/(r-1)^2 \rceil < n/(r-2)$ vertices, and hence any tree partition would require at least $r-1$ trees.

The tree partition conjecture for $r = 2$ is equivalent to the fact that either a graph or its complement is connected, an old remark of Erdős and Rado. For $r = 3$ the conjecture was proved by Erdős, Gyárfás, and Pyber [**1**]. It was proved by Hajnal (see [**4**]) that the tree partition number for r-edge-coloured infinite complete graphs is at most r. Rado [**5**] proved that an r-edge-coloured countably infinite complete graph may be partitioned into r monochromatic, possibly one-way infinite, paths, generalising a result of Erdős, who had proved this statement for $r = 2$.

A problem that is related to tree partitioning is finding the *tree cover number* for r-edge-coloured complete graphs, the minimum number k such that the vertices of an r-edge-coloured complete graph can be covered by k (not necessarily disjoint) monochromatic trees. It is clear that this number is at most r, as the set of monochromatic stars centred at any vertex give a cover, and in the example described previously it is at least $r-1$. As shown by Gyárfás [**3**], proving that the tree cover number is $r-1$ is equivalent to proving a conjecture of Lovász and Ryser (see, *e.g.* Füredi [**2**]), which states that an r-partite intersecting hypergraph has a transversal of size at most $r-1$. This was proved by Tuza for $r \leq 5$ in [**6**].

§2. Proof of Theorem 1

Let integers r and n as in the statement of the theorem be given. We may assume that $r \geq 4$. Suppose the edges of K^n have been coloured with colours from $[r] = \{1, \ldots, r\}$.

We start with some definitions. Let $s_1 = \lceil (n-1)/r \rceil$, and define the integers s_i $(2 \leq i \leq r)$ by letting

$$s_i = \left\lfloor \frac{1}{r-i+1} \left(1 - \frac{1}{r}\right)^2 s_{i-1} \right\rfloor. \tag{1}$$

It is clear that the s_i form a non-increasing sequence. A straightforward but somewhat tedious calculation shows that $s_r \geq r^3 \log n$, since n is assumed to be sufficiently large with respect to r. (Throughout this note, log will denote the natural logarithm.)

Let B be a complete bipartite subgraph of our K^n with a specified bipartition $V(B) = X \cup Y$. Suppose that $X = \{x_1, \ldots, x_t\}$, where $1 \leq t \leq r$, and $|Y| \geq s_t$. We say that B is an *anchor* graph if the colouring of K^n is such that the colour of the edge $x_i y$ $(y \in Y)$ depends only on x_i, and this colour is distinct for each x_i $(1 \leq i \leq t)$. We refer to t as the *index* of B. Thus, an anchor graph of index t is formed by t monochromatic stars in K^n, all of distinct colours, and with the same set of leaves.

Let $1 \leq t \leq r$ be the largest integer for which K^n contains an anchor graph of index t. Fix an anchor graph $B \subset K^n$ of index t, and suppose without loss of generality that the edges $x_i y$ $(y \in Y)$ have colour i for all $1 \leq i \leq t$.

For $v \in K^n$, $U \subset V(K^n) \setminus \{v\}$, and $1 \leq i \leq r$, we let $\Gamma_i(v, U)$ be the set of vertices from U that are joined to v by edges of colour i. Put $d_i(v, U) = |\Gamma_i(v, U)|$. Observe that, by the maximality of t, we have $d_i(z, Y) < s_{t+1}$ for every $z \in Z = V(K^n) \setminus V(B)$ and every $t < i \leq r$.

We now use a simple greedy procedure to find a covering of Z by monochromatic stars with centres in Y. To be precise, suppose that, for some $q \geq 0$, vertices $y_1, \ldots, y_q \in Y$, colours $i_1, \ldots, i_q \in [t]$, and pairwise disjoint sets $Z_1, \ldots, Z_q \subset Z$ have been defined such that, for all $1 \leq k \leq q$, we have

(i) $Z_k \subset \Gamma_{i_k}(y_k, Z)$,

(ii) putting $Z'_0 = Z$ and $Z'_k = Z \setminus \bigcup_{j=1}^k Z_j$, for $1 \leq k \leq q$ we have

$$|Z'_k| \leq |Z'_{k-1}| \exp\left\{-\frac{1}{tr}\right\}. \tag{2}$$

If $Z = \bigcup_{k=1}^q Z_k$, we have found the desired covering. Thus suppose that $Z'_q = Z \setminus \bigcup_{k=1}^q Z_k \neq \emptyset$. Condition ($ii$) applied recursively gives that

$$|Z'_q| \leq |Z| \exp\{-q/tr\} < n\mathrm{e}^{-q/tr},$$

and hence, since $Z'_q \neq \emptyset$, we deduce that $q < tr \log n$. As observed above, $s_r \geq r^3 \log n$, whence

$$s_r > rq. \tag{3}$$

Therefore $Y'_q = Y \setminus \{y_1, \ldots, y_q\} \neq \emptyset$ since $|Y| \geq s_t \geq s_r > rq \geq q$. Now to continue the procedure, we let y_{q+1} and i_{q+1} be such that $d_{i_{q+1}}(y_{q+1}, Z'_q)$ is maximal over all choices of $y_{q+1} \in Y'_q \neq \emptyset$ and $i_{q+1} \in [t]$. Put $Z_{q+1} = \Gamma_{i_{q+1}}(y_{q+1}, Z'_q)$. We now check that ($i$) and ($ii$) hold for $k = q+1$. Clearly, the only work lies in checking (2). Let us first observe that the number of edges of colours $1, \ldots, t$ between Y'_q and Z'_q is at least

$$\sum_{z \in Z'_q} \sum_{1 \leq i \leq t} d_i(z, Y'_q) \geq |Z'_q| \big\{|Y| - q - (r-t)s_{t+1}\big\},$$

where s_{r+1} is taken to be 0. Thus

$$d_{i_{q+1}}(y_{q+1}, Z'_q) = \max\{d_i(y, Z'_q) \colon 1 \leq i \leq t,\ y \in Y'_q\} \geq \frac{1}{t}\left(1 - \frac{(r-t)s_{t+1}}{|Y| - q}\right)|Z'_q|.$$

Hence, putting $Z'_{q+1} = Z \setminus \bigcup_{k=1}^{q+1} Z_k$, we have

$$|Z'_{q+1}| \leq \left(1 - \frac{1}{t}\left(1 - \frac{(r-t)s_{t+1}}{|Y| - q}\right)\right)|Z'_q| \leq |Z'_q| \exp\left\{-\frac{1}{t}\left(1 - \frac{(r-t)s_{t+1}}{|Y| - q}\right)\right\}.$$

But $1 - (r-t)s_{t+1}/(|Y| - q) \geq 1 - (r-t)s_{t+1}/(1 - 1/r)s_t \geq 1/r$, where the first inequality follows from (3) and the second follows from (1). Therefore $|Z'_{q+1}| \leq |Z'_q| \exp\{-1/tr\}$, as required.

The considerations above show that our procedure terminates with a covering of Z by monochromatic stars with centres $y_1, \ldots, y_{q_0}$, colours $i_1, \ldots, i_{q_0}$, and sets of leaves $Z_1, \ldots, Z_{q_0}$. To complete our tree partition, we define trees $T_1, \ldots, T_t$ of colours $1, \ldots, t$ as follows. The tree T_i ($1 \leq i \leq t$) has vertex set $V(T_i) = \{x_i\} \cup \bigcup(\{y_k\} \cup Z_k)$ and edge set $E(T_i) = \bigcup(\{x_i y_k\} \cup \{y_k z \colon z \in Z_k\})$, where the unions range over all $k \in [q]$ with $i_k = i$. Clearly $\bigcup_{i=1}^t V(T_i) = X \cup Y_0 \cup Z$, where $Y_0 = \{y_1, \ldots, y_{q_0}\}$. It can be easily seen that the vertices in $Y \setminus Y_0$ may be added to T_1 as leaves to give a tree partition of K^n as required. This completes the proof of the theorem.

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