On Richardson's Model on the Hypercube

B. Bollobás[1, 2*] and Y. Kohayakawa[3†]

[1]Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

[2]Department of Mathematics,
Louisiana State University, Baton Rouge, LA 70803, USA

[3]Instituto de Matemática e Estatística, Universidade de São Paulo,
Caixa Postal 20570, 01452–990 São Paulo, SP, Brazil

Abstract. The n-cube Q^n is the graph on the subsets of $\{1,\ldots,n\}$ where two such vertices are adjacent if and only if their symmetric difference is a singleton. Fill and Pemantle [5] started the study of several percolation processes on Q^n and obtained many asymptotic results for $n\to\infty$. As an application of these results, they investigated the contact process with no recoveries in Q^n, also known as Richardson's model for the spread of a disease. They obtained that, in this model, the cover time T_n of Q^n starting from a single infected vertex is bounded in probability: they proved that, with probability tending to 1 as $n\to\infty$, one has $(1/2)\log(2+\sqrt{5})+\log 2+o(1)=1.414\ldots+o(1)\le T_n\le 4\log(4+2\sqrt{3})+6+o(1)=14.040\ldots+o(1)$. In this note we substantially improve this upper bound by showing that one in fact has $T_n\le 1+\log 2+o(1)=1.693\ldots+o(1)$ in probability.

* Part of this work was done while this author was visiting the University of São Paulo, supported by FAPESP under grant 92/3169–8.

† Research partially supported by FAPESP under grant 93/0603–1 and by CNPq under grant 300334/93–1.

§0. Introduction

The n-dimensional cube Q^n is the graph whose vertices are the subsets of $[n] = \{1, \ldots, n\}$ where two such vertices are adjacent if and only if their symmetric difference is a singleton. Here we are interested in a stochastic process on Q^n, known as Richardson's model for the spread of a disease. In this model, one vertex of Q^n is at first *infected* and the disease evolves by transmission of the infection according to i.i.d. Poisson processes associated to the edges of Q^n. Every time our Poisson clock associated to an edge xy goes off, if one of x or y is infected at that time the other vertex becomes infected. The natural questions concerning this model regard the manner in which the disease spreads, and in particular the time it takes for all the vertices of Q^n to become infected. The latter question has been recently studied by Fill and Pemantle [5], who proved that this *covering time* is bounded in probability. Our aim in this note is to give an improved upper bound for this covering time.

One may study Richardson's model on Q^n by studying first-passage percolation on this graph where the passage times on the edges are i.i.d. exponential random variables. In fact, we shall study Richardson's model only looking at first-passage percolation. Let $W = (W_e)_{e \in E(Q^n)}$ be a family of independent exponential random variables each with mean 1. If x and y are two vertices of Q^n, we write $d(x, y) = d_{Q^n}(x, y)$ for the distance between x and y in Q^n, and we set

$$d_W(x, y) = \min_P \{\ell(P, W) \colon P \text{ an } x\text{—}y \text{ path in } Q^n\},$$

where $\ell(P, W) = \sum_{e \in E(P)} W_e$ is the *W-length* of the path P. The result in [5] concerning Richardson's model on Q^n is that, for a fixed $x_0 \in Q^n$, with probability $1 - o(1)$ as $n \to \infty$, we have $(1/2)\log(2 + \sqrt{5}) + \log 2 + o(1) = 1.414\ldots + o(1) \leq \max_{y \in Q^n} d_W(x_0, y) \leq 4\log(4 + 2\sqrt{3}) + 6 + o(1) = 14.040\ldots + o(1)$. A corollary of our main result, Theorem 5, is that this upper bound can be improved to $1 + \log 2 + o(1) = 1.693\ldots + o(1)$.

Indeed, we shall show that with probability $1 - o(1)$ as $n \to \infty$, (i) for all fixed $x_0 \in Q^n$ we have that $\max_{y \in Q^n} d_W(x_0, y) \leq 1 + \log 2 + o(1)$, and (ii) $\operatorname{diam}(Q^n, W) = \max\{d_W(x, y) \colon x,\, y \in Q^n\} \leq 1 + 2\log 2 + o(1)$. The methods used in the proof Theorem 5 are reminiscent of certain techniques first used in [3, 4].

The organisation of this note is as follows. In §1 we present some definitions and preliminary results, and in §2 we prove the main technical lemma

(Lemma 4) that is needed in the proof of Theorem 5. This theorem is proved in §3. We close this note with some remarks and further problems.

§1. Preliminaries

Throughout this note, we let $W = (W_e)_{e \in E(Q^n)}$ be a family of independent exponential random variables each of mean 1. For this section and the next, we assume that a constant $0 < \varepsilon < 1/6$ has been fixed.

One extremely simple but important idea in our analysis of (Q^n, W) is that we may easily decompose W_e $(e \in E(Q^n))$ into a collection of independent exponential random variables. This decomposition is then used to handle several events independently. Let $X = (X_e)$, $Y = (Y_e)$, and $Z = (Z_e)$ be three families of independent exponential r.vs, where the X_e have mean $(1-2\varepsilon)^{-1}$, and the Y_e and Z_e have mean ε^{-1}. Writing $a \wedge b$ for the minimum of a and b, if $W'_e = X_e \wedge Y_e \wedge Z_e$ $(e \in E(Q^n))$, clearly W'_e is an exponential r.v. with mean 1. Thus we may study (Q^n, W) by analysing (Q^n, X), (Q^n, Y), and (Q^n, Z).

All the asymptotics below refer to $n \to \infty$. Moreover, we very often tacitly assume that n is large enough in the estimates that follow. As is usual in the theory os random graphs, we use the terms 'almost surely' and 'almost every' to mean 'with probability tending to 1 as $n \to \infty$'. For clarity, we shall denote $\emptyset \in Q^n$ by x_0. The neighbourhood of $x \in Q^n$ is denoted by $\Gamma(x) = \Gamma_{Q^n}(x)$. For graph-theoretical terminology not defined here, we refer the reader to [1].

Our first two lemmas concern (Q^n, X).

Lemma 1. *For all $x \in Q^n$, let $N_x = N_x(\varepsilon, X) \subset \Gamma_{Q^n}(x)$ be defined by $N_x = \{y \in \Gamma_{Q^n}(x) \colon X_{xy} \le \log 2 + 3\varepsilon\}$ if $x \ne x_0$, and $N_{x_0} = \{y \in \Gamma_{Q^n}(x_0) \colon X_{x_0 y} \le \varepsilon\}$. Then there is a constant $c_1 = c_1(\varepsilon) > 0$ such that with probability $1 - o(1)$ we have $|N_x| \ge c_1 n$ for all $x \in Q^n$.*

Proof. Fix an arbitrary vertex $x \ne x_0$ in Q^n. Then, for a fixed $y \in \Gamma_{Q^n}(x)$,

$$\mathrm{P}(X_{xy} \ge \log 2 + 3\varepsilon) = \exp\{-(\log 2 + 3\varepsilon)(1-2\varepsilon)\} \le (1-\alpha)/2,$$

for some $\alpha = \alpha(\varepsilon) > 0$. From standard bounds for the tail of the binomial distribution, we see that for a suitable constant $c'_1 > 0$ we almost surely have $|N_x| \ge c'_1 n$ for all $x \ne x_0$. Similarly, for a suitable constant $c''_1 > 0$, we have that $|N_{x_0}| \ge c''_1 n$ almost surely. Thus we may take $c_1 = c'_1 \wedge c''_1$. □

We now describe the results from [5] concerning first-passage percolation on Q^n. For all $n \ge 1$, let $x_0 = x_0^{(n)} = \emptyset \in Q^n$ and $y_0 = y_0^{(n)} = [n] \in Q^n$.

Consider (Q^n, W), and let

$$T_0 = \inf\{T \in \mathrm{R}: d_W(x_0, y_0) \leq T \text{ with probability } 1 - o(1)\}. \qquad (1)$$

Equivalently, T_0 is the smallest T $(0 \leq T \leq \infty)$ such that, for any given $\delta > 0$, we have $\limsup_n \mathrm{P}(d_W(x_0, y_0) \geq T + \delta) = 0$ Thus T_0 is the first-passage percolation time between two opposite vertices in Q^n. Fill and Pemantle [5] have shown that $T_0 \leq 1$, and an argument due to Durrett given in [5] gives that $T_0 \geq \log(1+\sqrt{2}) = .881\cdots$. Our next lemma brings in the parameter T_0 into our problem. Before we can state this lemma we need some definitions. A *subcube* of Q^n is a subgraph induced in Q^n by a set of the form

$$Q_{S,A} = Q(S, A) = \{x \in Q^n : x \cap S = A\},$$

where $A \subset S \subset [n]$. In the sequel we shall denote the subcube induced by $Q_{S,A}$ by $Q_{S,A} = Q(S, A)$ as well, since this will not cause any confusion. The *dimension* $\dim(Q_{S,A})$ of $Q_{S,A}$ is $n - |S|$. For an integer $k \geq 0$ and a set X, we write $X^{(k)}$ for the set of all k-element subsets of X.

Given x and $y \in Q^n$ we define the subcube $\langle x, y\rangle$ *spanned* by x and y by

$$\langle x, y\rangle = Q\big((x \bigtriangleup y)^{\mathrm{c}}, x \setminus (x \bigtriangleup y)\big) = Q\big((x \bigtriangleup y)^{\mathrm{c}}, y \setminus (x \bigtriangleup y)\big),$$

where $u^{\mathrm{c}} = [n] \setminus u$ and as usual $\bigtriangleup$ denotes symmetric difference. If $x \subset y$ then $\langle x, y\rangle$ equals the 'interval' $[x, y]$ in $Q^n = \mathcal{P}([n])$ regarded as a partial order under inclusion, that is $\langle x, y\rangle = [x, y] = \{z \in Q^n : x \subset z \subset y\}$. If $v \in \langle x, y\rangle$, let us call the pair $(\langle x, y\rangle\,; v)$ a *rooted cube* with *root* v. For the rest of this note, we let $k_0 = \lceil \log n \rceil$, and set

$$\mathcal{T} = \{(x, y, z) \in Q^n \times Q^n \times Q^n : d(x, y) \geq 2k_0 + 1,\ d(x, z) = 2k_0,\ z \in \langle x, y\rangle\}.$$

For a triple $(x, y, z) \in \mathcal{T}$ let $\mathcal{F}_{x,y,z}$ be the family

$$\big\{(\langle v, w\rangle\,; v) : v \in \langle x, z\rangle\,,\ d(x, v) = k_0,\ w = v \bigtriangleup y \bigtriangleup z\big\}$$

of rooted subcubes of $\langle x, y\rangle$. In visualising $\mathcal{F}_{x,y,z}$, it might be helpful to note that the map $\varphi_{y,z} : u \mapsto u \bigtriangleup (y \bigtriangleup z)$ gives a natural isomorphism between the subcubes $\langle x, z\rangle$ and $\langle \varphi_{y,z}(x), \varphi_{y,z}(z)\rangle = \langle \varphi_{y,z}(x), y\rangle$. Moreover, note that the rooted subcubes $(\langle v, w\rangle\,; v)$ in $\mathcal{F}_{x,y,z}$ are translates of the rooted cube $(\langle x, \varphi_{y,z}(x)\rangle\,; x) = (\langle x, x \bigtriangleup y \bigtriangleup z\rangle\,; x)$. In fact, we may clearly regard Q^n as $\langle x, z\rangle \times \langle x, \varphi_{y,z}(x)\rangle$. Let us say that $(\langle v, w\rangle\,; v) \in \mathcal{F}_{x,y,z}$ is *inexpensive* if $d_X(v, w)$ is at most $(T_0 + \varepsilon)/(1 - 2\varepsilon)$, and *expensive* otherwise. In the sequel we write $m_0 = \binom{2k_0}{k_0}$. Rather crudely we have $n^{1.38} \leq m_0 \leq n^{1.39}$.

Lemma 2. *There is a constant $c_2 = c_2(\varepsilon) > 0$ such that the following holds almost surely. For all $(x, y, z) \in \mathcal{T}$ the number of rooted subcubes in $\mathcal{F}_{x,y,z}$ that are inexpensive is at least $c_2 m_0 = c_2\binom{2k_0}{k_0}$.*

Proof. Let $(x, y, z) \in \mathcal{T}$ be given. Fix v, $w \in Q^\ell$ ($\ell \geq 1$) with $d(v, w) = \ell$, and let $U = (U_e)_{e \in E(Q^\ell)}$ be a family of independent exponential r.vs of mean 1. By the definition of T_0, we have that for any given fixed $\delta > 0$ we have $\lim_\ell \mathrm{P}(d_U(v, w) \geq T_0 + \delta) = 0$. It follows that there is an integer $\ell_0 = \ell_0(\varepsilon)$ such that $\mathrm{P}\big(d_U(v, w) \leq (T_0 + \varepsilon)/(1 - 2\varepsilon)\big) \geq 1/2$ if $\ell \geq \ell_0$. Moreover, there clearly exists a constant $c_2' = c_2'(\varepsilon) > 0$ such that the probability that a rooted cube $(\langle v, w\rangle\,; v)$ is inexpensive is at least c_2' if $1 \leq \dim\langle v, w\rangle \leq \ell_0$. Let us now set $c_2 = (1/2)(c_2' \wedge 1/2)$.

Let $(\langle v_i, w_i\rangle\,; v_i)$ $(1 \leq i \leq m_0)$ be the m_0 rooted cubes in $\mathcal{F}_{x,y,z}$. Note that the $\langle v_i, w_i\rangle$ are pairwise vertex-disjoint and hence the events that $(\langle v_i, w_i\rangle\,; v_i)$ $(1 \leq i \leq m_0)$ should be inexpensive are independent. Thus the number J of inexpensive rooted cubes in $\mathcal{F}_{x,y,z}$ has binomial distribution $\mathrm{Bi}(m_0, p)$ with parameters $m_0 = \binom{2k_0}{k_0}$ and $p \geq 2c_2$. It now follows that $\mathrm{P}(J < c_2 m_0) \leq \exp\{-pm_0/8\} \leq \exp\{-n^{1.3}\} = o(|\mathcal{T}|^{-1})$, as required. $\square$

Our next result is a simple lemma concerning (Q^n, Y). Recall that $k_0 = \lceil \log n\rceil$. If x, $y \in Q^n$ are such that $d(x, y) = 2\ell$, let $L_{x,y}$ denote the *middle layer* $\{z \in \langle x, y\rangle : d(x, z) = d(z, y) = \ell\}$ of $\langle x, y\rangle$.

Lemma 3. *Let $V_0 = V_0(Y)$ be the set of vertices z of Q^n that are incident to at least $(\varepsilon^2/3)n$ edges e with $Y_e \leq \varepsilon$. Then almost surely for all x, $y \in Q^n$ with $d(x, y) = 2k_0$ at most n vertices in the middle layer $L_{x,y}$ of $\langle x, y\rangle$ fail to be in V_0.*

Proof. Let $x = x_0 = \emptyset$ and fix $y \in [n]^{(2k_0)}$. For any edge $e \in E(Q^n)$ we have $\mathrm{P}(Y_e \leq \varepsilon) = 1 - \exp(-\varepsilon^2) \geq \varepsilon^2/2$, and hence for fixed $z \in L_{x,y}$ the probability that z does not belong to V_0 is at most $p_0 = \exp\{-\varepsilon^2 n/36\}$. Thus the probability that we have $|L_{x,y} \setminus V_0| \geq n$ is at most

$$\binom{m}{n}\exp\{-\varepsilon^2 n^2/36\} \leq \binom{n^{1.4}}{n}\exp\{-\varepsilon^2 n^2/36\} \leq \exp\{-\Omega(n^2)\}.$$

Thus the probability that there exist x, $y \in Q^n$ with $d(x, y) = 2k_0$ such that $|L_{x,y} \setminus V_0| \geq n$ is at most $4^n \exp\{-\Omega(n^2)\} = o(1)$. $\square$

Let $V_0 = V_0(Y)$ be defined as in Lemma 3. For all $z \in V_0$ we let $N_z' = N_z'(\varepsilon, Y) = \{z' \in \Gamma_{Q^n}(z) : Y_{zz'} \leq \varepsilon\}$. Thus $|N_z'| \geq (\varepsilon^2/3)n$ for any $z \in V_0$.

§2. The main lemma

In this section we study (Q^n, Z), and prove a lemma (Lemma 4) that will be fundamental for the proof of our main result, Theorem 5. We stress that the argument in the proof below is quite crude, and that with a little more patience one may prove a stronger result. However, such a stregthening of this lemma would not, as far as we can see, improve Theorem 5. Before turning to our lemma, recall that $0 < \varepsilon < 1/6$ has been fixed.

Lemma 4. *Set $K = \lfloor (\varepsilon^6 n/650)^{1/7} \rfloor$, and let $m^2 = \omega n^{11/7} \leq n^2$, where $\omega = \omega(n) \to \infty$ as $n \to \infty$. Let $x = x_0 = \emptyset$ and fix $y \subset [n]$ with $1 \leq k = |y| \leq K$. Let $S \subset \Gamma_{Q^n}(x)$ and $T \subset \Gamma_{Q^n}(y)$ with $|S|, |T| \geq m$ be given. Then in (Q^n, Z) there is an S—T path P of Z-length $\ell(P, Z) = \sum_{e \in E(P)} Z_e$ at most ε with probability at least $1 - \mathrm{e}^{-\omega n/\log \omega}$.*

Proof. For all $e \in E(Q^n)$ and $2 \leq j \leq K + 1$, let $Z_e^{(j)}$ be an exponential r.v. with mean K/ε, and assume that all these variables are independent. Set $Z^{(j)} = (Z_e^{(j)})_{e \in E(Q^n)}$ for $2 \leq j \leq K + 1$. Clearly Z_e and $\bigwedge_j Z_e^{(j)}$ have the same distribution for all $e \in E(Q^n)$. Let $t_0 = \varepsilon/(2K + k - 2)$. Then for all $2 \leq j \leq K + 1$ we have

$$p = \mathrm{P}\{Z_e^{(j)} \leq t_0\} = 1 - \exp\left\{\frac{-\varepsilon^2}{K(2K + k - 2)}\right\} \geq \frac{\varepsilon^2}{2K(2K + k - 2)} \geq \frac{\varepsilon^2}{6K^2}.$$

For all $2 \leq j \leq K + 1$, let $H_j = H_j(Z^{(j)}) \subset Q^n$ be the spanning subgraph of Q^n whose edge-set is $\{e \in E(Q^n) \colon Z_e^{(j)} \leq t_0\}$. Note that the H_j $(2 \leq j \leq K + 1)$ are K independent random elements of $\mathcal{G}(Q^n, p)$, the space of random spanning subgraphs of Q^n whose edges are independently present with probability p. Let us set $G_j = H_2 \cup \cdots \cup H_j$ $(2 \leq j \leq K + 1)$. We now claim the following.

Claim. In G_{K+1} there is an S—T path of length at most $2K + k - 2$ with probability at least $1 - \exp\{-\omega n/\log \omega\}$.

Note that a path as in the claim above has Z-length at most $t_0(2K+k-2) = \varepsilon$, and hence to prove our lemma it suffices to prove this claim. To verify this claim we start by setting $S_1 = \{v \in S \colon v \not\subset y\}$ and $T_1 = \{v \in T \colon y \subset v\}$. Clearly $|S_1|, |T_1| \geq m - k \geq m/2$. We shall think of Q^n as $Q^{n-k} \times Q^k$ in the following way. For each $v \subset y^{\mathrm{c}} = [n] \setminus y$, let

$$Q_v = \langle v, v \triangle y \rangle = \langle v, v \cup y \rangle = \{z \subset [n] \colon z \cap y^{\mathrm{c}} = v\}. \tag{2}$$

Note that Q_v is a cube of dimension $k = |y|$, and that $Q = \langle x_0, y^c\rangle = \{v \in Q^n : v \subset y^c\}$ is a cube of dimension $n-k$. Also, we may naturally regard Q^n as $\langle x_0, y^c\rangle \times \langle x_0, y\rangle$.

Write $n_1 = |y^c| = n - k$ and note that clearly $n_1 = (1+o(1))n$. Let us set

$$S_2' = \{z \in (y^c)^{(2)} : z \in \Gamma_{Q^n}(S_1) \text{ and } z \cup y \in \Gamma_{Q^n}(T_1)\}.$$

A moment's thought shows that $|S_2'| \geq \binom{m/2}{2}$. Thus, setting

$$S_2 = \{z \in (y^c)^{(2)} : z \in \Gamma_{H_2}(S_1) \text{ and } z \cup y \in \Gamma_{H_2}(T_1)\},$$

quite crudely we have $\mathrm{E}(|S_2|) \geq \binom{m/2}{2}p^2 \geq m^2p^2/9$. Hence, with probability at least $1 - \exp\{-m^2p^2/1800\} = 1 - \exp\{-\Omega(\varepsilon^{4/7}\omega n)\}$, we have $|S_2| \geq m^2p^2/10$.

For the rest of the argument, we condition on $|S_2| \geq m^2p^2/10 \geq (1/5)(mp/n)^2\binom{n_1}{2}$. In the sequel, if $U \subset \langle x_0, y^c\rangle$, we write $U \vee y$ for $\{u \cup y : u \in U\}$, namely the translate of U by the map $u \mapsto u \cup y$, which is contained in the cube $\langle y, [n]\rangle$. Let us now fix $3 \leq j \leq K$ and condition on the existence of $S_2 \subset (y^c)^{(2)}, \ldots, S_{j-1} \subset (y^c)^{(j-1)}$ for which the following conditions hold for all $2 \leq i < j$.

(*i*) $|S_i| \geq (4/5)2^{-i}\left(mp^{i-1}/n\right)^2\binom{n_1}{i}$,

(*ii*) for all $z \in S_i$ there is an S_1—z path of length $i-1$ in $G_i = H_2 \cup \cdots \cup H_i$,

(*iii*) for all $z \in S_i$ there is a T_1—$(z \cup y)$ path of length $i-1$ in $G_i = H_2 \cup \cdots \cup H_i$.

Now let $S_j' = \Gamma_{Q^n}(S_{j-1}) \cap (y^c)^{(j)} = \{z' \in (y^c)^{(j)} : z \subset z' \text{ for some } z \in S_{j-1}\}$. Then, by the local LYM inequality (see for instance [2], §3), we have that

$$|S_j'| \geq |S_{j-1}|\binom{n_1}{j-1}^{-1}\binom{n_1}{j} \geq \frac{4}{5}2^{-j+1}\left(\frac{mp^{j-2}}{n}\right)^2\binom{n_1}{j}.$$

We now set $S_j = \{z \in S_j' : z \in \Gamma_{H_j}(S_{j-1}) \text{ and } z \cup y \in \Gamma_{H_j}(S_{j-1} \vee y)\}$. Then very crudely we have that

$$\mathrm{E}(|S_j|) \geq |S_j'|p^2 \geq \frac{4}{5}2^{-j+1}\left(\frac{mp^{j-1}}{n}\right)^2\binom{n_1}{j},$$

and so with probability at least

$$1 - \exp\left\{-\frac{1}{5}2^{-j}\left(\frac{mp^{j-1}}{n}\right)^2\binom{n_1}{j}\right\},$$

condition (i) above holds for $i=j$, that is $|S_j| \geq (4/5)2^{-j}\left(mp^{j-1}/n\right)^2\binom{n_1}{j}$. Note that by the definition of S_j conditions (ii) and (iii) above hold for $i=j$. Let

$$S''_K = \{z \in (y^{\mathrm{c}})^{(K)} : d_{G_K}(S,z),\ d_{G_K}(T, z\cup y) \leq K-1\},$$

where d_{G_K} denotes the distance in the graph G_K. Then, using the argument above for $3 \leq j \leq K$ in turn, we see that with probability at least

$$P_0 = 1 - \sum_{3\leq j\leq K} \exp\left\{-\frac{1}{5}2^{-j}\left(\frac{mp^{j-1}}{n}\right)^2\binom{n_1}{j}\right\}$$

we have

$$|S''_K| \geq \frac{4}{5}2^{-K}\left(\frac{mp^{K-1}}{n}\right)^2\binom{n_1}{K}. \tag{3}$$

To estimate P_0, note that if $3 \leq j \leq K$ then

$$\begin{aligned} A = 2^{-j}\left(\frac{mp^{j-1}}{n}\right)^2\binom{n_1}{j} &\geq \frac{m^2}{n^2p^2}\left(\frac{p^2n_1}{2j}\right)^j \\ &\geq \frac{m^2}{n^2p^2}\left\{\left(\frac{p^2n_1}{6}\right)^3 \wedge \left(\frac{p^2n_1}{2K}\right)^K\right\}, \end{aligned}$$

since the function $f(x) = (B/x)^x$ $(x>0)$ is log-concave. However,

$$\left(\frac{p^2n_1}{2K}\right)^K \geq \left(\frac{1}{73}\frac{\varepsilon^4 n}{K^5}\right)^K = \left(\Omega(n^{2/7})\right)^K \geq \left(\frac{p^2n_1}{6}\right)^3,$$

and hence $A \geq (1/216 + o(1))m^2np^4 = \Omega(\omega n^{10/7})$. Thus, very crudely,

$$P_0 \geq 1 - K\exp\{-\Omega(\omega n^{10/7})\} \geq 1 - \exp\{-\omega n\}. \tag{4}$$

Let us now condition on (3). To finish the proof of our claim, we shall show that with very high probability there is an S''_K—$(S''_K \vee y)$ path of length k in H_{K+1}. We shall do this using an extremely crude argument. Consider the family $\mathcal{F} = \{Q_v : v \in S''_K\}$ of k-cubes as defined in (2). Recall that Q_v is a translate of $\langle x_0, y\rangle$ by v, and note that in particular the Q_v $(v \in S''_K)$ are pairwise disjoint. Consider for each $v \in S''_K$ a fixed v—$(v\cup y)$ path $P_v \subset Q_v$ of length k, and note that the probability P_1 that there should not be an S''_K—$(S''_K \vee y)$ path of length k in H_{K+1} is at most $(1-p^k)^{|S''_K|} \le (1-p^K)^{|S''_K|} \le \exp\{-p^K|S''_K|\}$. However, as (3) holds,

$$p^K|S''_K| \ge \frac{4}{5}2^{-K}p^K\left(\frac{mp^{K-1}}{n}\right)^2\binom{n_1}{K}$$
$$\ge \frac{4}{5}\frac{m^2}{p^2n^2}\left(\frac{p^3n}{3K}\right)^K \ge \frac{4}{5}\left(\frac{m}{pn}\right)^2\left(\frac{650}{648}\right)^K,$$

which is much larger than ωn. Thus $P_1 \le \mathrm{e}^{-\omega n}$. This bound and inequality (4) finish the proof of the claim, and hence Lemma 4 is proved. □

§3. The main result

Recall that $W = (W_e)$ is a collection of independent exponential r.vs each with mean 1. Also, recall that T_0 is the first-passage percolation time between two diametrically opposite vertices in Q^n when the passage times are given by W (cf. (1) in Section 2).

Theorem 5. *The following hold with probability* $1-o(1)$ *as* $n \to \infty$.
(i) *Writing* $x_0 = \emptyset \in Q^n$, *we have* $\max_{y\in Q^n} d_W(x_0, y) \le T_0 + \log 2 + o(1)$,
(ii) $\operatorname{diam}(Q^n, W) \le T_0 + 2\log 2 + o(1)$.

Proof. (i) Let $0 < \varepsilon < 1/6$ be fixed. We regard W_e as $X_e \wedge Y_e \wedge Z_e$ ($e \in E(Q^n)$), where X_e, Y_e, and Z_e are as in Section 2, *i.e.* the X_e are exponential with mean $(1-2\varepsilon)^{-1}$, the Y_e and Z_e are exponential with mean ε^{-1}, and all these variables are independent. Let us now consider the following events. (We keep the notation introduced in Section 2.)

- (a) For all $x \in Q^n$ we have $|N_x| = |N_x(\varepsilon, X)| \ge c_1 n$,
- (b) for all $(x, y, z) \in \mathcal{T}$, the number of inexpensive rooted subcubes in $\mathcal{F}_{x,y,z}$ is not smaller than $c_2 m_0 = c_2\binom{2k_0}{k_0}$,
- (c) for all x, $y \in Q^n$ with $d(x, y) = 2k_0$, we have $|L_{x,y} \setminus V_0| = |L_{x,y} \setminus V_0(Y)| \le n$.

Note that by Lemmas 1, 2, and 3 conditions (a), (b), and (c) above hold almost surely. For the rest of the argument we condition on (Q^n, X) and (Q^n, Y) satisfying (a), (b), and (c).

Let $S_{x_0} = N_{x_0} = N_{x_0}(\varepsilon, X) = \{y \in \Gamma_{Q^n}(x_0) \colon X_{x_0 y} \le \varepsilon\}$. For every $x \in Q^n - x_0$, set $S_x = N'_x = N'_x(\varepsilon, Y)$ if $x \in V_0 = V_0(Y)$, and $S_x = N_x = N_x(\varepsilon, X)$ otherwise. For a fixed pair x, $y \in Q^n$ with $d(x, y) \le$

$K = \lfloor (\varepsilon^6 n/650)^{1/7} \rfloor$, if we let $S = S_x$ and $T = S_y$, Lemma 4 tells us that $d_Z(S,T) \leq \varepsilon$ with probability at least $1 - \exp\{-n^{5/4}\}$. Thus (d) below holds with probability $1 - o(1)$.

(d) For all x, $y \in Q^n$ with $d(x,y) \leq K$ we have

$$d_W(x,y) \leq \begin{cases} 3\varepsilon & \text{if } x,\, y \in V_0 \cup \{x_0\} \\ \log 2 + 5\varepsilon & \text{if } x \in V_0 \cup \{x_0\} \text{ or } y \in V_0 \cup \{x_0\} \\ 2\log 2 + 7\varepsilon & \text{otherwise.} \end{cases}$$

We now condition on (Q^n, W) satisfying (d), and claim that consequently we have $\max\{d_W(x_0,y) \colon y \in Q^n\} \leq T_0 + \log 2 + 3\varepsilon T_0 + 12\varepsilon$. First note that it follows from (d) that $\max\{d_W(x_0,y) \colon y \in Q^n,\, d(x_0,y) \leq K\}$ is as small as claimed, and hence let $y \in Q^n$ be such that $d(x_0,y) > K \geq 2k_0$. Pick $z \in \langle x_0, y\rangle$ with $d(x_0,z) = 2k_0$, and note that then $(x_0,y,z) \in \mathcal{T}$. Since we are conditioning on (b) and (c) above, we may find $v \in L_{x_0,y}$ such that $v \in V_0$, $w = v \bigtriangleup y \bigtriangleup z \in V_0$, and $(\langle v,w\rangle\, ; v)$ is an inexpensive rooted cube. But then

$$\begin{aligned} d_W(x_0,y) &\leq d_W(x_0,v) + d_W(v,w) + d_W(w,y) \\ &\leq 3\varepsilon + \frac{T_0 + \varepsilon}{1 - 2\varepsilon} + \log 2 + 5\varepsilon \leq T_0 + \log 2 + 3\varepsilon T_0 + 12\varepsilon. \end{aligned}$$

Theorem 5(i) follows by letting $\varepsilon \to 0$.

(ii) Minor modifications to the above argument gives a proof of Theorem 5(ii). □

The result above, coupled with the upper bound $T_0 \leq 1$ for the first-passage percolation time T_0 established by Fill and Pemantle [5], gives the following corollary.

Corollary 6. *The following hold with probability* $1 - o(1)$ *as* $n \to \infty$.
(i) *Writing* $x_0 = \emptyset \in Q^n$, *we have* $\max_{y \in Q^n} d_W(x_0,y) \leq 1 + \log 2 + o(1)$,
(ii) $\operatorname{diam}(Q^n, W) \leq 1 + 2\log 2 + o(1)$. □

§4. Concluding remarks and open problems

The best bounds so far for the first-passage percolation time T_0 in Q^n, defined in §1, are the ones given in [5], namely $.881\ldots = \log(1+\sqrt{2}) \leq T_0 \leq 1$. We believe that percolation happens at a sharply defined time.

Conjecture 7. *For all* $\delta > 0$ *we have* $\lim_{n\to\infty} \mathrm{P}\{d_W(x_0, y_0) \leq T_0 - \delta\} = 0$. □

We also believe that the analogous phenomenon happens in Richardson's model. As a starting point, it would be interesting to settle the following problem.

Problem 8. *Do the following two assertions hold with probability* $1 - o(1)$ *as* $n \to \infty$?

(i) *Writing* $x_0 = \emptyset \in Q^n$, *one has* $\max_{y\in Q^n} d_W(x_0, y) = T_0 + \log 2 + o(1)$.

(ii) *One has* $\mathrm{diam}(Q^n, W) = T_0 + 2\log 2 + o(1)$. □

References

[1] Bollobás, B., *Graph Theory—An Introductory Course*, Springer–Verlag, New York 1979, *viii* + 180pp.

[2] Bollobás, B., *Combinatorics*, Cambridge University Press, Cambridge 1986, *xii* + 177pp.

[3] Bollobás, B., Kohayakawa, Y., Łuczak, T., On the diameter and radius of random subgraphs of the cube, *Random Structures and Algorithms*, to appear.

[4] Bollobás, B., Kohayakawa, Y., Łuczak, T., Connectivity properties of random subgraphs of the cube, *Random Structures and Algorithms*, to appear.

[5] Fill, J.A., Pemantle, R., Percolation, first-passage percolation, and covering times for Richardson's model on the n-cube, *The Annals of Applied Probability* **3** (1993), 593–629.