The Hitting Time of Hamilton Cycles in Random Bipartite Graphs

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Abstract. Frieze showed that the limit distributions of Hamilton cycles and minimum degree two coincide for random bipartite graphs. We strengthen this by showing that the hitting times of having minimum degree at least two and being Hamiltonian coincide for almost every random bipartite graph process. The method used in our proof includes a new way of finding Hamilton cycles in random bipartite graphs.

1. Introduction

Pósa [10] and Korshunov [8] showed that a.e. G_p is Hamiltonian if $p = c(\log n)/n$, where $c > 0$ is some absolute constant; this came close to answering completely a question of Erdős and Rényi [4], who raised the problem of computing the 'minimal' $p = p(n)$ for which a.e. random graph G_p of order n has a Hamilton cycle. Based on the work of Pósa, Komlós and Szemerédi [7] succeeded in determining the threshold function for Hamilton cycles (see also Korshunov [9]); they proved the very pleasing fact that it coincides with the threshold function for minimum degree at least two. In fact, as proved in [2], a much more refined result is true: not only do the threshold functions of these two properties coincide, but so do their hitting times for almost every graph process.

In this note we are interested in Hamilton cycles in random bipartite graphs. Frieze [6] determined the limit distribution for Hamilton cycles in such graphs, thus showed that the threshold functions for Hamilton cycles and minimum degree two coincide in this case as well. In spite of the fact that such a result is not very surprising, the proof is rather complicated: it is certainly much more difficult than the proof of the results in [7] or [9]. Our aim in this note is to sharpen this theorem of Frieze by showing that the properties of being Hamiltonian and having minimum degree two do not only have the same threshold function, but their hitting times coincide almost surely as well.

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It might be at first a little surprising that one needs any new idea to determine the threshold function of being Hamiltonian for bipartite graphs, however no simple-minded application of the known techniques is enough: they break down when one wants to apply Pósa's lemma owing to a certain parity problem. Our method of overcoming this difficulty is different from Frieze's; being simpler, it will allow us to sharpen his result as described above.

The notation and terminology used in this note are standard (see [3]) but for the sake of convenience we define some of the terms we shall use. We start with a model of random subgraphs of a graph. Let a graph H be given and assume that H has $m \geq 1$ edges. Let $0 \leq p \leq 1$ and $0 \leq t \leq m$ be given. Let us first define the model $\mathcal{G}(H,p)$ of random spanning subgraphs of H as follows. A random element $G_p \in \mathcal{G}(H,p)$ is obtained by selecting its edges randomly from the edge-set of H, where the probability of an arbitrary fixed edge of H being selected is p, and all these selections are independent from one another. More precisely, for an arbitrary fixed spanning subgraph G of H,

$$\mathsf{P}(G_p = G) = p^{e(G)}(1-p)^{m-e(G)},$$

where as usual $e(G)$ denotes the number of edges of G. We now define the space $\mathcal{G}(H,t)$; this is simply the probability space on the spanning subgraphs of H with exactly t edges where all of them are equiprobable.

Analogously to a random graph process (see [3], p. 38), we define a *random H-process*, or simply an *H-process*, as a Markov chain $\widetilde{G} = (G_t)_0^m$ whose states are spanning subgraphs of H. The process starts with the empty graph and, for $1 \leq t \leq m$, the graph G_t is obtained from G_{t-1} by the random addition to it of an edge of H not present in G_{t-1}, with all the possible new edges equiprobable. Thus G_t has precisely t edges and $G_m = H$. Let $\widetilde{\mathcal{G}}(H)$ be the set of all $m!$ random H-processes. Then it is a probability space all of whose elements are equiprobable; we shall usually write $\widetilde{G}$ for a random element of it.

Suppose that Q is a non-trivial monotone increasing property of the subgraphs of H; that is, the graph H has it, the empty subgraph of H does not have it, and if $G \subset H$ is a subgraph of H that has it then any subgraph of H that contains G has it as well. Then for all H-processes $\widetilde{G} = (G_t)_0^m$ we call the time τ for which Q appears the *hitting time* of Q:

$$\tau = \tau_Q = \tau_Q(\widetilde{G}) = \tau(\widetilde{G}; Q) = \min\{t \geq 0 : G_t \text{ has } Q\}.$$

Let us now go back to bipartite graphs by letting H above be the complete balanced bipartite graph $K^{n,n}$. Let $0 \leq p = p(n) \leq 1$ be given. The most basic model of *random bipartite graphs* is simply $\mathcal{G}(K^{n,n}, p)$. We shall call a $K^{n,n}$-process a *random bipartite graph process*, or briefly a *bipartite graph process*. Our main result of this note is that the hitting times of having minimum degree two and of being Hamiltonian coincide in a.e. such process; formally, for a.e. $\widetilde{G} \in \widetilde{\mathcal{G}}(K^{n,n})$,

$$\tau(\widetilde{G}; G \text{ is Hamiltonian}) = \tau(\widetilde{G}; \delta(G) \geq 2), \tag{1}$$

where as customary $\delta(G)$ denotes the minimum degree of G.

Let us outline the organisation of this note. In Section 2 we give some preliminaries concerning models of random graphs; in particular, we introduce several spaces of random graphs that will be needed in the proof of our result. We also describe the various relationships between these spaces and briefly discuss why they are needed. Our main result (Theorem 11) is given in the next section; it follows soon after Lemma 8, which is the key result needed in its proof. In Section 4 we discuss a related problem concerning certain special k-factors.

2. Preliminaries

We shall mostly be dealing with balanced bipartite graphs and therefore it is going to be convenient for us to fix once and for all two disjoint n-sets, say $A = A_n$ and $B = B_n$, as the vertex classes of our bipartite graphs. More precisely, the term 'bipartite graph' will mean a spanning subgraph of the complete bipartite graph $K^{n,n}$ whose vertex classes are A and B.

As mentioned in the introduction, the most basic model of random bipartite graphs is $\mathcal{G}(K^{n,n}, p)$. As an illustration of one of the ideas involved in the proof of (1), let us very roughly sketch how one might prove that a.e. $G_p \in \mathcal{G}(K^{n,n}, p)$ is Hamiltonian, provided $p = p(n)$ is large enough.

Given a random element $G_p \in \mathcal{G}(K^{n,n}, p)$, we first prove that, as p is large enough, it a.s. has a 2-factor F. We then select a largest component Q of F, which is obviously a cycle. If Q is a Hamilton cycle we are done, hence we assume the contrary. We now use the fact that G_p is a.s. connected: let $Q' \neq Q$ be a component of F which is connected to Q by an edge in G_p. The cycles Q and Q' and any edge between them give us a path P in G_p of order strictly larger than the order of Q. Note that we now have a spanning subgraph of G_p that is a collection of cycles and a path. If we can extend our path P to a longer path in G_p, we do so. Otherwise we note that, by adding a very small number of edges to G_p, we can get a cycle Q'' for which $V(Q'') = V(P)$. We now have a new 2-factor F' whose largest component is strictly larger than the largest component of F. We complete the proof by iterating this process, until we find a Hamilton cycle.

Unfortunately, the proof of (1) will involve a slightly more complicated model than

$\mathcal{G}(K^{n,n},p)$; in fact, by Lemma 1 below, it will suffice to prove that a.e. random bipartite graph is Hamiltonian according to the model $\mathcal{G}_{y,\underline{r}}(n,p;\geq 2)$, which we shall now introduce. This is a model of random *edge-coloured* bipartite graphs. Let $0 < p = p(n) < 1$ and $1 \leq k \leq n$ be given. (In Sections 2 and 3 we shall always have $k = 2$; in Section 4, we shall consider arbitrary k.) The model $\mathcal{G}_{y,\underline{r}} = \mathcal{G}_{y,\underline{r}}(n,p;\geq k)$ consists of bipartite graphs whose edges are coloured yellow and red. To define a random element of $\mathcal{G}_{y,\underline{r}}$, first choose an element G_y of $\mathcal{G}(K^{n,n},p)$. Let $x_1,\ldots,x_s$ be the vertices of degree less than k in G_y. For each $i = 1,\ldots,s$, randomly add to G_y an edge x_iy_i, where the vertex y_i belongs to the vertex class not containing x_i and $x_iy_i \notin E(G_y)$. Colour the edges in G_y yellow and the edges added to G_y red. This yellow-red edge-coloured bipartite graph $G_{y,\underline{r}}$ is a random element of $\mathcal{G}_{y,\underline{r}}$. The model $\mathcal{G}_{b,\underline{r}}(n,p;\geq k)$ is defined analogously; thus its elements are random bipartite graphs whose edges are coloured blue and red: the blue edges are generated first with probability p and then the red ones are added as before.

We shall always choose p in such way that the minimum degrees of both G_y and G_b are almost surely one. Thus the red edges that we add to them when we generate $G_{y,\underline{r}}$ and $G_{b,\underline{r}}$ guarantee that we obtain graphs of minimum degree two almost surely. In fact, it might be of some help to keep in mind that the red edges are used always solely for the purpose of making the minimum degree of our graphs almost surely two: hence the notation that distinguishes red. We shall use the bipartite version of a lemma of Bollobás [2].

Lemma 1. *Let $k \geq 1$ be fixed and let Q be a non-trivial monotone increasing property of bipartite graphs such that every graph having Q has minimum degree at least k. Let*

$$p = (\log n + (k-1)\log\log n - \omega(n))/n, \tag{2}$$

where $\omega(n) \to \infty$ and $\omega(n) \leq \log\log\log n$. If a.e. graph in $\mathcal{G}_{y,\underline{r}} = \mathcal{G}_{y,\underline{r}}(n,p;\geq k)$ has Q, then

$$\tau(\widetilde{G};Q) = \tau(\widetilde{G};\delta(G)\geq k)$$

for a.e. bipartite graph process $\widetilde{G}$. □

It should now be clear that our aim is to prove that a.e. $G_{y,\underline{r}} \in \mathcal{G}_{y,\underline{r}} = \mathcal{G}_{y,\underline{r}}(n,p;\geq k)$ is Hamiltonian, where $k = 2$ and p satisfies (2). Let us set

$$p = (\log n + \log\log n - \log\log\log n)/n; \tag{2'}$$

from here onwards, we tacitly assume that p is given by (2′). We shall use the powerful colouring technique of Fenner and Frieze [5] in order to establish that a.e. $G_{y,\underline{r}}$ is Hamiltonian.

At the beginning of this section, we sketched a proof of the fact that a.e. $G_p \in \mathcal{G}(K^{n,n},p)$ is Hamiltonian if p is large enough. Fenner and Frieze's method can be applied to rid that proof of its recursive character, thereby substantially simplifying it and allowing one to derive a sharper result. (For instance, compare the proofs of Theorems VIII.9 and VIII.11 in [3].) The key ingredient of the colouring method is the use of *two* certain models of edge-coloured graphs. Since we are already looking at a model of such graphs, namely $\mathcal{G}_{y,\underline{r}}$, our use of Fenner and Frieze's technique will be slightly complicated by the fact that we shall have to introduce two further models of edge-coloured bipartite graphs. The models we shall need are defined as follows.

Let us start by setting $p_g' = (n\log\log n)^{-1}$ and $p_b' = 1 - p_g'$. We now define the model $\mathcal{G}_{b,g,\underline{r}} = \mathcal{G}_{b,g,\underline{r}}(n,p,p_b',p_g';\geq 2)$ consisting of bipartite graphs whose edges are coloured blue, green and red. To define a random element $G_{b,g,\underline{r}}$ of $\mathcal{G}_{b,g,\underline{r}}$, pick an element $G_{y,\underline{r}}$ of $\mathcal{G}_{y,\underline{r}}$ and recolour its yellow edges blue with probability p_b' and green with probability p_g', the colours being chosen independently for each edge. We denote the probability in $\mathcal{G}_{b,g,\underline{r}}$ by $\mathbf{P}_{b,g,\underline{r}}$.

Given a random element $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$, let us denote by $G_{b,\widehat{g},\underline{r}}$ the graph obtained from $G_{b,g,\underline{r}}$ by the deletion of its green edges. We define the space $\mathcal{G}_{b,\widehat{g},\underline{r}}$, consisting of random blue-red edge-coloured bipartite graphs, by requiring that the map $\mathcal{G}_{b,g,\underline{r}} \to \mathcal{G}_{b,\widehat{g},\underline{r}}$ given by $G_{b,g,\underline{r}} \mapsto G_{b,\widehat{g},\underline{r}}$ should be measure preserving.

Although in principle we are now ready to start our proof, it will be convenient to introduce, merely for technical reasons, two further spaces of graphs. Instead of working directly with the models $\mathcal{G}_{b,g,\underline{r}}$ and $\mathcal{G}_{b,\widehat{g},\underline{r}}$, it will be easier to consider approximations to these spaces, denoted $\mathcal{G}_{b,\underline{r},g}$ and $\mathcal{G}_{b,\underline{r}}$ respectively, defined as follows.

We start with $\mathcal{G}_{b,\underline{r}}$. Set

$$\begin{aligned} p_b &= p_b'p \\ &= (1-p_g')p \\ &= \left(1-(n\log\log n)^{-1}\right)(\log n + \log\log n - \log\log\log n)/n. \end{aligned}$$

Our space $\mathcal{G}_{b,\underline{r}}$ is simply $\mathcal{G}_{b,\underline{r}}(n,p_b;\geq 2)$; we denote the probability in this space by $\mathbf{P}_{b,\underline{r}}$.

Let us now define the model $\mathcal{G}_{b,\underline{r},g} = \mathcal{G}_{b,\underline{r},g}(n,p_b,p_g;\geq 2)$, which very closely approximates $\mathcal{G}_{b,g,\underline{r}}$. Let us set $p_g = pp_g'/(1-p_b)$. To define a random element $G_{b,\underline{r},g} \in \mathcal{G}_{b,\underline{r},g}$, we pick an element $G_{b,\underline{r}}$ from $\mathcal{G}_{b,\underline{r}}$ and an element H from $\mathcal{G}(K^{n,n},p_g)$ independently of each other and then set

$$G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H.$$

We keep the colours of the edges in $G_{b,\underline{r}}$ and colour the new edges coming from H green. The probability in the space $\mathcal{G}_{b,\underline{r},g}$ is denoted $\mathbf{P}_{b,\underline{r},g}$. Note that the map $\phi : \mathcal{G}_{b,\underline{r},g} \to \mathcal{G}_{b,\underline{r}}$ given by $G_{b,\underline{r},g} \mapsto G_{b,\underline{r}} = G_{b,\underline{r},\widehat{g}}$ is measure preserving.

The following lemma simplifies the computations and justifies the introduction of the last two spaces of random graphs.

Lemma 2. *Let Q be a property of blue-green-red edge-coloured bipartite graphs. Then a.e. $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ has Q if and only if a.e. $G_{b,\underline{r},g} \in \mathcal{G}_{b,\underline{r},g}$ has Q.* □

Lemma 2 follows easily from the two observations below. We denote by $V_1(G)$ the set of vertices of degree one in G. Also, given a coloured graph $G^*_{b,g,r}$, we denote the graph obtained by deleting its red edges by $G^*_{b,g,\widehat{r}}$, and the one obtained by further dropping the green ones by $G^*_{b,\widehat{g},\widehat{r}}$. The graph obtained by deleting its blue edges is denoted $G^*_{\widehat{b},g,r}$.

Lemma 3. *Let $G^*_{b,g,r}$ be a fixed bipartite graph whose edges are coloured blue, green and red in such a way that the following conditions hold:*

(i) $\delta\left(G^*_{b,\widehat{g},\widehat{r}}\right) = \Delta\left(G^*_{\widehat{b},g,r}\right) = 1,$

(ii) $\delta\left(G^*_{b,g,r}\right) = 2,$

(iii) $V_1\left(G^*_{b,g,\widehat{r}}\right) = V_1\left(G^*_{b,\widehat{g},\widehat{r}}\right).$

Then

$$\mathsf{P}_{b,g,\underline{r}}(G_{b,g,\underline{r}} = G^*_{b,g,r}) = (1+o(1))\mathsf{P}_{b,\underline{r},g}(G_{b,\underline{r},g} = G^*_{b,g,r}).$$ □

Lemma 4. *A.e. $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ satisfies (i), (ii) and (iii) of Lemma 3.* □

In the proof of the fact that a.e. $G_{y,\underline{r}} \in \mathcal{G}_{y,\underline{r}}$ is Hamiltonian, we shall need the following purely graph-theoretical lemma on 2-factors. We omit its proof, although we remark that one can easily deduce it from the max-flow min-cut theorem of Ford and Fulkerson (for instance, see [1], p. 47). Before giving the lemma, we introduce some further notation. Given a bipartite graph G and a subset U of its vertices, we denote by $\Gamma(U)$ the set of vertices of G adjacent to some vertex in U. Also, the set of vertices of G that are adjacent to exactly one vertex in U is denoted by $\Gamma_{=1}(U)$. We set $\Gamma_{\geq 2}(U) = \Gamma(U) \setminus \Gamma_{=1}(U)$. Thus the vertices in $\Gamma_{\geq 2}(U)$ are adjacent to at least two vertices in U.

Note that if a bipartite graph G has a 2-factor then, for all $U \subset V(G)$ entirely contained in one of its vertex classes, we have that

$$|\Gamma_{\geq 2}(U)| + \frac{1}{2}|\Gamma_{=1}(U)| \geq |U|. \tag{3}$$

Lemma 5. *Let G be a bipartite graph. The following conditions are equivalent:*

(i) G has a 2-factor.

(ii) For all $U \subset A$, relation (3) holds.

(iii) For all $U \subset B$, relation (3) holds. □

We shall need the following definition. Let G be a bipartite graph. We say that G is *2-expanding*, or simply *expanding*, if for all $U \subset A$ and all $U \subset B$

$$|\Gamma(U)| \geq 2|U|,$$

provided $|U| \leq 2n/5$. We shall use the following corollary to Lemma 5.

Corollary 6. *Let G be an expanding bipartite graph. Suppose that for all $S \subset A$ and all $S \subset B$ we have*

$$|\Gamma_{\geq 2}(S)| \geq |S|,$$

provided $|S| = \lceil 2n/5 \rceil$. Then, if G does not have a 2-factor then there is a set $U_0 \subset V(G)$, $U_0 \subset A$ or $U_0 \subset B$, for which (3) fails and $|U| \leq 3n/5$.

Proof. Suppose G does not have a 2-factor. By Lemma 5 and the symmetry between A and B, we may and shall assume that there exists a set $U \subset A$ for which (3) fails. Assume that $|U| > 3n/5$, since otherwise we are done. It is easy to check that (3) must fail for $U_0 = B \setminus \Gamma_{\geq 2}(U)$ as well. Choose $S \subset U$ with $|S| = \lceil 2n/5 \rceil$. Then

$$|U_0| = n - |\Gamma_{\geq 2}(U)| \leq n - |\Gamma_{\geq 2}(S)| \leq n - |S| \leq \lfloor 3n/5 \rfloor. \qquad \square$$

3. The main result

Not surprisingly, our proof of the main result of this note is based on Pósa's lemma [10] (see also [3], Lemma VIII.4). Given a graph G and a path P of G, we shall call P *strongly maximal* if (i) G has no cycle whose vertex set is $V(P)$ and (ii) G has no path longer than P in which the vertices of P appear contiguously, not necessarily in the same order as they do in P.

We are now ready to state a simple corollary of Pósa's lemma; it is the obvious analogue of Lemma VIII.5 in [3] for bipartite graphs.

Lemma 7. *Let G be an expanding bipartite graph. Set $u = \lfloor 2n/5 \rfloor$. Suppose G has a strongly maximal path P of even order. Then there are u distinct vertices $y_1, \ldots, y_u$ in A and sets $Y_1, \ldots, Y_u \subset B$ such that $|Y_i| \geq u$ and G has no y_i–Y_i edges, $1 \leq i \leq u$. Furthermore, for all $i = 1, \ldots, u$ and $z_i \in Y_i$, the graph G has a path Q whose endvertices are y_i and z_i, and $V(Q) = V(P)$.* $\square$

The key lemma in the proof of (1) is the following result.

Lemma 8. *A.e. $G_{b,\underline{r}} \in \mathcal{G}_{b,\underline{r}} = \mathcal{G}_{b,\underline{r}}(n, p_b; \geq 2)$ is connected, expanding and has a 2-factor.*

Proof. We assume throughout this proof that n is large enough. Let us first note that

$$(\log n + \log\log n - 2\log\log\log n)/n < p_b < (1+o(1))(\log n)/n.$$

(*a*) *A.e.* $G_{b,\Gamma}$ *is connected.*

This is entirely routine: one shows that a.e. $G_b \in \mathcal{G}(K^{n,n}, p_b)$ has no isolated vertices and, in fact, is a.s. connected.

(*b*) *A.e.* $G_{b,\Gamma}$ *is expanding.*

Set $R = (\log n)^{2/3}$. We call a subset U of the vertices of a bipartite graph G a *u-obstruction* if $|U| = u$, either $U \subset A$ or $U \subset B$, and

$$|\Gamma(U)| < 2|U| + R.$$

Moreover, if the graph $G[U \cup \Gamma(U)]$ induced by $U \cup \Gamma(U)$ in G is connected, we say that the u-obstruction U is *connected*.

Claim. A.e. $G_b \in \mathcal{G}(K^{n,n}, p_b)$ *has no connected u-obstructions for* $2 \le u \le 2n/5$.

We shall prove this by considering two ranges for u.

Case 1. 'Large' u: $(1/13)\log n \le u \le 2n/5$.

Let $m_u = m_u(G_b)$ denote the number of connected u-obstructions of G_b. Then, using the fact that the number of labelled trees on t vertices is t^{t-2}, we have that

$$\begin{aligned}
\mathsf{E}(m_u) &\le \sum_{v=1}^{2u+\lceil R\rceil-1} 2\binom{n}{u}\binom{n}{v}(u+v)^{u+v}(1-p)^{u(n-v)}p^{u+v-1} \\
&\le 2\frac{n}{\log n}\sum \left(\frac{en}{u}\right)^u \left(\frac{en}{v}\right)^v (u+v)^{u+v} e^{-u(n-v)(\log n)/n}\left(\frac{2\log n}{n}\right)^{u+v} \\
&\le n\sum (2e\log n)^{u+v}\frac{(u+v)^{u+v}}{u^u v^v} e^{-u(\log n)(1-v/n)} \\
&\le n\sum (4e\log n)^{u+v} e^{-2u(\log n)/11} \\
&\le n(3u)(4e\log n)^{4u} n^{-2u/11} \\
&\le 3n^2((4e\log n)^4 n^{-2/11})^u \\
&\le n^{-u/6}.
\end{aligned}$$

Thus

$$\mathsf{E}\left[\sum_{u=\lceil(1/13)\log n\rceil}^{\lfloor 2n/5\rfloor} m_u\right] \le 2n^{-(1/78)\log n} = o(1),$$

and hence a.e. G_b has no connected u-obstruction for $(1/13)\log n \le u \le 2n/5$.

Case 2. 'Small' u: $2 \le u < (1/13)\log n$.

For a vertex v of G_b, let $d_b(v)$ denote its degree in G_b. We show that a.e. G_b is such that

$$d_b(x) + d_b(y) > \lfloor (1/3)\log n \rfloor = w, \tag{3}$$

whenever x and y are vertices at distance two in G_b.

This will imply the claim in this case, since then we have that $|\Gamma(U)| \geq (1/6)\log n$ for any U contained in a vertex class of G_b with $|U| \geq 2$ and $G_b[U \cup \Gamma(U)]$ connected. Hence, if $2 \leq u = |U| < (1/13)\log n$ and $G_b[U \cup \Gamma(U)]$ is connected, then

$$|\Gamma(U)| \geq (1/6)\log n > 2|U| + R,$$

showing that U is not a connected $|U|$-obstruction.

Let us now show that (3) holds for a.e. G_b. Let $m = m(G_b)$ denote the number of triples of vertices (x, y, z) where x and y are at distance two, have z as a common neighbour, and the sum of the degrees of x and y is at most $w = \lfloor (1/3)\log n \rfloor$. It is enough for us to show that for a.e. G_b we have $m = m(G_b) = 0$. The following crude estimates for the expectation of m will suffice:

$$\begin{aligned}
\mathsf{E}(m) &= 2n^2(n-1)p_b^2 \sum_{v=0}^{w} \binom{2(n-1)}{v} p_b^v (1-p_b)^{2(n-1)-v} \\
&\leq \frac{5}{2} n^3 p_b^2 \left(\frac{2enp_b}{w}\right)^w e^{-2np_b} e^{(w+2)p_b} \\
&\leq 3n^3 \left(\frac{\log n}{n}\right)^2 \left[\frac{2en(\log n)/n}{(1/3)\log n}\right]^{(1/3)\log n} n^{-2} \left[1 + \frac{\log\log n}{\log n}\right]^{(1/3)\log n} \\
&\leq (\log n)^3 n^{-1+(1/3)\log(6e)} \\
&< n^{-0.06},
\end{aligned}$$

hence $\mathsf{E}(m) = o(1)$, and so a.s. $m = 0$, as required. This completes the proof of Case 2 and so of our claim.

Let us now continue with the proof of (b). Given a red-blue edge-coloured graph $G_{b,\underline{r}}$, let G_b denote the graph obtained by the deletion of its red edges. It is straightforward to check that a.e. $G_{b,\underline{r}}$ is such that (i) its minimum degree is two, (ii) it has at most $2(\log n)^{1/2}$ red edges, (iii) no two of its red edges are at distance less than or equal to 2, and (iv) vertices at distance one from a red edge have degree at least $2 + R$ in G_b. By our claim above, we know that a.e. $G_{b,\underline{r}}$ is such that (v) G_b has no connected u-obstruction for $2 \leq u \leq 2n/5$.

To complete the proof of (b), it suffices to show that if a red-blue edge-coloured graph $G_{b,\underline{r}}$ satisfies (i)–(v), then it is expanding. Suppose then that, contrary to this assertion, there is a graph $G = G_{b,\underline{r}}$ satisfying (i)–(v) such that

$$|\Gamma(U)| < 2|U|,$$

for some set U contained in one of its vertex classes and satisfying $|U| \leq 2n/5$. Without loss of generality, we may assume that $U \subset A$ and that $G[U \cup \Gamma(U)]$ is connected. Let $E_r \subset$

$E(G)$ be the red edges of G. Let the components of $G[U \cup \Gamma(U)] - E_r$ be $G_1, \ldots, G_\omega$. Let $C_1, \ldots, C_s$ be the components G_i for which

$$|V(G_i) \cap B| \geq 2|V(G_i) \cap A|, \tag{4}$$

and $C'_1, \ldots, C'_t$ be the remaining ones $(s + t = \omega)$. We clearly have that $t \geq 1$. Note that $|V(C'_i) \cap A| = 1$ for all $1 \leq i \leq t$, since G_b satisfies (v) above. Thus, each C'_i is a single edge $u_i v_i$ $(u_i \in U)$, say, and u_i is incident to a red edge, but v_i is not, by (iii). In particular, by (ii),

$$t \leq 2(\log n)^{1/2}.$$

Note that if a little more than (4) is true for some component C_{i_0}, then we get a contradiction. More precisely, suppose that there is an i_0 for which

$$|V(C_{i_0}) \cap B| \geq 2|V(C_{i_0}) \cap A| + R \tag{5}$$

holds. Then, as $R = (\log n)^{2/3} > t$, we have that

$$\begin{aligned}
|\Gamma(U)| &= \sum_i |V(C'_i) \cap B| + \sum_i |V(C_i) \cap B| \\
&\geq t + (2|V(C_{i_0}) \cap A| + R) + \sum_{i \neq i_0} |V(C_i) \cap B| \\
&> 2t + 2|V(C_{i_0})| + \sum_{i \neq i_0} |V(C_i) \cap B| \\
&\geq 2|U|,
\end{aligned}$$

contradicting the choice of U. Hence we assume that for no i_0 relation (5) holds, and therefore, by (v), that $|V(C_i) \cap A| = 1$, for all $1 \leq i \leq s$.

We now claim that $s = 0$. Let us assume the contrary. As $G[U \cup \Gamma(U)]$ is connected, we may assume without loss of generality that there is a red edge joining $u_1 \in V(C'_1)$ to a vertex in $V(C_1) \cap B$. As the degree in G_b of the unique vertex in $V(C_1) \cap A$ is less than $2 + R$, this contradicts (iv). Hence $s = 0$.

We now claim that $t = 1$. Indeed, if $t \geq 2$ then $G[U \cup \Gamma(U)]$ cannot be connected since the vertices in $\Gamma(U) \cap B = \{v_1, \ldots, v_t\}$ are not incident to red edges.

Thus $s = 0$ and $t = 1$. Now we simply note that this contradicts the choice of U and (i). This completes the proof that if $G = G_{b,\underline{r}}$ satisfies (i)–(v), then it is expanding. As we saw above, this shows that (b) holds.

(c) *A.e.* $G_{b,\underline{r}}$ *has a 2-factor.*

Let G be a bipartite graph. We now call a set $U \subset V(G)$ a *weak u-obstruction* if $|\Gamma_{\geq 2}(U)| \leq |U|$, and either $U \subset A$ or $U \subset B$. Corollary 6 implies that if G is expanding and has no weak u-obstruction for $2n/5 \leq u \leq 3n/5$, then G has a 2-factor.

Since we already know that a.e. $G_{b,\underline{r}}$ is expanding, in order to prove that a.s. such a graph has a 2-factor, it is enough to show that a.e. $G_{b,\underline{r}}$ has no weak u-obstruction for $2n/5 \le u \le 3n/5$. We shall show that this is in fact the case even for a.e. $G_b \in \mathcal{G}(K^{n,n}, p_b)$. Let $m_u = m_u(G_b)$ denote the number of weak u-obstructions of G_b. Let $2n/5 \le u \le 3n/5$ and set $u = \eta n$. Then $2/5 \le \eta \le 3/5$ and

$$\begin{aligned}
\mathsf{E}(m_u) &\le \binom{n}{u}^2 \left((1-p_b)^u + up_b(1-p_b)^{u-1}\right)^{n-u} \\
&\le \left(\frac{e}{\eta}\right)^{2\eta n} \left[\frac{4}{3}up_b e^{-p_b u}\right]^{n-u} \\
&\le \left(\frac{e}{\eta}\right)^{2\eta n} \left[\frac{3}{2}\eta(\log n)^{1-\eta/2} n^{-\eta}\right]^{(1-\eta)n} \\
&\le \left[\left(\frac{e}{\eta}\right)^{2\eta} \left(\frac{3}{2}\eta\right)^{1-\eta}\right]^{n} (\log n)^{(1-\eta/2)(1-\eta)n} n^{-\eta(1-\eta)n} \\
&\le (\log n)^{13n/25} n^{-6n/25}.
\end{aligned}$$

Hence

$$\mathsf{E}\left[\sum_{u=\lceil 2n/5\rceil}^{\lfloor 3n/5\rfloor} m_u\right] \le (\log n)^{13n/25} n^{1-6n/25} = o(1),$$

and therefore a.e. G_b has no weak u-obstruction for $2n/5 \le n \le 3n/5$, as required. □

By applying Lemma 2, we conclude the next result. As always, given $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$, we denote by $G_{b,\widehat{g},\underline{r}}$ the graph obtained from $G_{b,g,\underline{r}}$ by the deletion of its green edges.

Lemma 9. *A.e. $G_{b,g,\underline{r}}$ in $\mathcal{G}_{b,g,\underline{r}}$ is such that $G_{b,\widehat{g},\underline{r}}$ is connected, expanding and furthermore has a 2-factor.*

Proof. Let us say that a fixed blue-green-red edge-coloured $G^*_{b,g,r}$ satisfies property Q if $G^*_{b,\widehat{g},r}$, the graph obtained by the deletion of its green edges, is connected, expanding and has a 2-factor. As the map $G_{b,\underline{r},g} \in \mathcal{G}_{b,\underline{r},g} \mapsto G_{b,\underline{r},\widehat{g}} \in \mathcal{G}_{b,\underline{r}}$ is measure preserving, Lemma 8 implies that a.e. $G_{b,\underline{r},g}$ has Q. Hence, by Lemma 2, a.e. $G_{b,g,\underline{r}}$ has Q. □

Before we can state our last lemma, we need to introduce a further piece of notation. Let G be a graph. A subgraph F of G is called an *almost 2-factor* if it spans the whole of G and all its vertices have degree 2 except for possibly two of them, which may have degree 1. Thus F is either a 2-factor of G or it is a spanning subgraph of G whose components are all cycles except for one, which is a path. We set

$$\ell(F) = \begin{cases} e(Q) & \text{if } F \text{ has a component } Q \\ & \text{that is a path} \\ \max\{e(C) : C \text{ a cycle in } F\} & \text{otherwise,} \end{cases}$$

where, as usual, the number of edges of a graph H is denoted by $e(H)$. Finally, given a graph G that has an almost 2-factor, we set

$$L(G) = \max\{\ell(F) : F \text{ an almost 2-factor of } G\}.$$

If G has no almost 2-factor, we set $L(G) = 0$. We can now state our last lemma.

Lemma 10. *A.e.* $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ *is such that* $L(G_{b,g,\underline{r}}) = L(G_{b,\widehat{g},\underline{r}})$.

Proof. Recall that $G_{b,g,\underline{r}}$ can be generated from a randomly chosen $G_{y,\underline{r}}$ by an appropriate random recolouring of the yellow edges by blue and green. Assume that $G_{b,g,\underline{r}}$ has been generated from $G_{y,\underline{r}} \in \mathcal{G}_{y,\underline{r}}$, with $G_{y,\underline{r}}$ having an almost 2-factor. Let F be an almost 2-factor of $G_{y,\underline{r}}$ such that $\ell(F) = L(G_{y,\underline{r}}) = L(G_{b,g,\underline{r}})$. Let F' be the component of F that is a path if such exists, otherwise let it be a longest cycle of F. Thus $\ell(F) = e(F')$. Note that $L(G_{b,g,\underline{r}}) = L(G_{b,\underline{r}})$ if no edge of F' has been assigned green when we generated $G_{b,g,\underline{r}}$ from $G_{y,\underline{r}}$. As $e(F') \le n$, we see that the probability that all edges of F' have been coloured blue is at least

$$(p_b')^n = (1 - p_g')^n = (1 - (n \log\log n)^{-1})^n = 1 - o(1),$$

and the result follows. □

We are finally ready to prove our main result.

Theorem 11. *Let* τ_2, τ_{2F} *and* τ_H *be the hitting times of having minimum degree at least two, having a 2-factor and the property of being Hamiltonian, respectively. Then a.e. bipartite graph process* $\widetilde{G}$ *is such that*

$$\tau_2(\widetilde{G}) = \tau_{2F}(\widetilde{G}) = \tau_H(\widetilde{G}).$$

In particular, if $p = (\log n + \log\log n + c_n)/n$ *then*

$$\lim_{n\to\infty} \mathbf{P}(G_p \in \mathcal{G}(K^{n,n}, p) \text{ is Hamiltonian}) = \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-2e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to +\infty. \end{cases}$$

Proof. By Lemma 1, it is enough to prove that a.e. $G_{y,\underline{r}} \in \mathcal{G}_{y,\underline{r}}(n, p; \ge 2)$ is Hamiltonian, where

$$p = (\log n + \log\log n - \log\log\log n)/n;$$

equivalently, that a.e. $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ is Hamiltonian. Let us denote by $\mathcal{H}^c$ the subset of $\mathcal{G}_{b,g,\underline{r}}$ consisting of the non-Hamiltonian coloured graphs, and set

$$\mathcal{G}^{\mathrm{N}}_{b,g,\underline{r}} = \{G_{b,g,\underline{r}} : G_{b,\widehat{g},\underline{r}} \text{ is connected, expanding, has a 2-factor, and furthermore } L(G_{b,g,\underline{r}}) = L(G_{b,\widehat{g},\underline{r}})\}.$$

Thus $\mathcal{G}^{\rm N}_{b,g,\underline{r}}$ is the set of graphs $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ satisfying the conditions in Lemmas 9 and 10. By these lemmas, it follows that $\mathsf{P}_{b,g,\underline{r}}(\mathcal{G}^{\rm N}_{b,g,\underline{r}}) = 1 - o(1)$. Hence, in order to show that $\mathsf{P}_{b,g,\underline{r}}(\mathcal{H}^c) = o(1)$, it suffices to prove that

$$\mathsf{P}_{b,g,\underline{r}}(\mathcal{G}^{\rm exc}_{b,g,\underline{r}}) = o(1), \tag{6}$$

where $\mathcal{G}^{\rm exc}_{b,g,\underline{r}} = \mathcal{H}^c \cap \mathcal{G}^{\rm N}_{b,g,\underline{r}}$. Note that, by Lemma 2, this is equivalent to showing that

$$\mathsf{P}_{b,\underline{r},g}(\mathcal{G}^{\rm exc}_{b,g,\underline{r}}) = o(1), \tag{7}$$

which we shall now prove.

Recall that an element $G_{b,\underline{r},g} \in \mathcal{G}_{b,\underline{r},g}$ can be generated by randomly choosing $G_{b,\underline{r}} \in \mathcal{G}_{b,\underline{r}}$ and $H \in \mathcal{G}(K^{n,n}, p_g)$ (independently from each other), and then setting $G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H$, where we keep the colours of the edges in $G_{b,\underline{r}}$, and colour the ones in H but not in $G_{b,\underline{r}}$ green. Also, recall that the map $\phi : G_{b,\underline{r},g} \in \mathcal{G}_{b,\underline{r},g} \mapsto G_{b,\widehat{\underline{r},g}} \in \mathcal{G}_{b,\underline{r}}$ is measure preserving. Let us set $\mathcal{G}^{\rm exc}_{b,\underline{r}} = \phi(\mathcal{G}^{\rm exc}_{b,g,\underline{r}})$. We then have that

$$\mathsf{P}_{b,\underline{r},g}(\mathcal{G}^{\rm exc}_{b,g,\underline{r}}) \le \mathsf{P}_{b,\underline{r}}(\mathcal{G}^{\rm exc}_{b,\underline{r}}) \max \mathsf{P}(H : G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H \in \mathcal{G}^{\rm exc}_{b,g,\underline{r}}),$$

where the maximum is taken over all $G_{b,\underline{r}} \in \mathcal{G}^{\rm exc}_{b,\underline{r}}$. Hence (7) follows from the claim below.

Claim. *For all* $G_{b,\underline{r}} \in \mathcal{G}^{\rm exc}_{b,\underline{r}}$,

$$P_0 = P_0(G_{b,\underline{r}}) = \mathsf{P}(H : G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H \in \mathcal{G}^{\rm exc}_{b,g,\underline{r}}) = o(1). \tag{8}$$

Fix $G_{b,\underline{r}} \in \mathcal{G}^{\rm exc}_{b,\underline{r}}$, and assume that H is such that $G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H \in \mathcal{G}^{\rm exc}_{b,g,\underline{r}}$. We now claim that there are $\lfloor 2n/5 \rfloor^2$ edges of $K^{n,n}$ that cannot appear in H. Indeed, let F be an almost 2-factor of $G_{b,\underline{r}}$ that has a component C such that $L(G_{b,\underline{r}}) = e(C)$. As $G_{b,\underline{r}}$ is connected, we see that C is a strongly maximal path. Applying Lemma 7, we conclude that $G_{b,\underline{r}}$ is either Hamiltonian, or else it is such that if any of the $\lfloor 2n/5 \rfloor^2$ edges $y_i z_i$ of Lemma 7 is in H, then $L(G_{b,\underline{r},g}) > L(G_{b,\underline{r}})$. Since we are assuming that $G_{b,\underline{r},g}$ is not Hamiltonian, this implies that P_0 is at most

$$(1-p_g)^{\lfloor 2n/5 \rfloor^2} \le \exp\left[-\frac{1}{2} \left(\frac{\log n}{n^2 \log\log n} \right) \left\lfloor \frac{2n}{5} \right\rfloor^2 \right] = o(1),$$

where we used that

$$p_g \ge \frac{1}{2} \left(\frac{\log n}{n^2 \log\log n} \right). \qquad \square$$

4. A related problem

Let $k \ge 1$ be fixed and let F be a k-regular graph. We say that F is *decomposable* if either k is even and F is the disjoint union of $k/2$ Hamilton cycles, or else k is odd and F is the disjoint union of a 1-factor and $(k-1)/2$ disjoint Hamilton cycles.

An obvious necessary condition for a graph G to have a decomposable k-factor is that its minimum degree should be at least k. The following natural question arises. Let $k \geq 1$ be fixed. Let $p = p(n)$ be such that a.e. $G_p \in \mathcal{G}(K^{n,n}, p)$ has minimum degree k. Is it then true that a.e. $G_p \in \mathcal{G}(K^{n,n}, p)$ has a decomposable k-factor? Or, in fact, do the hitting times of minimum degree at least k and of the existence of a decomposable k-factor coincide for a.e. bipartite graph process? In this section we outline a proof of the fact that the hitting times do coincide almost surely. We shall in fact just state the necessary lemmas and sketch the proof.

The case $k = 1$ is that the hitting times of a 1-factor and minimum degree at least one coincide; this may be found in [3] as Theorem VII.1. The case $k = 2$ is Theorem 11 in this note; we may therefore assume that $k \geq 3$.

For this section we set

$$p = (\log n + (k-1)\log\log n - \log\log\log n)/n,$$

$$p'_g = (n \log\log n)^{-1},$$

$$p'_b = 1 - p'_g$$

and

$$p_b = p'_b p.$$

(We have only redefined p; the other probabilities have been changed accordingly.) We shall again make use of the models of bipartite random graphs introduced in Section 2; more precisely, we shall need the following spaces of random bipartite graphs: $\mathcal{G}_{b,g,\underline{r}} = \mathcal{G}_{b,g,\underline{r}}(n, p, p'_b, p'_g; \geq k)$, $\mathcal{G}_{b,\underline{r}} = \mathcal{G}_{b,\underline{r}}(n, p_b; \geq k)$, and $\mathcal{G}_{b,\underline{r},g} = \mathcal{G}_{b,\underline{r},g}(n, p_b, p_g; \geq k)$. The key lemma is the following.

Lemma 12. *Let $k \geq 3$ be fixed. Then a.e. $G_{b,\underline{r}} \in \mathcal{G}_{b,\underline{r}}$ is such that if F_0 is any $(k-2)$-factor of $K^{n,n}$, then $G_{b,\underline{r}} - E(F_0)$ is connected, expanding and has a 2-factor.* □

Again by Lemma 2, this immediately gives us the following result, which is the analogue of Lemma 9 that we shall need.

Lemma 13. *Let $k \geq 3$ be fixed. Then a.e. $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ is such that if F_0 is any $(k-2)$-factor of $K^{n,n}$, then $G_{b,\widehat{g},\underline{r}} - E(F_0)$ is connected, expanding and has a 2-factor.* □

Arguing in the same way as in the proof of Lemma 10, we conclude the following.

Lemma 14. *Let $k \geq 3$ be fixed. Then a.e. $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ is such that*

$$L(G_{b,g,\underline{r}} - E(F_0)) = L(G_{b,\widehat{g},\underline{r}} - E(F_0)),$$

for any $(k-2)$-factor F_0 of $G_{b,g,\underline{r}}$. □

We now state the main result of this section and give an outline of its proof.

Theorem 15. *Let $k \geq 1$ be fixed. Let τ_k, τ_{kF} and τ_{dkF} be the hitting times of having minimal degree at least k, having a k-factor, and having a decomposable k-factor,* *respectively. Then a.e. bipartite graph process $\widetilde{G}$ is such that*

$$\tau_k(\widetilde{G}) = \tau_{kF}(\widetilde{G}) = \tau_{dkF}(\widetilde{G}). \tag{9}$$

In particular, if $p = (\log n + (k-1)\log\log n + c_n)/n$ then

$$\lim_{n\to\infty} \mathsf{P}(G_p \in \mathcal{G}(K^{n,n}, p) \text{ has a decomposable } k\text{-factor}) = \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-2e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to +\infty. \end{cases}$$

Proof. As the proof does not need a substantial new idea, we shall only sketch it. We use induction on k. As observed above, we know that our result holds if $k = 1, 2$. Let us assume that $k \geq 3$ and that (9) holds for smaller values of k. Again by Lemma 1, it suffices to show that a.e. $G_{b,g,\underline{r}} \in \mathcal{G}_{b,g,\underline{r}}$ has a decomposable k-factor. Analogously to the proof of Theorem 11, let us denote by $\mathcal{H}^c$ the subset of $\mathcal{G}_{b,g,\underline{r}}$ consisting of the graphs that do not have a decomposable k-factor.

By applying the induction hypothesis, one can check that a.e. $G_{b,g,\underline{r}}$ has a decomposable $(k-2)$-factor. We now set

$$\mathcal{G}^{\mathrm{N}}_{b,g,\underline{r}} = \{G_{b,g,\underline{r}} : G_{b,g,\underline{r}} \text{ has a decomposable } (k-2)\text{-factor } F_0 \text{ such that } G_{b,\widehat{g},\underline{r}} - E(F_0) \text{ is connected, expanding, has a 2-factor, and furthermore } L(G_{b,g,\underline{r}} - E(F_0)) = L(G_{b,\widehat{g},\underline{r}} - E(F_0))\}.$$

As in the proof of Theorem 11, but this time using Lemmas 13 and 14, we are left with proving that

$$\mathbf{P}_{b,\underline{r},g}(\mathcal{G}^{\mathrm{exc}}_{b,g,\underline{r}}) = o(1), \tag{10}$$

where again $\mathcal{G}^{\mathrm{exc}}_{b,g,\underline{r}} = \mathcal{H}^c \cap \mathcal{G}^{\mathrm{N}}_{b,g,\underline{r}}$.

We now argue exactly as in the proof of Theorem 11; in particular, the following claim completes the proof.

Claim. For all $G_{b,\underline{r}} \in \mathcal{G}^{\mathrm{exc}}_{b,\underline{r}}$,

$$P_0 = P_0(G_{b,\underline{r}}) = \mathsf{P}(H : G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H \in \mathcal{G}^{\mathrm{exc}}_{b,g,\underline{r}}) = o(1). \tag{11}$$

Fix $G_{b,\underline{r}} \in \mathcal{G}^{\mathrm{exc}}_{b,\underline{r}}$, and assume that H is such that $G_{b,\underline{r},g} = G_{b,\underline{r}} \cup H \in \mathcal{G}^{\mathrm{exc}}_{b,g,\underline{r}}$. Let F_0 be a decomposable $(k-2)$-factor of $G_{b,\underline{r},g}$. Since $G_{b,\underline{r},g}$ does not have a decomposable k-factor, we know that $G_{b,\underline{r},g} - E(F_0)$, and hence $G_{b,\underline{r}} - E(F_0)$, is not Hamiltonian. We now claim that there are $\lfloor 2n/5 \rfloor^2 - 2n$ edges of $K^{n,n}$ that cannot appear in H. Indeed, let F be an almost 2-factor of $G_{b,\underline{r}} - E(F_0)$ that has a component C such that $L(G_{b,\underline{r}} - E(F_0)) = e(C)$. As $G_{b,\underline{r}} - E(F_0)$ is connected, we see that C is a strongly maximal path. Applying Lemma 7, we conclude that $G_{b,\underline{r}} - E(F_0)$ is either Hamiltonian, or else it is such that if any of the $\lfloor 2n/5 \rfloor^2$ edges $y_i z_i$ of Lemma 7, except the ones in F_0, is in H, then $L(G_{b,\underline{r},g} - E(F_0)) > L(G_{b,\underline{r}} - E(F_0))$. Since we are assuming that $G_{b,\underline{r}} - E(F_0)$ is not Hamiltonian, this implies that P_0 is at most

$$(1-p_g)^{\lfloor 2n/5 \rfloor^2 - 2n} \leq \exp\left\{ -\frac{1}{2}\left(\frac{\log n}{n^2 \log\log n}\right)\left[\left\lfloor \frac{2n}{5} \right\rfloor^2 - 2n\right]\right\} = o(1),$$

where we used that

$$p_g \geq \frac{1}{2}\left(\frac{\log n}{n^2 \log\log n}\right).$$

□

References

[1] Bollobás, B., *Graph Theory—An Introductory Course*, GTM, Springer-Verlag, New York, Heidelberg, Berlin, 1979, $x + 180$pp.

[2] Bollobás, B., The evolution of sparse graphs, *in* Graph Theory and Combinatorics, *Proc. Cambridge Combinatorial Conf. in honour of Paul Erdős* (Bollobás, B., ed.). Academic Press 1984, pp. 35–57.

[3] Bollobás, B., *Random Graphs*, Academic Press, London, 1985, $xvi + 447$pp.

[4] Erdős, P., Rényi, A., On the evolution of random graphs, *Bull. Inst. Int. Statist. Tokyo* **38** (1961), 343–347.

[5] Fenner, T.I., Frieze, A.M., On the existence of Hamilton cycles in a class of random graphs, *Discrete Math.* **45** (1983), 301–305.

[6] Frieze, A.M., Limit distribution for the existence of Hamilton cycles in a random bipartite graph, *Europ. J. Comb.* **6** (1985), 327-334.

[7] Komlós, J., Szemerédi, E., Limit distributions for the existence of Hamilton cycles in a random graph, *Discrete Math.* **43** (1983), 55–63.

[8] Korshunov, A.D., Solution of a problem of Erdős and Rényi on Hamilton cycles in non-oriented graphs, *Soviet Mat. Dokl.* **17** (1976), 760–764.

[9] Korshunov, A.D., A solution of a problem of P. Erdős and A. Rényi about Hamilton cycles in non-oriented graphs (in Russian), *Metody Diskr. Anal. Teoriy Upr. Syst., Sb. Trudov Novosibirsk* **31** (1977), 17–56.

[10] Pósa, L., Hamilton circuits in random graphs, *Discrete Math.* **14** (1976), 359–364.