The regular points of simple functions

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1 Introduction

In a recent article in this journal [7], Ioffe and Lewis discuss alternative notions of differentiability for non smooth functions f. They consider piecewise linear functions due to their relation with semi-algebraic functions, which have received much attention in the optimization literature recently. This article extends their work and describes the geometric topology behind their results. Therefore, reading [7] is an important preliminary step to understand the relevance of the present work for optimization.

An earlier analysis of regular points of non smooth functions was presented in Marston Morse's article [9]. We explain the relation of Morse's work with Ioffe and Lewis' for piecewise linear functions. We extend the two dimensional results of Ioffe and Lewis to dimension $n \leq 4$ and describe the subtle topological geometric question involved in going to higher dimensions. Theorem 1 in the next section presents a clean picture for piecewise linear functions $f : \mathbb{R}^n \to \mathbb{R}$ if $n \leq 4$. It shows that all the concepts discussed by Morse, Ioffe and Lewis are equivalent for such n. Unfortunately we hit a wall at $n = 5$ and things get messy. The analysis of $n = 5$ or greater leads us to the very hard problems of Schöenflies and Poincaré, which are unsolved in the piecewise linear context.

Some questions raised by our analysis are at least as hard as the Schöenflies problem but some are simpler and a natural continuation of this work would be to relate our results to the work of Edwards and Cannon regarding suspensions of homology spheres [3]. Unfortunately this study would take us too far from our expertise and, we believe, from the interest of the readers of this journal. Therefore, we leave this task to professional geometric topologists. In the next section we present five definitions of regularity, or non criticality, and a list of lemmas and theorems relating them among themselves and to geometric topological concepts. The exposition is dry, but we have not found a way to avoid the technicalities. The lemmas and theorems stated in section 2 are proved in the next two sections and in the appendix we correct a minor detail in Morse's work.

In resume, this article is one more step towards understanding regularity for piecewise linear and semi algebraic functions. We showed that for such functions the relations between various concepts of regularity can be expressed in geometric topological terms. The complete understanding of these relations is beyond what is currently known in geometric topology. In fact, by understanding regularity for piecewise linear functions in more depth we may even obtain new results in geometric topology.

We would like to thank Robert Daverman and Martin Scharlemann for helping us to learn geometric topology. They have already spared the reader from enough misconceptions due to our lack of experience with this field and certainly deserve no blame for the ones that may still have been left in this work.

2 Ordinarity, regularity and geometric topology

In this section we present Morse's and Ioffe and Lewis' definitions of regularity and explain that for piecewise linear functions they all can be understood in terms of the topology of some sets which are similar to spheres in many respects. Our main result is theorem 1, which shows the power of the geometric topological approach for $n \leq 4$. On the other hand, the restriction on n in this theorem also exposes the limitations of the current knowledge in geometric topology. In fact, neither we nor the experts in geometric topology know at this time if these sets are actually homeomorphic to spheres for $n > 4$.

Morse's work was motivated by the fact that given a smooth function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $z \in \mathbb{R}^n$ with $\nabla f(z) \neq 0$ there exists $\varepsilon > 0$ and a diffeomorphism [1] $h : \mathbb{B}^n(\varepsilon) \mapsto h(\mathbb{B}^n(\varepsilon)) \subset \mathbb{R}^n$ such that

$$h(0) = z \qquad \text{and} \qquad f(h(x)) = f(z) + x_n \quad \text{for } x \in \mathbb{B}^n(\varepsilon)\,. \tag{1}$$

Based on (1), Morse meant[2] to propose something like:

Definition 1 *Let M be a topological n manifold and let $f : M \mapsto \mathbb{R}$. A point $z \in M$ is topologically ordinary, or T-ordinary, if there exists $\varepsilon > 0$ and an homeomorphism $h : \mathbb{B}^n(\varepsilon) \mapsto h(\mathbb{B}^n(\varepsilon)) \subset M$ as in (1). If M and h are piecewise linear then we say that z is PL-ordinary.*

For a smooth $f : \mathbb{R}^n \mapsto \mathbb{R}$ the existence of a diffeomorphism h as in (1) can be proved by applying the box flow theorem to the vector field $v(x) = -\nabla f(x)$. This led Morse [9] [10] to propose

Definition 2 *Let X be a metric space and $f : X \mapsto \mathbb{R}$. A point $z \in X$ is downards-regular if there exists $\sigma > 0$ and a continuous function [3] $\Phi : \mathbb{B}(z,\sigma) \times [0,\sigma] \mapsto X$ such that $\Phi(x,0) = x$ and $f(\Phi(x,t)) < f(x)$.*

Similarly, Ioffe and Lewis state

Definition 3 *Let X be a metric space and $f : X \mapsto \mathbb{R}$. A point $z \in X$ is Morse-regular if there exists $\sigma > 0$ and a continuous function $\Phi : \mathbb{B}(z,\sigma) \times [0,\sigma] \mapsto X$ such that $\Phi(x,0) = x$ and $f(\Phi(x,t)) \leq f(x) - \sigma t$.*

Definition 3 is a bit unsatisfactory because it allows us to "speed up" the deformation at will, i.e., it does not control how fast we move along the trajectories $T(x) = \{\Phi(x,t),\ t \in [0,\sigma]\}$. This lack of control yields this lemma:

Lemma 1 *In locally compact spaces Morse-regularity is equivalent to downards-regularity.*

A stronger version of Morse-regularity follows from the concept of "weak slope" introduced in [5] and [6]:

Definition 4 *Let X be metric space and $f : X \mapsto \mathbb{R}$. A point $z \in X$ is deformationally regular from below if there exists $\sigma > 0$ and a continuous function $\Phi : \mathbb{B}(z,\sigma) \times [0,\sigma] \mapsto X$ such that*

$$\operatorname{dist}(\Phi(x,t),x) \leq t/\sigma \qquad \text{and} \qquad f(\Phi(x,t)) \leq f(x) - \sigma t. \tag{2}$$

If z is T-ordinary and h is the homeomorphism in equation (1), then, for $e_n = (0,..,0,1)^t \in \mathbb{R}^n$, the deformation

$$\Phi(x,t) = h\big(h^{-1}(x) - t e_n\big) \tag{3}$$

shows that z is Morse-regular. Moreover, if h is Lipschitz then the function Φ in equation (3) shows that z is also deformationally regular. Therefore, ordinarity is an stronger requirement than regularity.

The point $z = 0 \in \mathbb{R}$ is T-ordinary for $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x^3$, because $h(x) = \sqrt[3]{x}$ satisfies the condition (1). However, 0 is not PL-ordinary for f, because the only homeomorphism h which satisfies equation (1) is the h above, and it is not piecewise linear. Therefore, even for polynomials, PL-ordinarity is an strictly stronger requirement than T-ordinarity. The same example shows that deformational regularity is strictly stronger than Morse-regularity for polynomials. However, for piecewise linear functions and $1 \leq n \leq 4$ we prove

Theorem 1 *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be piecewise linear. If $n \leq 4$ and z is downards-regular for f then z is PL-ordinary.*

The extension of 1 to $n > 4$ is related to the classic Schöenflies problem [1] [8]. This problem regards subsets S of the sphere [4] $\mathbb{S}^m$ which are homeomorphic to $\mathbb{S}^{m-1}$. Schöenflies asked whether there exists a homeomorphism $h : \mathbb{S}^m \mapsto \mathbb{S}^m$ such that $h(S) = \mathbb{E}^{m-1} = \{x \in \mathbb{S}^m \text{ with } x_{m+1} = 0\}$, i.e., whether S can be flattened to the equator $\mathbb{E}^{m-1}$. Since the 1920's it is known that the answer is negative in general. In the 1960's a positive answer was provided by Brown [1] [2] under the condition of local of flatness of S.

[1] $\mathbb{B}^n(r) = \{x \in \mathbb{R}^n \text{ with } \|x\|_\infty \leq r\}$.

[2] Appendix A explains why, strictly speaking, Morse's definition in [9] is flawed.

[3] In a metric space X, $\mathbb{B}(x,r) = \{y \in X \text{ with } \operatorname{dist}(y,x) \leq r\}$. If $X = \mathbb{R}^n$ we define $\operatorname{dist}(x,y) = \|x - y\|_\infty$.

[4] $\mathbb{S}^m = \big\{x \in \mathbb{R}^{m+1} \text{ with } \|x\|_\infty = 1\big\}$.

Definition 5 *A subset S of a topological n-manifold M is locally flat if for every $x \in S$ there exists a neighborhood V of x and a homeomorphism $h : V \mapsto h(V) \subset \mathbb{R}^n$ such that $h(S \cap V) \subset \mathbb{R}^{n-1} \times \{0\}$.*

Unfortunately, Brown's answer is not enough for us, because we need a piecewise linear h. The version of the Schöenflies problem which is relevant to our discussion is conjecture 1 below. This conjecture is known to be correct for $m \neq 4$ but is status for $m = 4$ is unknown [11]. In fact, deciding whether conjecture 1 is correct for $m = 4$ is considered to be a very hard task by the geometric topology community.

Conjecture 1 *Let S be a subcomplex of a rectlinear triangulation of $\mathbb{S}^m$. If S is piecewise linearly homeomorphic to $\mathbb{S}^{m-1}$ and locally flat then there exists a piecewise linear homeomorphism $h : \mathbb{S}^m \mapsto \mathbb{S}^m$ such that $h(S) = \mathbb{E}^{m-1}$.*

The next theorem relates conjecture 1 to our discussion.

Theorem 2 *T-ordinarity implies PL-ordinarity for all piecewise linear functions $f : \mathbb{R}^5 \mapsto \mathbb{R}$ if and only if conjecture 1 is correct for $m = 4$.*

Besides $\mathbb{S}^n$, the spheres relevant to our discussion are

$$\mathbb{S}(z,r) = \{x \in \mathbb{R}^n \text{ with } \|x - z\|_\infty = r\}$$

and the sets S are

$$S_r = f^{-1}(f(z)) \cap \mathbb{S}(z,r), \tag{4}$$

for r small enough so that $g(x) = f(x) - f(z)$ is homogeneous in $\mathbb{B}(z,r)$. When z is T-ordinary, as in definition 1, we assume that $S_r \subset h(\mathbb{B}^n(\varepsilon))$. When z is downards-regular, as in definition 2, we also assume that $S_r \subset \mathbb{S}(z,\sigma)$. This smallness of r will be implicit whenever we mention S_r. The next lemmas show that our S_r's are homeomorphic to spheres, or almost, and illustrate their importance.

Lemma 2 *Suppose $n \geq 2$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear. If z is downards-regular then S_r has the same homology as $\mathbb{S}^{n-2}$.*

Lemma 3 *Suppose $n \geq 4$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear. If z is T-ordinary then S_r is simply connected.*

Lemma 4 *Suppose $n \geq 2$, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear and $z \in \mathbb{R}^n$. If S_r is locally flat and piecewise linearly homeomorphic to $\mathbb{S}^{n-2}$ then z is T-ordinary.*

Lemma 5 *Suppose $n \geq 2$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear. A point $z \in \mathbb{R}^n$ is PL-ordinarity for f if and only if there exists a piecewise linear homeomorphism $h : \mathbb{S}(z,r) \mapsto \mathbb{S}^{n-1}$ such that $h(S_r) = \mathbb{E}^{n-2}$.*

The concept of homology in lemma 2 is described formally and in detail in [4]. Informally, a set has the same homology as $\mathbb{S}^m$ if it passes various tests to be homeomorphic to $\mathbb{S}^m$, but may fail two of them: it may not be a manifold or, if $m \geq 2$, it may not be simply connected. If S has the same homology as $\mathbb{S}^m$, is a manifold and, when $m \geq 2$, is simply connected, then the famous Poincaré theorem implies that S is homeomorphic to $\mathbb{S}^m$.

Lemma 2 shows that in order to extend theorem 1 we must prove that the S_r are simply connected manifolds, explicitly or implicitly. The relevance of the simply connectedness of sets S_r is illustrated by the next theorem.

Theorem 3 *Consider a piecewise linear $f : \mathbb{R}^5 \mapsto \mathbb{R}$ and a downards-regular point $z \in \mathbb{R}^5$. If the set S_r is simply connected then z is T-ordinary.*

This theorem leads naturally to

Question 1 *Suppose $f : \mathbb{R}^m \mapsto \mathbb{R}$ be piecewise linear and z is downards-regular. Is the set S_r simply connected?*

Question 2 *Suppose $f : \mathbb{R}^m \mapsto \mathbb{R}$ be piecewise linear and z is deformationally regular. Is S_r simply connected?*

Affirmative answers to these questions would allow us to use these general results:

Theorem 4 *If theorem 1 is correct for all piecewise linear functions $f : \mathbb{R}^6 \mapsto \mathbb{R}$ and question 1 has a positive answer for $m = 7, 8, \dots, n$ then theorem 1 holds for every piecewise linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$.*

Theorem 5 *If deformational regularity implies PL-ordinarity for every piecewise linear function $f : \mathbb{R}^6 \mapsto \mathbb{R}$ and question 2 has a positive answer for $m = 7, 8, \dots, n$ then deformational regularity implies PL-ordinarity for every piecewise linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$.*

Theorem 6 *If the piecewise linear Poincaré conjecture is correct for $m = 4$ and question 1 has an affirmative answer for $m = 5, 6, \dots, n$ then theorem 1 holds for every piecewise linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$.*

Theorem 7 *If the piecewise linear Poincaré conjecture is true for $m = 4$ and question 2 has an affirmative answer for $m = 5, 6, \dots, n$ then deformational regularity implies PL-ordinarity for piecewise linear functions $f : \mathbb{R}^n \mapsto \mathbb{R}$.*

3 Proofs of the theorems

Here we prove the theorems in the introduction. We use the next lemmas, which are proved in the next section.

Lemma 6 *Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear and homogeneous and $z = 0$ is downards-regular for f. If $|\varepsilon|$ is small then the set $L_\varepsilon = \left\{ x \in \mathbb{S}^{n-1} \text{ with } f(x) \leq \varepsilon \right\}$ is contractible.*

Lemma 7 *Suppose $n \geq 2$, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear, $z \in \mathbb{R}^n$ and $w \in S_r$, where S_r is the set in equation (4). Consider the hyperplane H which contains w and is orthogonal to $w - z$. If z is downards-regular for f then w is downards-regular for the restriction of f to H. If z is deformationally regular for f then w is deformationally regular for the restriction of f to H.*

Corollary 1 *Suppose $n \geq 2$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is piecewise linear. If theorem 1 holds for all piecewise linear functions $g : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ and $z \in \mathbb{R}^n$ is downards-regular for f then S_r is a locally flat $(n-2)$ piecewise linear sub manifold of $\mathbb{S}(z, r)$. Analogously, if deformational regularity implies PL-ordinarity for every piecewise linear function $g : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ and $z \in \mathbb{R}^n$ is deformationally regular then S_r is a locally flat $(n-2)$ piecewise linear sub manifold of $\mathbb{S}(z, r)$.*

Proof of theorem 1. Let us assume that $z = 0$ and f is homogeneous. The proof is by induction in n, starting with $n = 1$. In this case f must be strictly monotone, because 0 is not a local minimizer and lemma 6 implies that 0 is not a local maximizer either (otherwise L_0 would be equal to $\mathbb{S}^0$, which is not contractible). Therefore, there exist a, b with the same sign such that $f(x) = ax$ for $x \leq 0$ and $f(x) = bx$ for $x \geq 0$. If $a, b > 0$ then the homeomorphism $h : \mathbb{R} \mapsto \mathbb{R}$ given by $h(x) = x/a$ for $x < 0$ and $h(x) = x/b$ for $x \geq 0$ shows that 0 is PL-ordinary. If $a, b < 0$ then $h : \mathbb{R} \mapsto \mathbb{R}$ given by $h(x) = x/b$ for $x < 0$ and $h(x) = x/a$ for $x \geq 0$ shows that 0 is PL-ordinary.

For $2 \leq n \leq 4$, let us assume that theorem 1 holds for $(n-1)$ and prove it for n. Applying corollary 1 to $(n-1)$ we conclude that S_1 is a locally flat piecewise linear $(n-2)$ sub manifold of $\mathbb{S}^{n-1}$. Lemma 2 shows that S_1 has the same homology as $\mathbb{S}^{n-2}$ and, for $n = 4$, lemma 3 show that S_1 is simply connected. These observations allows to use the piecewise linear Poincaré theorem to conclude that S_1 is piecewise linearly homeomorphic to $\mathbb{S}^{n-2}$. Since conjecture 1 is valid for $n - 1 \leq 3$ we conclude from lemma 5 that z is PL-ordinary. $\square$

Proof of theorem 2. Suppose z is T-ordinary. Lemma 3 implies that S_r is simply connected and corollary 1 implies that S_r is a piecewise linear manifold. Therefore, the piecewise linear Poincaré theorem for $m = 3$ implies that S_r is piecewise linearly homeomorphic to $\mathbb{S}^3$. Corollary 1 also shows that S_r is locally flat and if conjecture 1 is correct for $m = 4$ then S can be flattened by a piecewise linear homeomorphism and lemma 5 implies that z is PL-ordinary. In resume, if conjecture 1 is correct for $m = 4$ then T-ordinarity implies PL-ordinarity for $n = 5$.

We now show that if T-ordinarity implies PL-ordinarity for $n = 5$ then conjecture 1 is correct for $m = 4$. Suppose $S \subset \mathbb{S}^4$ is a sub complex of a rectilinear a triangulation T of $\mathbb{S}^4$ such that S is piecewise linearly homeomorphic to $\mathbb{S}^3$ and locally flat. Consider unique the piecewise linear homogeneous function $f : \mathbb{R}^5 \mapsto \mathbb{R}$ such that $f(x) = 0$ for $x \in S$ and $f(v) = 1$ for the vertices of T which are not in S. Applying lemma 4 to f we conclude that $z = 0$ is T-ordinary and, under the assumption that T-ordinarity implies PL-ordinarity, we conclude that $z = 0$ is PL-ordinary.

Lemma 5 shows that S can be flattened by a piecewise linear homeomorphism. This is the statement of conjecture 1 for $m = 4$ and we are done. $\square$

Proof of theorem 3. Take $n = 5$. Theorem 1 and corollary 1 show that S_r is a piecewise linear $(n-2)$ manifold locally flat in $\mathbb{S}^{n-1}$. Lemma 2 shows that S_r has the same homology as $\mathbb{S}^{n-2}$ and Lemma 3 show that S_r is simply connected. Therefore, the piecewise linear Poincaré's theorem for $m = n-2$ implies that S_r is piecewise linearly homeomorphic to $\mathbb{S}^{n-2}$ and lemma 4 yields the thesis of theorem 3. $\square$

Proofs of theorems 4-7. It is known [11] that if the piecewise linear Poincaré conjecture is correct for $m = 4$ then conjecture 1 also hold for $m = 4$. Therefore, if the piecewise linear Poincaré's conjecture is correct for $m = 4$ then the proof of theorem 3 provides an inductive argument to prove theorems 6 and 7. Theorems 4 and 5 can be proved in the same way because the piecewise linear Poincaré conjecture and conjecture 1 are valid for $m > 4$. $\square$

3.1 Proofs of the lemmas and corollary

In this section we prove the lemmas and corollaries stated in the introduction and in the previous section.

Proof of lemma 1. We only need to show that downards-regularity implies Morse-regularity. Suppose the neighborhood V in definition 2 is compact. To prove lemma 1 it suffices to show that there exists $\tau > 0$ and a continuous function $h : [0,\tau] \mapsto [0,\sigma]$ such that $f(\Phi(x,h(t))) \leq f(x) - t$. A first attempt to build h would be to consider the inverse of $g : [0,\sigma] \mapsto \mathbb{R}$ given by

$$g(t) = \inf\{\, f(x) - f(\Phi(x,s)),\ \ x \in V,\ s \in [t,\sigma]\ \}.$$

The function g is continuous, increases monotonically, $g(t) > 0$ for $t > 0$ by the compactness of V and $g(0) = 0$ and $g(t) \leq f(x) - \Phi(x,t)$. Unfortunately, g's inverse may not exist because it may be constant in some intervals. To fix g, define

$$t_n = \sup\{t \in [0,\sigma] \text{ with } g(t) \leq g(\sigma)/n\}.$$

The sequence $\{t_n, n \in \mathbb{N}\}$ decreases monotonically to 0 and we can define $q : [0,\sigma] \mapsto [0, g(t_2)]$ by $q(0) = 0$ and

$$q(t) = g(t_{k+2}) + \frac{t - t_{k+1}}{t_k - t_{k+1}} \left(g(t_{k+1}) - g(t_{k+2})\right).$$

The function q is continuous, strictly increasing, $q(0) = 0$ and $q(t) \leq g(t)$ for $t \in [0,\sigma]$. Thus, its inverse exists, is continuous and satisfies $q(s) \leq g(s) \leq f(x) - f(\Phi(x,s))$. Replacing s by $q^{-1}(t)$ for $t \in [0, g(t_2)]$ we obtain $f\big(\Phi(x, q^{-1}(t))\big) \leq f(x) - t$. $\square$

Proof of lemma 2. Lemma 2 follows from lemma 6 and the next one, which we prove at the end of this section:

Lemma 8 *Suppose $f : \mathbb{S}^m \mapsto \mathbb{R}$ is piecewise linear and $z, w \in \mathbb{S}^m$ are such that $f(z) = 0$ and $f(w) < 0$. If for all ε with $|\varepsilon|$ small $L_\varepsilon = \{x \in \mathbb{S}^m \text{ with } f(x) \leq \varepsilon\}$ is contractible then $f^{-1}(0)$ has the same homology as $\mathbb{S}^{m-1}$.* $\square$

Proof of lemma 3. Assume that $z = 0$ and $f(z) = 0$ and consider h and ε as in equation (1). Since h is an homeomorphism there exists $\delta > 0$ such that $\mathbb{B}^n(\delta) \subset h(\mathbb{B}^n(\varepsilon))$. Let us denote by A the interior of $\mathbb{B}^n(\delta)$ and define C as $A \cap f^{-1}(0)$. Equation (1) shows that h^{-1} maps C into a subset $D = h^{-1}(C)$ of the $(n-1)$ dimensional hyperplane $H = \{x \in \mathbb{R}^n \text{ with } x_n = 0\}$. Since h is a homeomorphism and A is open, $D = H \cap h^{-1}(A)$ is an open neighborhood of 0 in H. By f's homogeneity, if $y \in C$ and $\lambda \in [0,1]$ then $\lambda y \in C$. This shows that C is contractible. As a consequence $D = h^{-1}(C)$ is contractible. Since D is an open subset set of H and H has dimension $n - 1 \geq 3$, we have that $D - \{0\}$ is simply connected. As a consequence, $C - \{0\} = h^{-1}(D - \{0\})$ is also simply connected. Notice that $C - \{0\}$ is homeomorphic to $Z \times (0,\delta)$ and $\pi_1(Z) = \pi_1(Z) \times \{0\} = \pi_1(C - \{0\}) = \{0\}$. Therefore, Z is simply connected. $\square$

Proof of lemma 4. Let us assume that $z = 0$, f is homogeneous and $r = 1$. In particular, $S_r = S_1$ and $\mathbb{S}(z,r) = \mathbb{S}^{n-1}$. The weak piecewise linear Schöenflies theorem in page 47 of [11] implies that there exists an homeomorphism $g : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$ and $s, z \in \mathbb{S}^{n-1} - S$ such that $g(S) = \mathbb{E}^{n-2}$ and g is piecewise linear in $\mathbb{S}^{n-1} - \{s,z\}$, in the sense that there exist locally finite triangulations T of $\mathbb{S}^{n-1} - \{s,z\}$ and U of $\mathbb{S}^{n-1} - \{g(s), g(z)\}$ such that g is affine in each simplex of $\sigma \in T$ and $g(\sigma) \in U$. By refining T and U if necessary, we can also assume that $q : \mathbb{S}^{n-1} - \{g(s), g(z)\} \mapsto \mathbb{R}$ given by $q(x) = f\big(g^{-1}(x)\big)$ is linear in each simplex of U. Let $h : \mathbb{B}^n \mapsto \mathbb{R}^n$ be the unique homogeneous function such that $h(g(s)) = sg(s)_n/f(s)$, $h(g(z)) = zg(z)_n/f(z)$, g is piecewise linear in $|U|$ and if v is a vertex of U then $h(v) = v$ if $v \in \mathbb{E}^{n-2}$ and $h(v) = v_n g^{-1}(v)/f\left(g^{-1}(v)\right)$ if $v \notin \mathbb{E}^{n-2}$. The continuity of g at s,z implies that h is continuous. The same argument used in lemma 5 shows that the restriction of h to $|U|$ is an homeomorphism and it follows that h is an homeomorphism in $\mathbb{S}^{n-1} = |U| \cup \{g(s), g(z)\}$. Finally, by the construction of h and the homogeneity of f we have that $f(h(x)) = x_n$ if x is a vertex of U or $x \in \{g(s), g(z)\}$. Since q is piecewise linear this implies that $f(h(x)) = x_n$ for all $x \in \mathbb{S}^{n-1}$. By homogeneity, h satisfies equation (1) for all $x \in \mathbb{B}^n$ and $z = 0$ is T-ordinary. $\square$

Proof of lemma 5. Given a simplicial complex T and a sub complex $\sigma \subset T$, the link of σ in T is

$$\mathrm{lk}(\sigma, T) = \{\tau \in T \text{ such that } \tau \cap \sigma = \emptyset \text{ and } \mu \cup \sigma \in T \text{ for some } \mu \in S\} \tag{5}$$

and the convex hull of σ is denoted by $|\sigma|$. For $v \in \mathbb{R}^n - \{0\}$, we define $\overline{v} = v/\|v\|_\infty$ and if $\sigma = \left\{v^1, \ldots, v^k\right\} \subset \mathbb{R}^n - \{0\}$ then we write $\overline{\sigma} = \{\overline{v}^1, \ldots, \overline{v}^k\}$. We assume that $z = 0$, f is homogeneous and T is rectilinear simplicial complex with $|T| = \mathbb{S}^{n-1}$ such that f is linear in each simplex $\sigma \in T$. We also assume that f does not change signs in the edges of T, ie. if $\{a, b\} \in T$ then $f(a)$ and $f(b)$ do not have opposite signs.

We now show that if $z = 0$ is PL-ordinary then $S_1 = \mathbb{S}^{n-1} \cap f^{-1}(0)$ can be piecewise linearly flattened to $\mathbb{E}^{n-2}$. Let ε and h be as in definition 1. By reducing ε if necessary, we may assume that there exists a triangulation T_h of $\mathbb{S}^{n-1}(\varepsilon)$ such that h is linear in $|\{0\} \cup \sigma|$ for each $\sigma \in T_h$ and $\overline{h(\sigma)}$ is contained in one face of $\mathbb{S}^{n-1}$. Therefore, for each $(n-1)$ dimensional simplex $\sigma = \left\{v^1, \ldots, v^n\right\} \in T_h$ there exists a nonsingular matrix A_σ such that $h(x) = A_\sigma x$ for $x \in |\{0\} \cup \sigma|$ and a vector c_σ such that $f(y) = c_\sigma^t y$ for $y \in h(\sigma)$. Thus, equation (1) leads to

$$f(h(x)) = c_\sigma^t A_\sigma x = x_n \tag{6}$$

for $x \in |\{0\} \cup \sigma|$. Since the interior of $|\{0\} \cup \sigma|$ is not empty equation (6) implies that $c_\sigma^t A_\sigma = e_n$. Since f does not change signs across the edges of T, we have that either $v_n^i \geq 0$ for $i = 1, \ldots, n$ or $v_n^i \leq 0$ for $i = 1, \ldots, n$. For $\sigma \in T_h$ consider $g_\sigma : |\overline{\sigma}| \mapsto \mathbb{R}^n$ given by

$$g_\sigma(x) = A_\sigma \overline{V}_\sigma \overline{D}_\sigma \overline{V}_\sigma^{-1} x,$$

where $\overline{V}$ is the matrix with kth column equals to $\overline{v}^k$ and $\overline{D}^\sigma$ is the diagonal matrix with kth diagonal entry equals to $d_{kk}^\sigma = 1/\|A_\sigma \overline{v}_k\|_\infty$. Notice that

$$g_\sigma(\overline{v}^k) = A_\sigma \overline{V}_\sigma \overline{D}^\sigma \overline{V}_\sigma^{-1} v^k = A_\sigma \overline{V}_\sigma \overline{D}^\sigma e_k = \frac{1}{\left\|A_\sigma \overline{v}^k\right\|_\infty} A_\sigma \overline{V}_\sigma e_k = \frac{1}{\left\|A_\sigma \overline{v}^k\right\|_\infty} A_\sigma \overline{v}^k \in \mathbb{S}^{n-1}. \tag{7}$$

By convexity and the fact $\overline{h(\sigma)}$ is contained in one face of $\mathbb{S}^{n-1}$ we conclude that $g_\sigma(\overline{\sigma}) \subset \mathbb{S}^{n-1}$. Using that $x \in \overline{\sigma}$ if and only if $x = \overline{V}_\sigma y(x)$ for $y(x)$ in the unit simplex of $\mathbb{R}^n$ we get

$$g_\sigma(x) = A_\sigma \overline{V}_\sigma \overline{D}^\sigma y(x).$$

Now, if $\delta > 0$ is small enough then $\delta \overline{V}_\sigma \overline{D}^\sigma y(x) \in |\{0\} \cup \sigma|$ and, as a consequence,

$$\delta g_\sigma(x) \in h(|\{0\} \cup \sigma|). \tag{8}$$

Therefore,

$$f(g_\sigma(x)) = \frac{1}{\delta} f(\delta g_\sigma(x)) = \frac{1}{\delta} \delta c_\sigma^t A_\sigma \overline{V}_\sigma \overline{D}^\sigma y(x) = e_n^t \overline{V}_\sigma \overline{D}^\sigma y(x) = \sum_{k=1}^n \overline{v}_n^k \overline{d}_{kk} y_k(x).$$

Since $\overline{v}_n^k$ have the same sign for all k and $\overline{d}_{kk} > 0$ we have that $f(g_\sigma(x)) = 0$ if and only if $y_k(x) = 0$ for all k such that $\overline{v}_n^k \neq 0$. Therefore, if $x \in |\overline{\sigma}|$ then

$$f(g_\sigma(x)) = 0 \iff x = \overline{V}_\sigma y(x) \in \mathbb{E}^{n-2}. \tag{9}$$

To prove that the $\{g_\sigma, \sigma \in T\}$ lead to a well defined piecewise linear map $g : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$ given by

$$g(x) = g_\sigma(x) \text{ for some } \sigma \text{ with } x \in |\overline{\sigma}|$$

it suffices to show that g_σ and g_τ coincide in $\overline{\sigma}\cap\overline{\tau}$, because they are linear in $|\overline{\sigma}\cap\overline{\tau}|$. But this is a direct consequence of the continuity of h. In fact, consider a vertex $v\in\sigma\cap\tau$. Since h is continuous at v we have that $A_\sigma v = A_\tau v$. This leads to $A_\sigma\overline{v} = A_\tau\overline{v}$ and equation (7) shows that $g_\sigma(\overline{v}) = g_\tau(\overline{v})$. Finally, to prove that g is a homeomorphism notice that each g_σ is injective and equation (8) and the injectivity of h imply that g is globally injective. Brouwer's invariance of domain theorem implies that $g\left(\mathbb{S}^{n-1}\right) = \mathbb{S}^{n-1}$. As a consequence, g is surjective and (9) leads to $g\left(\mathbb{E}^{n-2}\right) = \mathbb{S}^{n-1}\cap f^{-1}(0)$. Therefore, g^{-1} is a piecewise linear homeomorphism that flattens S_1 and we finished the first part of this proof.

Let us now show that if there exists a piecewise linear homeomorphism $g:\mathbb{S}^{n-1}\mapsto\mathbb{S}^{n-1}$ with $g\left(\mathbb{E}^{n-2}\right) = S_1$ then $z=0$ is PL-ordinary. Let T_S be a triangulation of $\mathbb{S}^{n-1}$ such that g is linear in each simplex $\sigma\in T_S$ and such that $g(\sigma)$ is contained in some simplex of T. As before, for each $(n-1)$ dimensional simplex $\sigma\in T_S$ there exists a non singular matrix A_σ and a vector c_σ such that, for $x\in|\sigma|$,

$$g(x) = A_\sigma x \qquad \text{and} \qquad f(g(x)) = c_\sigma^t A_\sigma x.$$

By hypothesis, $f(g(x)) = 0$ if and only if $x\in\mathbb{E}^{n-2}$. Therefore, $c_\sigma^t A_\sigma x = 0$ if and only $x\in\mathbb{E}^{n-2}$. Now, for each $\sigma\in T_S$ define $h_\sigma:|\{0\}\cup\sigma|\mapsto\mathbb{R}^n$ by

$$h_\sigma(x) = A_\sigma V_\sigma D^\sigma V_\sigma^{-1}x$$

where V_σ is the matrix with kth column equals to v^k and D^σ is the diagonal matrix with diagonal entry $d_{kk}^\sigma = 1$ if $v^k\in\mathbb{E}^{n-2}$ and $d_{kk}^\sigma = v_n^k/(c_\sigma^t A_\sigma v^k)$ if $v^k\notin\mathbb{E}^{n-2}$. It follows that if

$$\varepsilon = \min\left\{\frac{\delta\left\|v^k\right\|_\infty}{\left\|A_\sigma V_\sigma D^\sigma V_\sigma^{-1}v^k\right\|_\infty},\ \text{for } \sigma\in T_S \text{ and } v^k\in\sigma\right\},$$

and $v^k\in\sigma$ then $w^k = \varepsilon v^k/\left\|v^k\right\|_\infty$ is such that $h_\sigma(w^k)\in\mathbb{B}^n(\delta)\subset\bigcup_{\sigma\in T_S}|\{0\}\cup\sigma|$ and

$$f\Big(h_\sigma(\varepsilon w^k)\Big) = \frac{\varepsilon}{\|v^k\|_\infty}c_\sigma^t A_\sigma V_\sigma D^\sigma V_\sigma^{-1}v^k = \frac{\varepsilon v_n^k}{\|v^k\|_\infty} = w_n^k. \tag{10}$$

Since $f(h_\sigma(x))$ is linear and the w^k span $|\{0\}\cup\sigma|$ it follows that $f(h_\sigma(x)) = x_n$ for all $x\in|\{0\}\cup\sigma|\cap\mathbb{B}^n(\varepsilon)$. Finally, the same arguments used in the first part of this proof shows that

$$h(x) = h_\sigma(x) \text{ for some } \sigma\in T_S \text{ with } x\in|\{0\}\cup\sigma|$$

yields a well defined homeomorphism $h:\mathbb{B}^n(\varepsilon)\mapsto h(\mathbb{B}^n(\varepsilon))$. Equation (10) yields equation (1). □

Proof of lemma 6. Let $\sigma>0$ and $\Phi:\mathbb{B}^n(\sigma)\times[0,\sigma]\mapsto\mathbb{R}^n$ be such that $\Phi(x,0)=x$ and $f(\Phi(x,t))<f(x)$ and define $h:L_0\mapsto\mathbb{R}^n$ by

$$h(x,t) = \frac{\Phi((\sigma-t)x,t)}{\|\Phi((\sigma-t)x,t)\|_\infty}. \tag{11}$$

Notice that if $x\in L_0$ then $\Phi(x,0)=x\neq 0$ and if $t\in(0,\sigma]$ then

$$f(\Phi((\sigma-t)x,t)) < f((\sigma-t)x)\leq 0. \tag{12}$$

Therefore, Φ is well defined and continuous and $\Phi(L_0,[0,\sigma])\subset L_0$. Moreover, $w = h(x,\sigma) = \Phi(0,\sigma)/\|\Phi(0,\sigma)\|_\infty$ does not depend on x. Therefore, h contracts L_0 to w. For $\varepsilon>0$ small L_ε deformation retracts to L_0. Thus, L_ε is contractible for $\varepsilon\geq 0$ small. To handle $\varepsilon<0$ we take $\delta<0$ such that if $\varepsilon\in[\delta,0)$ then L_ε deformation retracts into L_δ. Equation (12) and $f(\Phi(x,0)) = f(x)\leq\delta$ for $x\in L_\delta$ imply that $\mu = \sup_{x\in L_\delta, t\in[0,\sigma]} f(h(x,t))<0$. We can contract L_ε for $\varepsilon\in[\mu,0)$ by first deformation retracting L_ε in L_δ and then using h to contract L_δ through L_μ. □

Proof of lemma 7. Let us assume that $z=0$, f is homogeneous, $r=1$ and w is a vertex of a triangulation T of $\mathbb{S}^{n-1}$. Let $\mathrm{lk}(w,T)$ be the link of w in T (see equation (5)). For each $\sigma\in\mathrm{lk}(w,T)$ let us denote by C_σ the cone

$$C_\sigma = \{\alpha x,\ \text{with } \alpha\in[0,\infty) \text{ and } x\in|\{w\}\cup\sigma|\}.$$

By refining T if necessary, we can assume that for each $\sigma \in T$ there exists $a_\sigma \in \mathbb{R}^n$ such that $x \in C_\sigma \Rightarrow f(x) = a_\sigma^t x$. In particular, if $\sigma \in \mathrm{lk}(w,T)$ then $0 = f(w) = a_\sigma^t w$. The ortogonal projection of x in H is

$$P(x) = x + \lambda(x)\, w \qquad \text{for} \qquad \lambda(x) = \frac{w^t(w-x)}{w^t w}. \tag{13}$$

If $\sigma = \{v^1, \ldots, v^k\} \in \mathrm{lk}(w,T)$ then $x \in C_\sigma$ if and only if $x = \beta_0(x)\, w + \sum_{i=1}^k \beta_i(x)\, v^i$ for $\beta_0(x), \ldots, \beta_i(x) \geq 0$ and

$$\lambda(x) = 1 - \beta_0(x) - \sum_{i=1}^k \beta_i(x)\, \frac{w^t v^i}{w^t w} \quad \text{and} \quad P(x) = \theta(x)\, w + \sum_{i=1}^k \beta_i(x)\, v^i \quad \text{for} \quad \theta(x) = 1 - \sum_{i=1}^k \beta_i(x)\, \frac{w^t v^i}{w^t w}.$$

Therefore, if $x \in C_\sigma$ and $\theta(x) \geq 0$ then $P(x) \in C_\sigma$ and

$$f(P(x)) = a_\sigma^t P(x) = a_\sigma^t\,(x + \lambda(x)\, w) = a_\sigma^t x + \lambda(x)\, a_\sigma^t w = a_\sigma^t x = f(x).$$

Applying this argument to all $\sigma \in \mathrm{lk}(w,T)$ we conclude that there exists $\delta > 0$ such that if $\|x - w\|_\infty \leq \delta$ then $f(P(x)) = f(x)$. As a consequence, if Φ is the deformation which assures the downards-regularity of w for f in definition 2 then $\tilde{\Phi}(x,t) = P(\Phi(x,t))$ certifies the downards-regularity of w for the restriction of f to H. The same argument applies to deformational regularity, because $\|Px - Py\|_\infty \leq \|P\|_\infty \|x - y\|_\infty$. $\square$

Proof of corollary 1. We prove the part of corollary 1 regarding downards-regularity. The same argument applies to deformational regularity. We assume that $z = 0$ is downards-regular, f is piecewise linear and homogeneous and $r = 1$. Given $w \in S_1 = \mathbb{S}^{n-1} \cap f^{-1}(0)$, let H_w be the $(n-1)$ dimensional hyperplane H_w which contains w and is ortogonal to w and let us call by f_w the restriction of f to H_w. Lemma 7 shows that w is downards-regular for f_w. By the hypothesis, theorem 1 applies to f_w and w is PL-ordinary for f_w. Let then $\varepsilon > 0$ and the piecewise linear homeomorphism $h_w : \mathbb{B}^{n-1}(\varepsilon) \mapsto H_w$ be as in definition 1 for w and f_w:

$$h_w(0) = w \qquad \text{and} \qquad f_w(h_w(x)) = x_{n-1}.$$

Since $w \in \mathbb{S}^{n-1}$ we have $|w_i| = 1$ for some i and there exist $\gamma_w \in (0,1)$ such that $|w_j| \leq \gamma_w$ if $|w_j| \neq 1$. By reducing ε, we may assume that if $w_j \neq 0$ then w_j and $h_w(x)_j$ have the same sign and

$$1/2 \leq \frac{h_w(x)_j}{w_j} = \left| \frac{h_w(x)_j}{w_j} \right| \leq 1 + \frac{1 - \gamma_w}{2} \leq 3/2 \tag{14}$$

and if $w_j = 0$ then $\left| h_w(x)_j \right| < 1$. As a consequence, the piecewise linear function $\rho : \mathbb{B}^{n-1}(\varepsilon) \mapsto \mathbb{R}$ given by

$$\rho_w(x) = \max\left\{ |h_w(x)_i| \ \text{for } i \text{ with } |w_i| = 1 \right\} \tag{15}$$

is such that $|1 - \rho_w(x)| \leq (1 - \gamma_w)/2$. Consider the piecewise linear function $g_w : \mathbb{B}^{n-1}(\varepsilon) \mapsto \mathbb{R}^n$ given by

$$g_w(x) = h_w(x) + (1 - \rho_w(x))\, w.$$

Since $\rho_w(0) = 1$ we have that $g_w(0) = w$ and we claim that $g_w(x) \in \mathbb{S}^{n-1}$ for all $x \in \mathbb{B}^{n-1}(\varepsilon)$. In fact, if $w_i = 0$ then $|g_w(x)_i| = |h_w(x)_i| < 1$. If $0 < |w_i| < 1$ then

$$|g_w(x)_i| = |w_i| \left| \frac{h_w(x)_i}{w_i} + 1 - \rho_w(x) \right| \leq \gamma_w \left(\frac{h_w(x)_i}{w_i} + |1 - \rho_w(x)| \right) \leq \gamma_w(2 - \gamma_w) \leq 1,$$

because $t(2-t) \leq 1$ for $t \in [0,1]$. If $|w_i| = 1$ then

$$|g_w(x)_i| = |w_i| \left| \frac{h_w(x)_i}{w_i} + 1 - \rho_w(x) \right| = \left| 1 + \left(\frac{h_w(x)_i}{w_i} - \rho_w(x) \right) \right| \leq 1,$$

because $-1 \leq \left(\frac{h_w(x)_i}{w_i} - \rho_w(x) \right) \leq 0$ by equations (14) and (15). Finally, there exists $i(x)$ such that $\left| w_{i(x)} \right| = 1$ and $\rho_w(x) = h_w(x)_{i(x)} / w_{i(x)}$ and for such $i(x)$

$$\left| g_w(x)_{i(x)} \right| = \left| w_{i(x)} \right| \left| \frac{h_w(x)_{i(x)}}{w_{i(x)}} + 1 - \rho_w(x) \right| = 1.$$

Therefore, g_w is a piecewise linear function from $\mathbb{B}^{n-1}(\varepsilon)$ to $\mathbb{S}^{n-1}$. The injectivity of h_w and the orthogonality of w and $h_w(x) - w$ imply that g_w is injective and Brower's invariance of domain implies that $g_w : \mathbb{B}^n(\varepsilon/2) \mapsto g_w(\mathbb{B}^n(\varepsilon/2))$ is a piecewise linear homeomorphism. Consider now the projection P and λ defined in equation (13). Notice that, since $w^t h_w(x) = 0$,

$$\lambda(g_w(x)) = \frac{w^t\,(w - h_w(x) - (1-\rho_w(x))\,w)}{w^t w} = \rho_w(x).$$

The same argument used in lemma 7 shows that if $\delta_w \in (0, \varepsilon/2)$ is small enough then, for $x \in \mathbb{B}^{n-1}(\delta)$,

$$f(g_w(x)) = f(h_w(x)) = x_{n-1}.$$

Therefore, $V_w = g_w\big(\mathbb{B}^{n-1}(\delta_w)\big)$ is a neighborhood of w in $\mathbb{S}^{n-1}$ and taking $h_w : V_w \mapsto \mathbb{R}^{n-1}$ as the inverse of the restriction of g_w to $\mathbb{B}^{n-1}(\delta_w)$ we get that $h_w(V_w \cap f^{-1}(0)) \subset \mathbb{R}^{n-2} \times \{0\}$. As a conclusion, for each $w \in S_1$ we have a neighborhood V_w of w in $\mathbb{S}^{n-1}$ and a piecewise linear homeomorphism $h_w : V_w \mapsto h_w(V_w)$ such that $h_w(V_w \cap S_1) \subset \mathbb{R}^{n-2} \times \{0\}$. This shows that S_1 is a locally flat, piecewise linear, $(n-2)$ sub manifold of $\mathbb{S}^{n-1}$. $\square$

Proof of lemma 8. All ε's below are small and positive and we define $A_\varepsilon = L_{-\varepsilon}$. The sets A_ε are not empty, because $w \in A_\varepsilon$, and proper, because $z \notin A_\varepsilon$. Alexander duality (see page 254 of [4]) applied to A_ε implies that $\tilde{H}_k(\mathbb{S}^m - A_\varepsilon) = \{0\}$ for all k. Since $\mathbb{S}^m - A_\varepsilon$ deformation retracts to $T = \{x \in \mathbb{S}^m \text{ with } f(x) \geq 0\}$ we obtain that $\tilde{H}_k(T) = \{0\}$ for all k. Moreover, $T \cap L_\varepsilon$ deformation retracts to $f^{-1}(0)$. Therefore, it suffices to show that $T \cap L_\varepsilon$ has the same homology as $\mathbb{S}^{m-1}$. The first set in the decomposition

$$\mathbb{S}^m = \{x \in \mathbb{S}^m \text{ with } f(x) > 0\} \cup \{x \in \mathbb{S}^m \text{ with } f(x) < \varepsilon\}$$

is contained in the interior of T and the second lies in the interior of L_ε and the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_{k+1}(T) \oplus \tilde{H}_{k+1}(L_\varepsilon) \to \tilde{H}_{k+1}(\mathbb{S}^m) \to \tilde{H}_k(T \cap L_\varepsilon) \to \tilde{H}_k(T) \oplus \tilde{H}_k(L_\varepsilon) \to \ldots$$

yields $0 \to \tilde{H}_{k+1}(\mathbb{S}^m) \to \tilde{H}_k(T \cap L_\varepsilon) \to 0$. Therefore, $H_k(T \cap L_\varepsilon) = H_{k+1}(\mathbb{S}^m) = H_k\big(\mathbb{S}^{m-1}\big)$. $\square$

A Morse's flawed definition

The definition of T-ordinary point by Morse in page 1 of [9] is flawed. He proposed the following:

Definition 6 *Let M be a topological n manifold and let $f : M \mapsto \mathbb{R}$ be a continuous function. A point $z \in X$ is topologically ordinary, or T-ordinary, if there exists an homeomorphism $h : \mathbb{B}^n \mapsto h(\mathbb{B}^n) \subset M$ as in (1).*

According to this definition no point $z \in \mathbb{R}$ is T-ordinary for the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by

$$f(x) = \frac{1}{9\pi} \arctan(x).$$

In fact, $f(\mathbb{R}) \subset (-2/9, 2/9)$ and there is no pair $y, z \in \mathbb{R}$ such that $f(y) = f(z) + 2/3$ and equation (1) cannot be satisfied by any $y = h(x)$ and z for $x = 2/3 \in \mathbb{B}^1$! This shows that, due to its lack of scale invariance, Morse's definition is misleading.

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